1. The metric of a line of massless particles

We know that the gravity theory that we start with, 11-D M theory, has gravitational waves, which if quantized gives massless quanta called gravitons. If we have a collection of gravitons, then what is the metric that they produce? For simplicity we assume that all gravitons are moving in the direction z, and that they are distributed uniformly along z. Thus the metric will be independent of z and since the distribution of particles remains the same at different times, it will also be independent of the time t.

1.1. The black string metric

Let us first find the metric of a line of *massive* particles, which are at *rest*. The metric of a point mass is the Schwarzschild metric

$$ds^2 = -(1-\frac{B}{r^d})dt^2 + \frac{dr^2}{(1-\frac{B}{r^d})} + r^2 d\Omega_{d+1}^2 \tag{1.1}$$

where B is a constant and the spacetime dimension is $d+3$. We can add on an extra dimension z to get a metric in $d+4$ dimensions

$$ds^2 = -(1-\frac{B}{r^d})dt^2 + \frac{dr^2}{(1-\frac{B}{r^d})} + r^2 d\Omega_{d+1}^2 + dz^2 \tag{1.2}$$

If (1.1) satisfied $R_{ab} = 0$, we can show that (1.2) does too. Since (1.2) is translationally invariant in the direction z, we guess that it represents the metric of massive particles distributed uniformly along z. Since the metric is symmetric under $z \to -z$ these particles are at rest. The metric (1.2) is called a *black string.*

1.2. Boosting the black string

Let us now view the metric (1.2) in a frame that is moving along the z direction

$$t = \cosh\beta\ t' + \sinh\beta\ z', \qquad z = \sinh\beta\ t' + \cosh\beta\ z' \tag{1.3}$$

This gives

$$\begin{aligned} ds^2 =& -[(1-\frac{B}{r^d})\cosh^2\beta - \sinh^2\beta]dt'^2 + [\cosh^2\beta - (1-\frac{B}{r^d})\sinh^2\beta]dz'^2 \\ &+ 2\cosh\beta\sinh\beta(\frac{B}{r^d})dt'dz' + \frac{dr^2}{(1-\frac{B}{r^d})} + r^2 d\Omega_{d+1}^2 \\ =& -[1-\frac{B}{r^d}\cosh^2\beta]dt'^2 + [1+\frac{B}{r^d}\sinh^2\beta]dz'^2 \\ &+ 2\cosh\beta\sinh\beta(\frac{B}{r^d})dt'dz' + \frac{dr^2}{(1-\frac{B}{r^d})} + r^2 d\Omega_{d+1}^2 \end{aligned} \tag{1.4}$$

To get to the limit where we have massless particles moving along z, we take

$$B \to 0, \quad \beta \to \infty, \qquad Be^{\beta} \to 4P = \text{finite, nonzero} \tag{1.5}$$

We then get

$$ds^2 = -dt'^2 + dz'^2 + \frac{P}{r^d}(dt' + dz')^2 + dr^2 + r^2 d\Omega^2_{d+1} \tag{1.6}$$

It is useful to introduce coordinates

$$x^+ = t' + z', \quad x^- = t' - z' \tag{1.7}$$

and also to use Cartesian coordinates to describe the flat metric on the remaining coordinates which brings (1.6) to the form

$$ds^2 = -dx^2 dx^- + \frac{P}{r^d}(dx^+)^2 + dx^i dx^i \tag{1.8}$$

where $i = 1 \ldots d+2$.

1.3. A more general metric

Note that the function $\frac{P}{r^d}$ appearing in (1.8) is a harmonic function with source at $r = 0$, which is a kind of behavior that we have encountered before in our study of brane solutions. We can actually get a more general class of solutions by starting with the ansatz

$$ds^2 = -dx^+ dx^- + f(x^+, x^i) + dx^i dx^i \tag{1.9}$$

The only nontrivial component of R_{ab} is

$$R_{++} = -\frac{1}{2} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^i} f \tag{1.10}$$

Thus to satisfy Einstein equations $R_{ab} = 0$ all we need to ensure is that f is harmonic in the x^i for each value of x^+. (Of course we will violate this condition where there are sources, as at $r = 0$ in (1.8).)

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Problem 1: Find all components R_{ab} for the metric (1.9) and verify (1.10).

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2. The D0 brane of string theory

Let us return to 11-D M theory, where we compactify the direction $\xi^{10} \equiv y$ to get 10-D string theory. Let gravitons of M theory run around the y direction, at the origin $x = 0$ of the 9 noncompact spatial directions. We let the distribution of gravitons be uniform in the y direction, and can thus use the metric (1.8) found above. We thus get

$$ds^2 = -dt^2 + dy^2 + \frac{P}{r^7}(dt + dy)^2 + dx^i dx^i \tag{2.1}$$

where $i = 1 \ldots 9$. We can rewrite this as

$$ds^2 = -\frac{1}{1 + \frac{P}{r^7}} dt^2 + (1 + \frac{P}{r^7})[dy + \frac{\frac{P}{r^7}}{1 + \frac{P}{r^7}} dt]^2 + dx^i dx^i \tag{2.2}$$

which allows us to pick up the parts of the 10-D decription

$$e^C = 1 + \frac{P}{r^7}, \quad e^{2\phi} = e^{\frac{3C}{2}} = (1 + \frac{P}{r^7})^{\frac{3}{2}} \tag{2.3}$$

$$A_t \equiv A_0 = \frac{\frac{P}{r^7}}{1 + \frac{P}{r^7}} \tag{2.4}$$

$$ds_{10}^2 = -\frac{1}{1 + \frac{P}{r^7}} dt^2 + dx^i dx^i \tag{2.5}$$

The gauge field (2.4) has only the part A_0, which at large distances from the origin is

$$A_0 \approx \frac{P}{r^7} \tag{2.6}$$

This is the potential expected from a pointlike electric charge in 10-D. We have thus found the object that carries electric charge under the gauge field A_a. This pointlike object will be called the D0 brane.

We can write the 10-D metric in the string frame

$$g^S_{ab} = e^{\frac{C}{2}} g_{ab} = e^{\frac{2\phi}{3}} g_{ab} \tag{2.7}$$

We get

$$ds_S^2 = -\frac{1}{(1 + \frac{P}{r^7})^{\frac{1}{2}}} dt^2 + (1 + \frac{P}{r^7})^{\frac{1}{2}} dx^i dx^i \tag{2.8}$$