1. The 9-11 flip

S-duality is a symmetry of type IIB string theory, where we find two different 1-branes – the NS1 and the D1 strings. We have no such symmetry for type IIA. But given that any T-duality on IIB gives IIA, we expect that there must be some reflection of the idea of S-duality in the IIA theory as well.

Take IIB string theory, with the coordinates $x^0, x^1 \dots x^9$. Compactify x^9 on a circle, so that we can do T-dualities.

Consider a NS1 string wrapped on x^9. Upon T_9 we get a momentum mode in IIA along x^9

$$T_9 : \quad NS1_9 \;\; (IIB) \;\rightarrow\; P_9 \;\; (IIA) \tag{1.1}$$

If on the other hand we had started with the S-dual of the NS1

$$S : \quad NS1_9 \;\; (IIB) \;\rightarrow\; D1 \;\; (IIB) \tag{1.2}$$

then we get

$$T_9 : \quad D1_9 \;\; (IIB) \;\rightarrow\; D0 \;\; (IIA) \tag{1.3}$$

The right hand sides of (1.1), (1.3) look more symmetrical when we note that IIA is best seen as a reduction of M theory compactified on a circle. For the 11-d M theory we use coordinates $x^0, x^1, \dots x^9, x^{11}$. (It is traditional to not use the symbol x^{10} for the last coordinate, in part because if we looked at Euclidean 10-d physics we would like to use coordinates $x^1 \dots x^{10}$.) Then we see

$$T_9 : \quad NS1_9 \;\; (IIB) \;\rightarrow\; P_9 \;\; (M) \tag{1.4}$$

$$T_9 : \quad D1_9 \;\; (IIB) \;\rightarrow\; P_{11} \;\; (M) \tag{1.5}$$

where in (1.5) we have recalled that the D0 brane of IIA is just a momentum mode along x^{11}.

We thus see that the action of S-duality in IIB results in an interchange of the *two* compact directions of M theory – i.e. we get a '9-11 flip'. More explicitly, suppose we start with any configuration of branes in 10-d IIB theory, and have one direction, say x^9 compact. T duality along x^9 will get us to IIA, which we will immediately regard as 11-d M theory. But the M theory has two compact directions, x^9 since this was compact in the starting IIB configuration, and x^{11}, which is compact anyway if we have IIA string theory. The result of the T-duality will thus be some brane configuration C_1 in M theory.

If however we first apply S-duality to the starting brane configuration in IIB, then T_9 will give us some other configuration C_2 in M theory. The claim then is that C_2 is obtained from C_1 by interchanging the coordinates x^9 and x^{11} in M theory. This may not look like a big change in M theory, but in going to IIA string theory we will break the symmetry between x^9, x^{11} since we will reduce along x^{11}. In terms of the branes obtained in IIA therefore we get quite a difference, as seen from (1.4),(1.5).

2. T-duality on the NS5 brane

We have obtained the effect of T-duality on all objects except the NS5 brane. Using the 9-11 flip we can now fill in this last step.

2.1. T-duality along the NS5 of IIB

Start with IIB, and compactify 5 directions $x^5, \ldots x^9$. Wrap a D5 brane on this T^5. We have

$$T_9: \quad D5_{56789} \;\; (IIB) \;\rightarrow\; D4_{5678} \;\; (IIA) \;=\; M5_{567811} \;\; (M) \tag{2.1}$$

If we first do an S-duality

$$S: \quad D5_{56789} \;\; (IIB) \;\rightarrow\; NS5_{56789} \;\; (IIB) \tag{2.2}$$

Then using the 9-11 flip we should get

$$T_9: \quad NS5_{56789} \;\; (IIB) \;\rightarrow\; M5_{56789} \;\; (M) \;=\; NS5_{56789} \;\; (IIA) \tag{2.3}$$

Thus we see that T-duality along the NS5 of IIB takes us to the NS5 of IIA, wrapped along the same directions.

2.2. T-duality perpendicular to the NS5 of IIB

Start with IIB, and compactify 6 directions $x^4, x^5, \ldots x^9$. Wrap a D5 on $x^5 \ldots x^9$. Then we have

$$T_9: \quad D5_{45678} \;\; (IIB) \;\rightarrow\; D6_{456789} \;\; (IIA) \tag{2.4}$$

But the D6 brane of IIA arises from a KK monopole in M theory. The compact circle x^{11} becomes fibered nontrivially over the angular S^2 in the directions x^1, x^2, x^3. We write this as

$$D6_{456789} \;\; (IIA) \;=\; KK_{11;123} \;\; (M) \tag{2.5}$$

If we first do an S-duality in IIB

$$S: \quad D5_{45678} \;\; (IIB) \; \rightarrow \; NS5_{45678} \;\; (IIB) \tag{2.6}$$

then we get by the 9-11 flip

$$T_9: \quad NS5_{45678} \;\; (IIB) \; \rightarrow \; KK_{9;123} \;\; (M) \tag{2.7}$$

The circle x^{11} is not involved in this KK monopole, and reduction to IIA thus gives a KK monopole in IIA,

$$T_9: \quad NS5_{45678} \;\; (IIB) \; \rightarrow \; KK_{9;123} \;\; (IIA) \tag{2.8}$$

The solution is homogeneous along the directions $x^4, \ldots x^8$. We see that if we T-dualize in a direction x^9 perpendicular to the NS5 of IIB then this direction x^9 gets fibered nontrivially over the directions transverse of the NS5, and we get a KK monopole solution of 10-d gravity in IIA.

