THE METRIC OF BRANES

We have seen that a very basic solution to Einstein's equations is the Schwarzschild metric: the metric produced by a point mass. This metric is spherically symmetric in the space directions. We can however imagine a source that is not pointlike but 'linelike', the metric will then have axial symmetry around this line. Similarily we can let the source be in the form of a 'sheet' or 'brane' of some dimensionality: let this dimension be called P. The worldvolume of this source will include time, and will thus be a $P+1$ dimensional hypersurface in spacetime.

While we can write down the solutions to such extended sources in general, a particular subclass of such solutions is of particular importance to the development of string theory. We can imagine that the brane carries a charge (of a kind to be specified in more detail below). Then the solutions will be the analogue of the Reissner-Nordstrom solutions rather than the Schwarzschild solution. We will in fact be interested in the special case where the solution is 'extremal' i.e. it has the maximal charge for its mass. This would thus be the analogue of the extremal Reissner Norstrom black hole. We will make an ansatz for the solution, find that the ansatz allows for a solution, and then analyze the remakable properties of the emerging solutions.

1. The metric produced by branes

Let us consider a spacetime that has a total dimension D. Consider the Ansatz for the metric

$$ds^2 = f(x)\sum_{a=1}^{q} dy_a dy_a + h(x)\sum_{i=1}^{p} dx_i dx_i \tag{1.1}$$

where

$$p + q = D \tag{1.2}$$

We assume that the time direction is among the directions y_a, but for convenience we take this to be a direction with Euclidean signature for the moment; later we can rotate this direction to Lorentzian signature since there is no dependence on the y_a in the coefficients in the metric. Note that we have also assumed here that the time direction will have a metric that is symmetrically related to the other directions of the brane worldvolume: thus if we find a solution with this ansatz then we will have a Lorentz symmetry among all the brane worldvolume directions.

2. The Reissner Nordstrom black hole

The metric of the Reissner Nordstrom hole in 3+1 spacetime dimensions is

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.1}$$

The extreme case has $Q = M$, which gives

$$ds^2 = -(1 - \frac{M}{r})^2 dt^2 + \frac{dr^2}{(1 - \frac{M}{r})^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.2}$$

The direction t is along the worldvolume of the point source, while the directins r, θ, ϕ are transverse to the worldvolume. As it stands this metric does not appear to be of the form (1.1), since the metric in the directions r, θ, ϕ is not manifestly flat. But we make a change of coordinates

$$r' = r - M \tag{2.3}$$

which gives

$$1 - \frac{M}{r} = \frac{1}{1 + \frac{M}{r'}} \tag{2.4}$$

$$dr = dr' \tag{2.5}$$

$$r^2 = r'^2(1 + \frac{M}{r'})^2 \tag{2.6}$$

and the metric becomes

$$ds^2 = -(1 + \frac{M}{r'})^{-2} dt^2 + (1 + \frac{M}{r'})^2 [dr'^2 + r'^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{2.7}$$

and the metric now indeed has the form of the ansatz (1.1).

3. Connections and curvature

The nonzero connection coefficients for the metric (1.1) are

$$\Gamma^a_{bi} = \frac{1}{2}\delta_{ab} f^{-1} f_{,i} \tag{3.1}$$

$$\Gamma^i_{ab} = -\frac{1}{2}\delta_{ab} h^{-1} f_{,i} \tag{3.2}$$

$$\Gamma^i_{jk} = \frac{1}{2}h^{-1}[\delta_{ij}h_{,k} + \delta_{ik}h_{,j} - \delta_{jk}h_{,i}] \tag{3.3}$$

Now we compute the components of the Ricci tensor. We have

$$R_{ab} = R^c{}_{acb} + R^i{}_{aib} \tag{3.4}$$

$$\begin{aligned} R^c{}_{acb} &= \Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^c_{c\mu}\Gamma^\mu_{ab} - \Gamma^c_{b\mu}\Gamma^\mu_{ac} \\ &= \Gamma^c_{ci}\Gamma^i_{ab} - \Gamma^c_{bi}\Gamma^i_{ac} \\ &= -\frac{1}{4}(q-1)\delta_{ab} f^{-1}h^{-1}f_{,i}f_{,i} \end{aligned} \tag{3.5}$$

$$\begin{aligned} R^i{}_{aib} &= \Gamma^i_{ab,i} - \Gamma^i_{ai,b} + \Gamma^i_{i\mu}\Gamma^\mu_{ab} - \Gamma^i_{b\mu}\Gamma^\mu_{ai} \\ &= \Gamma^i_{ab,i} + \Gamma^i_{ik}\Gamma^k_{ab} - \Gamma^i_{bc}\Gamma^c_{ai} \\ &= \delta_{ab}[-\frac{1}{2}(h^{-1}f_{,k})_{,k} - \frac{p}{4}h^{-2}f_{,k}h_{,k} + \frac{1}{4}h^{-1}f^{-1}f_{,k}f_{,k}] \end{aligned} \tag{3.6}$$

