1. The action for branes

In ordinary electromagnetism we have a potential A_a. The action of a point particle moving in this potential is given by

$$S = S_{DBI} + S_{CS} \tag{1.1}$$

Here

$$S_{DBI} = m \int ds \tag{1.2}$$

where ds is the length of a segment along the worldline of the particle. We can choose a parameter λ to label different points along the world line. The world line is then described by giving the spacetime coordinate x of the particle at each λ, i.e. the functions $x^a(\lambda)$. Then we have

$$ds = [\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}]^{\frac{1}{2}} d\lambda \tag{1.3}$$

and

$$S_{DBI} = m \int [\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}]^{\frac{1}{2}} d\lambda \tag{1.4}$$

The part S_{CS} represents the effect of the electromagnetic potential. Let q be the charge of the particle. Then

$$S_{CS} = q \int A_a(x) dx^a = q \int A_a(x) \frac{dx^a}{d\lambda} d\lambda \tag{1.5}$$

Thus for each small segment along the world line we take the vector dx^a describing the segment and contract this with the covariant vector A_a at the location x of the segment.

Now suppose that instead of a point particle we had a 1-dimensional extended object, i.e. a string. Then time evolution causes the string to sweep out a 2-dimensional *world sheet*. We need to find the analogue of the action (1.1). We can parametrize the points on the world sheet by two parameters ξ^0, ξ^1 where we chosen the notation to reflect that one of the directions along the sheet will be spacelike and one timelike (the 'evolution'). The worldsheet configuration is described by functions $x^a(\xi)$, which tells us where in spacetime each point on the worldsheet sits. The change in the spacetime coordinates between ξ and $\xi + d\xi$ is

$$dx^a = \frac{\partial x^a}{\partial \xi^i} d\xi^i \tag{1.6}$$

The distance between these points is then given by

$$ds^2 = \frac{\partial x^a}{\partial \xi^i} d\xi^i \frac{\partial x^b}{\partial \xi^j} d\xi^j g_{ab}(x) \tag{1.7}$$

where g_{ab} is the metric in spacetime. From (1.7) we can find the 'induced metric' on the worldsheet which gives the distance between points on the worldsheet, where the distance is just the one 'induced' from the distance between the points in the embedding spacetime

$$ds^2 = \bar{g}_{ij} d\xi^i d\xi^j, \qquad \bar{g}_{ij} = \frac{\partial x^a}{\partial \xi^i} \frac{\partial x^b}{\partial \xi^j} g_{ab}(x) \tag{1.8}$$

The action (1.2) measures the length of the worldline; the natural generalization will be to compute the area of the worldsheet. This is given by

$$S_{DBI} = T \int d^2\xi \sqrt{-\bar{g}} \tag{1.9}$$

where T is a constant, called the 'tension', which gives the mass of the string per unit length.

Let us now turn to generalizing S_{CS}. The analogue of segments of the worldline are 'plaquettes' of the worldsheet. Take a point ξ on the worldsheet and consider the plaquette formed by the two vectors

$$(d\xi^i)_{(1)} = (d\xi^0, 0), \quad (d\xi^i)_{(2)} = (0, d\xi^1) \tag{1.10}$$

These vectors represent spacetime displacements

$$(dx^a)_1 = \frac{\partial x^a}{\partial \xi^0} d\xi^0, \qquad (dx^a)_2 = \frac{\partial x^a}{\partial \xi^i} d\xi^1 \tag{1.11}$$

The potential is now given by A_{ab}, with

$$A_{ab} = -A_{ba} \tag{1.12}$$

The contribution of the plaquette is $A_{ab}(dx^a)_1(dx^b)_2$, and the action is

$$S_{CS} = q \int A_{ab}(x) \frac{\partial x^a}{\partial \xi^0} \frac{\partial x^b}{\partial \xi^1} d\xi^0 d\xi^1 \tag{1.13}$$

Recall that

$$\epsilon_{ij} = \sqrt{-\bar{g}} \hat{\epsilon}_{ij}, \qquad \epsilon^{ij} = \frac{1}{\sqrt{-\bar{g}}} \hat{\epsilon}^{ij}, \qquad \hat{\epsilon}_{01} = 1, \quad \hat{\epsilon}^{01} = -1 \tag{1.14}$$

We can then write (1.13) as

$$S_{CS} = q \int d^2\xi \sqrt{-\bar{g}} [\frac{1}{2!} A_{ab}(x) \frac{\partial x^a}{\partial \xi^i} \frac{\partial x^b}{\partial \xi^j} \epsilon^{ij}] \tag{1.15}$$

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Problem 1: We can choose any set of coordinates ξ^i on the worldsheet, and the action S_{CS} shpuld be independent of this coordinate choice. Show that (1.15) is unchanged under $\xi^i \to \xi'^i$ if the condition (1.12) is satisfied.

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The generalization to higher branes is evident. A p-brane has p spatial directions, and its worldvolume is $p+1$ dimensional. We choose coordinates $\xi^0, \xi^1 \ldots \xi^p$ on the worldvolume. Then

$$S = S_{DBI} + S_{CS} \tag{1.16}$$

$$S_{DBI} = T_p \int d^{p+1}\xi \sqrt{-\bar{g}}, \quad \bar{g}_{ij} = \frac{\partial x^a}{\partial \xi^i} \frac{\partial x^b}{\partial \xi^j} g_{ab}(x) \tag{1.17}$$

$$S_{CS} = q \int d^{p+1}\xi \sqrt{-\bar{g}} [\frac{1}{(p+1)!} A_{a_1 \ldots a_{p+1}}(x) \frac{\partial x^{a_1}}{\partial \xi^{i_1}} \cdots \frac{\partial x^{a_{p+1}}}{\partial \xi^{i_{p+1}}} \epsilon^{i_1 \ldots i_{p+1}}] \tag{1.18}$$

where $A_{i_1 \ldots i_{p+1}}$ is totally antisymmetric in its indices.

