1. Cosmological expansion

From Einstein's equations we see that if there is any matter in our spacetime, the geometry will in general be curved. If we assume that the spacetime is spatially homogeneous, then this curvature will manifest itself in a uniform expansion of the spatial sections of the spacetime.

1.1. The metric

Let us consider the following simple form of the meric

$$ds^2 = -dt^2 + a^2(t)dx_i dx^i \tag{1.1}$$

where the index i runs over $1 \dots d$, and indices μ, ν will be taken to run over $0, 1 \dots d$ with $x^0 = t$. Let us write $A(t) \equiv a^2(t)$.

Note that we could have written the apparently more general metric

$$ds^2 = -b^2(t)dt^2 + a^2(t)dx_i dx^i \tag{1.2}$$

But we could define

$$dt' = b(t)dt, \quad \rightarrow \quad t' = \int dt b(t) \tag{1.3}$$

and the metric would reduce to the form (1.1).

It is easy to see that the connection coefficients are zero unless there is precisely one index 0 in $\Gamma^{\mu}_{\nu\lambda}$, since the only nonzero derivatives of the metric are time derivatives, and $g_{0i} = 0$, and g_{00} is a constant. Thus we get for the nonzero connection symbols

$$\Gamma^0_{ij} = \frac{1}{2}\delta_{ij}\dot{A} \tag{1.4}$$

$$\Gamma^i_{0j} = \frac{1}{2}A^{-1}\delta_{ij}\dot{A} \tag{1.5}$$

Let us compute the Ricci tensor. We get

$$\begin{aligned} R_{00} =& R^i{}_{0i0} = \Gamma^i_{00,i} - \Gamma^i_{0i,0} + \Gamma^i_{i\mu}\Gamma^\mu_{00} - \Gamma^i_{0\mu}\Gamma^\mu_{0i} = -\Gamma^i_{0i,0} + \Gamma^i_{i0}\Gamma^0_{00} - \Gamma^i_{0k}\Gamma^k_{0i} \\ &= -\delta^i_i \frac{1}{2}(\frac{\dot{A}}{A})^{\cdot} - \delta^i_k \frac{1}{2}\frac{\dot{A}}{A}\delta^k_i \frac{1}{2}\frac{\dot{A}}{A} = -\frac{d}{2}(\frac{\dot{A}}{A})^{\cdot} - \frac{d}{4}(\frac{\dot{A}}{A})^2 = -\frac{d}{2}\frac{\ddot{A}}{A} + \frac{d}{4}(\frac{\dot{A}}{A})^2 \end{aligned} \tag{1.6}$$

We also have

$$R_{ij} = R^0{}_{i0j} + R^k{}_{ikj} \tag{1.7}$$

By symmetry, we can write

$$R^0{}_{i0j} = Q\delta_{ij} \tag{1.8}$$

which gives after tracing over i, j

$$Qd = R^0{}_{i0i} = -R_{0i0i} = -AR^i{}_{0i0} = -AR_{00} \tag{1.9}$$

so that we get

$$R^0{}_{i0j} = \delta_{ij} A[\frac{1}{2}\frac{\ddot{A}}{A} - \frac{1}{4}(\frac{\dot{A}}{A})^2] \tag{1.10}$$

Now we compute

$$\begin{aligned} R^k{}_{ikj} =&\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{k\mu}\Gamma^\mu_{ij} - \Gamma^k_{j\mu}\Gamma^\mu_{ik} = \Gamma^k_{k0}\Gamma^0_{ij} - \Gamma^k_{j0}\Gamma^0_{ik} \\ =& \delta^k_k \frac{1}{2}\frac{\dot{A}}{A}\delta_{ij}\frac{1}{2}\dot{A} - \delta^k_j \frac{1}{2}\frac{\dot{A}}{A}\delta_{ik}\frac{1}{2}\dot{A} = \delta_{ij}\frac{(d-1)}{4}\frac{(\dot{A})^2}{A} \end{aligned} \tag{1.11}$$