Thus

$$R_{ab} = \delta_{ab}[-\frac{1}{2}(h^{-1}f_{,k})_{,k} - \frac{q-2}{4}f^{-1}h^{-1}f_{,k}f_{,k} - \frac{p}{4}h^{-2}f_{,k}h_{,k}] \tag{3.7}$$

Similarily.

$$R_{ij} = R^a{}_{iaj} + R^k{}_{ikj} \tag{3.8}$$

$$R^a{}_{iaj} = -\frac{q}{2}(f^{-1}f_{,i})_{,j} + \frac{q}{4}h^{-1}f^{-1}[f_{,i}h_{,j} + h_{,i}f_{,j} - \delta_{ij}f_{,k}h_{,k}] - \frac{q}{4}f^{-2}f_{,i}f_{,j} \tag{3.9}$$

$$\begin{aligned} R^k{}_{ikj} &= \frac{1}{2}[(h^{-1}h_{,i})_{,j} + (h^{-1}h_{,j})_{,i} - \delta_{ij}(h^{-1}h_{,k})_{,k}] - \frac{p}{2}(h^{-1}h_{,i})_{,j} \\ &\quad + \frac{p-2}{4}[h^{-2}h_{,i}h_{,j} - \delta_{ij}h^{-2}h_{,k}h_{,k}] \end{aligned} \tag{3.10}$$

$$\begin{aligned} R_{ij} = &-\frac{q}{2}(f^{-1}f_{,i})_{,j} + \frac{q}{4}h^{-1}f^{-1}[f_{,i}h_{,j} + f_{,j}h_{,i} - \delta_{ij}f_{,k}h_{,k}] - \frac{q}{4}f^{-2}f_{,i}f_{,j} \\ &+ \frac{1}{2}[(h^{-1}h_{,i})_{,j} + (h^{-1}h_{,j})_{,i} - \delta_{ij}(h^{-1}h_{,k})_{,k}] - \frac{p}{2}(h^{-1}h_{,i})_{,j} \\ &+ \frac{p-2}{4}[h^{-2}h_{,i}h_{,j} - \delta_{ij}h^{-2}h_{,k}h_{,k}] \end{aligned} \tag{3.11}$$

4. The stress tensor

Consider a q-form field $A_{\mu_1\dots\mu_q}$. Thus A is totally antisymmetric in its indices. From this gauge field we make the field strength

$$\begin{aligned} F_{\mu_1\mu_2\dots\mu_{q+1}} \equiv &\partial_{\mu_1}A_{\mu_2\dots\mu_q\mu_{q+1}} + (-1)^q\partial_{\mu_2}A_{\mu_3\dots\mu_{q+1}\mu_1} + \partial_{\mu_3}A_{\mu_4\dots\mu_{q+1}\mu_1\mu_2} \\ &+ \dots + (-1)^q\partial_{\mu_{q+1}}A_{\mu_1\dots\mu_q} \end{aligned} \tag{4.1}$$

Thus F is totally antisymmetric in its indices too. The action is

$$S = -\frac{1}{2} \int d^D \xi \frac{1}{(q+1)!} \sqrt{-g} F^2 \tag{4.2}$$

Thus the stress tensor is

$$T_{\mu\nu} = \frac{1}{q!} F_{\mu\mu_1 \ldots \mu_q} F_\nu{}^{\mu_1 \ldots \mu_q} - \frac{1}{2} g_{\mu\nu} \frac{1}{(q+1)!} F^2 \tag{4.3}$$

Let us consider the ansatz for the gauge field that will be compatible with our choice of ansatz for the metric. We will have

$$A_{12\ldots q} = A(x) \tag{4.4}$$

where the subscripts correspond to the coordinates y_a. Thus the field strength will be

$$F_{i12\ldots q} = A_{,i} \tag{4.5}$$

where $i = 1, \ldots p$ is an index for the x coordinates.