2. Field strengths and field equations

In ordinary electromagnetism we make the field strength F_{ab} from the potential A_a

$$F_{ab} = \partial_a A_b - \partial_b A_a, \qquad F_{ab} = -F_{ba} \tag{2.1}$$

F_{ab} satisfies the Bianchi identity as a consequence of its definition

$$F_{[ab;c]} = F_{[ab,c]} = 0 \tag{2.2}$$

where the square brackets represent antisymmetrization with weight unity

$$T_{[abc]} \equiv \frac{1}{3!}(T_{abc} + T_{bca} + T_{cab} - T_{bac} - T_{cba} - T_{acb}) \tag{2.3}$$

and the first step in (2.2) notes that terms involving the connection Γ^a_{bc} all cancel due to the asymmetry of F. The action for A_a is

$$S = -\frac{1}{2} \int d^d x \frac{1}{2!} F_{ab} F^{ab} \tag{2.4}$$

where we have chosen the overall normalization such that the kinetic term for a variable like A_i is $\int \frac{1}{2}(\partial_t A_i)^2$, similar to usual classical mechanics where the kinetic term is $\int dt \frac{1}{2} m(\dot{x})^2$. From this action we find the field equations

$$F_{ab}{}^{;b} = 0 \tag{2.5}$$

We can generalize the above to the potential $A_{a_1 \dots a_{p+1}}$. We write

$$\begin{aligned} F_{a_1 \dots a_{p+2}} = &\partial_{a_1} A_{a_2 a_3 \dots a_{p+2}} + (-1)^{p+1} \partial_{a_2} A_{a_3 \dots a_{p+2} a_1} \\ &+ \partial_{a_3} A_{a_4 \dots a_{p+2} a_1 a_2} + \dots (-1)^{p+1} \partial_{a_{p+2}} A_{a_1 a_2 \dots a_{p+1}} \end{aligned} \tag{2.6}$$

where we have $p+2$ terms in the sum and the factors $(-1)^p$ on alternate terms ensure that F is antisymmetric in all its indices. We have the Bianchi identity

$$F_{[a_1 \dots a_{p+2}; a_{p+3}]} = F_{[a_1 \dots a_{p+2}, a_{p+3}]} = 0 \tag{2.7}$$

The action, normalized as above, is

$$S = -\frac{1}{2} \int d^d x \frac{1}{(p+2)!} F_{a_1 \dots a_{p+2}} F^{a_1 \dots a_{p+2}} \tag{2.8}$$

The field equations are

$$F_{a_1 \dots a_{p+2}}{}^{;a_{p+2}} = 0 \tag{2.9}$$

We can define another tensor using F

$$\tilde{F}_{a_1 \dots a_{D-(p+2)}} = \frac{1}{(p+2)!} \epsilon_{a_1 \dots a_{D-(p+2)} b_1 \dots b_{p+2}} F^{b_1 \dots b_{p+2}} \tag{2.10}$$

From the *field equation* (2.9) of F we find the *Bianchi identity* for $\tilde{F}$

$$\tilde{F}_{[a_1 \dots a_{D-(p+2)}; a_{D-(p+1)}]} = \tilde{F}_{[a_1 \dots a_{D-(p+2)}, a_{D-(p+1)}]} = 0 \tag{2.11}$$

From the *Bianchi identity* (2.7) of F we get the *field equation* for $\tilde{F}$

$$\tilde{F}_{a_1 \dots a_{D-(p+2)}}{}^{;a_{D-(p+2)}} = 0 \tag{2.12}$$

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Problem 2: Prove that (2.9) gives (2.11) and that (2.7) gives (2.12).

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The relation (2.11) implies that in any simply connected region of spacetime we can find a potential $\tilde{A}_{a_1 \ldots a_{D-(p+3)}}$ such that

$$\tilde{F}_{a_1 \ldots a_{D-(p+2)}} = \partial_{a_1} \tilde{A}_{a_2 a_3 \ldots a_{D-(p+2)}} + \ldots (-1)^{D-(p+3)} \partial_{a_{D-(p+2)}} \tilde{A}_{a_1 a_2 \ldots a_{D-(p+3)}} \tag{2.13}$$

This potential would couple, in the manner given by (1.18), to a $D-(p+4)$ brane (which has a $D-(p+3)$ dimensional worldvolume). The $D-(p+4)$ brane is said to be related to the p brane by *electric-magnetic duality.* This term arises from the fact that if we start with 3+1 dimensional flat spacetime, and let a usual pointlike electric charge be the $p=0$ brane, then the dual brane is also $D-(p+4)=0$ dimensional (i.e. a pointlike particle) but a *magnetic monopole* instead of an electric charge. The electric charge produced nonzero components of the electric field $E_i = F_{0i}$. The relation (2.10) implies that for $\tilde{F}_{ab}$ the nonzero components are $\tilde{F}_{12} = F_{03}$, so that we have a *magnetic* field.

We summarize the above steps in the diagram

$$p\ brane \rightarrow A_{p+1} \rightarrow F_{p+2} \rightarrow \tilde{F}_{D-p+2} \rightarrow \tilde{A}_{D-(p+3)} \rightarrow [D-(p+4)]\ brane \tag{2.14}$$