Thus we get

$$R_{ij} = \delta_{ij}[\frac{(d-2)}{4}\frac{(\dot{A})^2}{A} + \frac{1}{2}\ddot{A}] \tag{1.12}$$

Thus the Ricci scalar is

$$\begin{aligned} R &= R_{00}g^{00} + R_{ij}g^{ij} = -R_{00} + A^{-1}R_{kk} \\ &= \frac{d}{2}\frac{\ddot{A}}{A} - \frac{d}{4}(\frac{\dot{A}}{A})^2 + \frac{(d-2)d}{4}(\frac{\dot{A}}{A})^2 + \frac{d}{2}\frac{\ddot{A}}{A} = d\frac{\ddot{A}}{A} + \frac{(d-3)d}{4}(\frac{\dot{A}}{A})^2 \end{aligned} \tag{1.13}$$

Thus the components of the Einstein tensor are

$$G_{00} = R_{00} - g_{00}\frac{1}{2}R = \frac{(d-1)d}{8}(\frac{\dot{A}}{A})^2 \tag{1.14}$$

In particular if we put $d = 3$, then we get

$$G_{00} = \frac{3}{4}(\frac{\dot{A}}{A})^2 = 3(\frac{\dot{a}}{a})^2 \tag{1.15}$$

Note the important fact that this expression does not have any second order derivatives of A; they have all canceled out. Thus if there were no matter, and we had $G_{00} = 0$, then we would be forced to $\dot{A} = 0$, which means that the Universe cannot expand. By contrast, if we had a secoond order derivative of A in G_{00}, then this equation would have allowed solutions that had nontrivial time dependence even in the absence of matter. Of course it is still possible that the other equations would have then disallowed such solutions, but here we see that the equation for G_{00} itself imposes the contraint that relates the expansion rate to T_{00} in general.

1.2. The stress-tensor

We assume that the matter is dust, at rest in the frame that we have chosen. Then the stress tensor is

$$T^{\mu\nu} = nmU^\mu U^\nu \equiv \rho_0 U^\mu U^\nu \tag{1.16}$$

where we have defined the mass density in the rest frame $\rho_0 = nm$. We have only the component U^0 for the 4-velocity, and from the normalisation of the 4-velocity we get

$$(U^0)^2 g_{00} = -1, \quad \rightarrow \quad U^0 = 1 \tag{1.17}$$

where we have made the choice of sign of the square root so as to have the 4-velocity point forwards in time.

Let us now investigate the conservation law for this stress tensor. We need

$$0 = T^{0\mu}_{;\mu} = T^{00}_{;0} + T^{0i}_{;i} = T^{00}_{,0} + \Gamma^i_{0i} T^{00} = T^{00}_{,0} + \frac{d}{2}\frac{\dot{A}}{A} T^{00} \tag{1.18}$$

This gives

$$0 = (\rho_0)_{,0} + d\frac{\dot{a}}{a}\rho_0 = a^{-d}(\rho_0 a^d)_{,0} \tag{1.19}$$

so that we have

$$\rho_0 a^d = C, \quad \rho_0 = \frac{C}{a^d} \tag{1.20}$$

1.3. The solution

Einsteins equation

$$G_{00} = 8\pi G T_{00} \tag{1.21}$$

now gives

$$\frac{(d-1)d}{2}(\frac{\dot{a}}{a})^2 = \frac{8\pi GC}{a^d} \tag{1.22}$$

or

$$a^{-1+d/2}\dot{a} = (\frac{16\pi GC}{d(d-1)})^{1/2} \tag{1.23}$$

which has the solution

$$a^{d/2} = (\frac{8\pi GC}{(d-1)})^{1/2}(t - t_0) \tag{1.24}$$

In particular for $d = 3$ we get

$$a(t) = (\frac{8\pi GC}{(d-1)})^{1/3}\ t^{2/3} \tag{1.25}$$

where we have chosen the origin of time to simplify the expression. Thus we see that we get an expanding Cosmology, with a rate of expansion that is governed by the latter density that we have at some given point of time that we may choose as a reference point. We could have obtained a contracting cosmology, if we had chosen the negative sign of the square root in passing from (1.22) to (1.23).