1. The Bianchi identity

We wish to prove the important relation

$$R_{ab[cd;e]} \equiv R_{abcd;e} + R_{abec;d} + R_{abde;c} = 0 \tag{1.1}$$

We prove it point by point for each point for spacetime. This is a tensor relation, so we can prove it in any coordinate system and then it will be valid in all coordinate systems. Thus go to a point x, and choose coordinates around this point such that

$$g_{ab}(x) = \eta_{ab}, \quad g_{ab,c}(x) = 0 \tag{1.2}$$

which implies that

$$\Gamma^a_{bc}(x) = 0 \tag{1.3}$$

Note however that in general

$$g_{ab,cd}(x) \neq 0 \tag{1.4}$$

The derivative in (1.1) is a covariant derivative

$$R_{abcd;e} = R_{abcd,e} + \text{terms involving } \Gamma \tag{1.5}$$

But from (1.3) we see that at x we have simply

$$R_{abcd;e} = R_{abcd,e} \tag{1.6}$$

Recall the explicit expression for the Riemann tensor

$$R_{abcd} = g_{aa'} R^{a'}{}_{bcd} = g_{aa'}(\Gamma^{a'}_{bd,c} - \Gamma^{a'}_{bc,d} + \Gamma^{a'}_{fc}\Gamma^f_{bd} - \Gamma^{a'}_{fd}\Gamma^f_{bc}) \tag{1.7}$$

Since $g_{aa',b}(x) = 0$, $\Gamma^a_{bc}(x) = 0$, we find that when we take the ordinary derivative $R_{abcd,e}$ we get only the following terms

$$\begin{aligned} R_{abcd,e} &= g_{aa'}(\Gamma^{a'}_{bd,ce} - \Gamma^{a'}_{bc,de}) \\ &= \frac{1}{2}(g_{ab,d} + g_{ad,b} - g_{bd,a})_{,ce} - \frac{1}{2}(g_{ab,c} + g_{ac,b} - g_{bc,a})_{,de} \\ &= \frac{1}{2}(g_{ad,bce} - g_{bd,ace} - g_{ac,bde} + g_{bc,ade}) \end{aligned} \tag{1.8}$$

Using (1.6) a little algebra immediately gives (1.1).

2. Relations arising from the Bianchi identity

We contract the indices a, c in (1.1)

$$g^{ac}(R_{abcd;e} + R_{abec;d} + R_{abde;c}) = 0 \tag{2.1}$$

Recall that ${g^{ab}}_{;c} = 0$, and that the operation of covariant differentiation $(;)$ satisfies the product rule. We thus get

$$(g^{ac}R_{abcd})_{;e} + (g^{ac}R_{abec})_{;d} + (g^{ac}R_{abde})_{;c} = R_{bd;e} - R_{be;d} + {R^a}_{bde;a} = 0 \tag{2.2}$$

Contracting the indices b, d we get

$$R_{;e} - 2{R_{be}}^{;b} = 0 \tag{2.3}$$

We define the *Einstein tensor* as

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R \tag{2.4}$$

Then from (2.3) we see that

$${G_{ab}}^{;b} = 0 \tag{2.5}$$

This property will be very important in writing the Einstein field equations for gravity.

3. Einstein field equations

In the absence of matter we had argued that the field equations determining the metric should be

$$R_{ab} = 0 \tag{3.1}$$

In the presence of matter, we are looking for equations that schematically have the form

$$\text{curvature} \;=\; \text{matter} \tag{3.2}$$

Matter is characterized by the energy momentum tensor T_{ab}. We might therefore try to write

$$R_{ab} = (\text{const})T_{ab} \quad (??) \tag{3.3}$$

But we have seen that conservation of energy-momentum gives

$${T_{ab}}^{;b} = 0 \tag{3.4}$$

But ${R_{ab}}^{;b}$ is *not* zero in general. Recall however from (2.5) that we *do* have ${G_{ab}}^{;b} = 0$. Thus we write

$$G_{ab} = 8\pi G\, T_{ab} \tag{3.5}$$

where we have written the constant as $8\pi G$; this will make sense later when we look at the Newtonian limit of Einstein's theory where the strength of gravitational interaction is described by Newton's gravitational constant G.