In this ansatz we get

$$T_{ab} = \delta_{ab} h^{-1} f^{-(q-1)} A_{,i} A_{,i} - \frac{1}{2} \delta_{ab} f h^{-1} f^{-q} A_{,i} A_{,i} = \frac{1}{2} \delta_{ab} h^{-1} f^{-(q-1)} A_{,i} A_{,i} \tag{4.6}$$

$$T_{ij} = A_{,i} A_{,j} f^{-q} - \frac{1}{2} \delta_{ij} h h^{-1} f^{-q} A_{,k} A_{,k} = f^{-q} [A_{,i} A_{,j} - \frac{1}{2} \delta_{ij} A_{,k} A_{,k}] \tag{4.7}$$

$$T = \frac{q+2-p}{2} f^{-q} h^{-1} A_{,k} A_{,k} \tag{4.8}$$

We will also need

$$T_{ab} - \frac{1}{p+q-2} g_{ab} T = \delta_{ab} h^{-1} f^{-(q-1)} A_{,i} A_{,i} \frac{p-2}{p+q-2} \tag{4.9}$$

$$T_{ij} - \frac{1}{p+q-2} g_{ij} T = f^{-q} [A_{,i} A_{,j} - \frac{q}{p+q-2} \delta_{ij} A_{,k} A_{,k}] \tag{4.10}$$

5. Comparing coefficients

For the solution we choose an ansatz

$$h = f^{\nu} \tag{5.1}$$

$$A = Cf^{\frac{q}{2}} \tag{5.2}$$

where the power in the second equation follows by looking at the powers of f in $G_{ij} = T_{ij}$.

Einstein's equations read (we have set the constant $8\pi G$ to unity)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \tag{5.3}$$

Thus we have

$$R(1 - \frac{D}{2}) = T, \quad R = \frac{2T}{2-D} \tag{5.4}$$

and the Einstein's equations can also be written as

$$R_{\mu\nu} = T_{\mu\nu} + \frac{1}{2}\frac{2T}{2-D} = T_{\mu\nu} - \frac{1}{D-2}T \tag{5.5}$$

We can already compare some coefficients:

5.1. $f_{,ij}$

First we examine the equation

$$G_{ij} = R_{ij} - \frac{1}{2}R = T_{ij} \tag{5.6}$$

With our ansatz for the solution, we see that only the first term on the LHS can have such a term. The coefficient is

$$f^{-1}[-\frac{q}{2} + \frac{2\nu}{2} - \frac{p\nu}{2}] \tag{5.7}$$

Thus we need

$$\nu = \frac{q}{2-p} \tag{5.8}$$

Let us verify that this works with our example of the Reissner-Nordstrom solution in 3+1 dimenions. We get

$$\nu = -1 \tag{5.9}$$

which is correct.

Note that we have

$$h_{,ij} = f^{\nu}_{,ij} = \nu[f^{\nu-1}f_{,i}]_{,j} = \nu f^{\nu-1}f_{,ij} + \nu(\nu-1)f^{\nu-2}f_{,i}f_{,j} \tag{5.10}$$

so that we get an extra contribution towards $f_{,i}f_{,j}$.

The equation $G_{ab} = T_{ab}$ does not have term $f_{,ij}$, so we get no more relations from the coefficient of such a term.

5.2. $f_{,kk}$

First we look at (5.6). Again the RHS does not have such a term. The LHS gives a coefficient

$$\delta_{ij} f^{-1}[-\frac{1}{2}\nu - \frac{1}{2}[-\frac{q}{2} - \frac{q}{2}\nu(2-p) - \nu\frac{p}{2}]] = \delta_{ij} f^{-1}[\frac{q}{4} + \frac{\nu}{4}[-2 + q(2-p) + p]] \qquad (5.11)$$

We do not need to require this to be identically zero. Rather, what we require is that

$$f = v^{\mu}, \quad f_{,k} = \mu v^{\mu-1} v_{,k}, \quad f_{,kk} = \mu(\mu-1)v^{\mu-2} v_{,k} v_{,k} + \mu v^{\mu-1} v_{,kk} \qquad (5.12)$$

and we require that

$$v_{kk} = 0 \qquad (5.13)$$

Then we will have

$$f_{kk} = \frac{\mu - 1}{\mu} f^{-1} f_{,k} f_{,k} \equiv \lambda f^{-1} f_{,k} f_{,k} \qquad (5.14)$$

Since we will also have $h = v^{\nu\mu}$, we will get

$$h_{,kk} = \tilde{\lambda} h^{-1} h_{,k} h_{,k} \qquad (5.15)$$

with

$$\tilde{\lambda} = \frac{\mu\nu - 1}{\mu\nu} \qquad (5.16)$$

Thus we get no new relation as such from here.