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Problem 1: Write down explicitly the 10-d string metric for the geometry $KK_{9;123}$.

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2.3. T-dualities on the NS5 of IIA

Since any T-duality squares to unity $T^2 = 1$ we get from (2.3)

$$T_9: \quad NS5_{56789} \;\; (IIA) \; \rightarrow \; NS5_{56789} \;\; (IIB) \tag{2.9}$$

To find the effect of a T-duality perpendicular to the NS5 we use the fact that T-dualities commute. Thus taking any direction along the NS5, say x^8, we have $T_8 T_9 T_8 = T_9$. Starting with $NS5_{45678}$ in IIA, the T_8 takes us to $NS5_{45678}$ in IIB. The T_9 then gives $KK_{9;123}$. The string metric of this KK monopole is independent of x^8, and the last T_8 thus gives no change to the metric. We thus get $KK_{9;123}$ in IIB. Summarizing,

$$T_9: \quad NS5_{56789} \;\; (IIA) \; \rightarrow \; NS5_{56789} \;\; (IIB) \tag{2.10}$$

$$T_9: \quad NS5_{45678} \;\; (IIA) \; \rightarrow \; KK_{9;123} \;\; (IIB) \tag{2.11}$$

3. Collecting the rules for T and S dualities

Let us collect together all the rules of T-duality. The branes of IIA are

$$\begin{array}{lccccccc} Type\ IIA: & & NS1 & & & NS5 & & \\ & D0 & & D2 & D4 & & D6 \end{array} \tag{3.1}$$

We also have the momentum mode P along any compact direction. The branes of IIB that we have discussed so far are

$$\begin{array}{lccc} Type\ IIB: & NS1 & & NS5 \\ & D1 & D3 & D5 \end{array} \tag{3.2}$$

and we again have the momentum modes P.

Under the T-duality T_k along the direction x^k we have the following:

(a) The string NS1 extended along x^k goes to a momentum mode along P_k along x^k, and P_k goes to $NS1$ along x^k. If the NS1 does not extend along x^k then it is unaffected by the duality, and similarly if P is not along x^k then it is not affected by the duality.

(b) If we started with a D-p-brane extending along x^k then we get a D-(p-1)-brane *not* extending along x^k; if we started with a D-p-brane not extended along x^k then we get a D-(p+1)-brane which *does* extend along x^k.

(c) If we start with NS5 along x^k then we get NS5 after the duality, if we start with NS5 *not* along x^k then we get a KK monopole solution in 10-d, with x^k giving the circle that is fibered nontrivially over the directions perpendicular to the NS5.

In IIB we have S-duality, which has the action on branes

$$\begin{aligned} NS1 &\leftrightarrow D1 \\ NS5 &\leftrightarrow D5 \end{aligned} \tag{3.3}$$

while D3 and momentum modes P go to themselves.

4. Duality chasing

The above dualities are very powerful, and map a given configuration of branes to a wide variety of other configurations. Finding these duality related systems is often called *duality chasing*. As an example we can start with a D5 in IIB, and map it to a momentum mode in IIB

$$\begin{aligned} D5_{56789}\ (IIB)\ &\to T_6T_7T_8T_9 \to D1_5\ (IIB)\ \to S \to NS1_5\ (IIB) \\ &\to T_5 \to P_5\ (IIA)\ \to T_6 \to P_5\ (IIB) \end{aligned} \tag{4.1}$$

We can also take a pair of branes, and perform dualities to map them to another pair. An example that we will need later is the following. Start with IIB, compactify $x^5 \dots x^9$, wrap D5 branes on $x^5 \dots x^9$ and D1 branes along x^5. This is called the D1-D5 system. We can map this by dualities to the NS1-P system, where the D5 branes turn into NS1 along x^5, and the D1 branes turn into P along x^5

$$\begin{aligned} D5_{56789}D1_5\ (IIB)\ &\to S \to NS5_{56789}NS1_5\ (IIB)\ \to T_5 \to NS5_{56789}P_5\ (IIA) \\ &\to T_6 \to NS5_{56789}P_5\ (IIB) \to S \to D5_{56789}P_5\ (IIB) \\ &\to T_6T_7T_8T_9 \to D1_5P_5\ (IIB) \to S \to NS1_5P_5\ (IIB) \end{aligned} \tag{4.2}$$

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Problem 2: Consider IIB string theory, and compactify $x^5 \dots x^9$. Wrap D5 branes on $x^5 \dots x^9$, wrap D1 branes on x^5, and take P along x^5. This is called the D1-D5-P system, and is of great interest in black hole physics.

For this system show that by duality maps we can take D1-D5-P into any of the other 5 permutations, e.g. we can map D1-D5-P to D5-P-D1.

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