5.3. $f_{,i} f_{,j}$

Let us examine (5.6). Only the first term on the LHS, and the RHS, can have such a term. We get from the LHS

$$\begin{aligned} &f^{-2}[\frac{q}{2} + \frac{(2\nu-1)q}{4} - \frac{(2-p)\nu^2}{2} + \frac{(p-2)\nu^2}{4}] + \nu(\nu-1)[\frac{2}{2} - \frac{p}{2}] \\ =&f^{-2}[\frac{(2\nu+1)q}{4} + \frac{3(p-2)\nu^2}{4} + \nu(\nu-1)\frac{2-p}{2}] \end{aligned} \qquad (5.17)$$

The RHS gives

$$f^{-2} C^2 \frac{q^2}{4} \qquad (5.18)$$

so we get the relation

$$(2\nu+1)q + 3(p-2)\nu^2 - 2\nu(\nu-1)(p-2) = C^2 q^2 \qquad (5.19)$$

This equation says

$$C^2 = -\frac{p+q-2}{q(p-2)} \qquad (5.20)$$

There is no other equation with this term.

6. $f_{,k}f_{,k}$

Let us first look at the equation

$$R_{ab} = T_{ab} - \frac{1}{q+p-2}T \tag{6.1}$$

We get

$$\frac{\nu}{2} - \frac{\lambda}{2} - \frac{q-2}{4} - \frac{p\nu}{4} = C^2 \frac{q^2}{4} \frac{p-2}{p+q-2} \tag{6.2}$$

From (1.2) we get

$$\frac{-q\nu}{4} + \frac{\nu^2}{2} - \frac{\tilde{\lambda}\nu^2}{2} - \frac{(p-2)\nu^2}{4} = -\frac{q}{p+q-2} C^2 \frac{q^2}{4} \tag{6.3}$$

The solution to these equations is

$$C^2 = -\frac{q+p-2}{q(p-2)} \tag{6.4}$$

$$\lambda = \frac{q}{2} + 1, \quad \rightarrow \quad \mu = -\frac{2}{q} \tag{6.5}$$

$$\tilde{\lambda} = \frac{4-p}{2}, \quad \rightarrow \quad \tilde{\mu} = \mu\nu = \frac{2}{p-2} \tag{6.6}$$

7. The solution

Thus we finally find that the metric produced by the branes is

$$ds^2 = v^{-\frac{2}{q}} dy_a dy_a + v^{\frac{2}{p-2}} dx_i dx_i \tag{7.1}$$

while the gauge potential is

$$A_{a_1 \ldots a_q} = [-\frac{p+q-2}{q(p-2)}]^{1/2} v^{-1} \tag{7.2}$$

Here $v(x)$ is any function of x that is harmonic with the flat metric on x, $v_{,ii} = 0$.

It may appear strange that A has an imaginary coefficient, but we note that we have here done a continuation to Euclidean signature

$$t = i\tilde{t} \tag{7.3}$$

where $\tilde{t}$ was the original time in the Lorentzian signature spacetime. When we write the result back in terms of $\tilde{t}$, we will also have

$$A_{\tilde{t}y_2\ldots y_q} = A_{ty_2\ldots y_q}\frac{\partial t}{\partial \tilde{t}} = iA_{ty_2\ldots y_q} \tag{7.4}$$

so that we will finally obtain a real gauge field in the Lorentzian signature spacetime.

Let us examine some particular solutions of this metric that will be relevant to string theory. In 11-d supergravity, the basic sources arise from the 2-brane and the 5-brane. The 2-brane has a 3-d worldvolume, so we have $q = 3$ and $p = 8$. The metric will thus be

$$ds^2 = v^{-\frac{2}{3}}dy_a dy_a + v^{\frac{1}{3}}dx_i dx_i \tag{7.5}$$

The 5-brane has a 6-d worldvolume, so we will have $q = 6$, $p = 5$, and the metric will be

$$ds^2 = v^{-\frac{1}{3}}dy_a dy_a + v^{\frac{2}{3}}dx_i dx_i \tag{7.6}$$