Equations of motion in general relativity

1. Gravitational waves

We have found a geometric quantity, the Riemann curvature tensor $R^a{}_{bcd}$, which describes the curvature of a manifold. In accord with our rough understanding of how the equations of motion should arise, we may try to write

$$R^a{}_{bcd} = matter \tag{1}$$

where the RHS is some function made out of the gravitational properties of the matter distribution. But we can see immediately that such an attempt would not succeed. Recall that in any relativistic theory, no signal can travel faster than the speed of light. Imagine two bodies of matter, quite well separated, exerting some gravitational force on each other. Let one of the bodies, which was say spherical to start with, collapse to a pancake form. Clearly the other body will feel a different gravitational force than before after things stabilize, but the change of force cannot be felt instantly - some object must carry the effect of the change from the first body to the other, at a speed less than or equal to the speed of light. This object must be part of the theory of gravitation, yet must travel over the intervening empty space where there is no matter. Because we are describing gravity by the deformation of spacetime, we see that there must be such propagating deformations even over regions where there is no matter.

But (1) would say that wherever there is no matter, the spacetime has vanishing Riemann curvature tensor, which means that the space is flat! Thus there can be no 'gravitational waves' which as we argued above, appear to be an essential feature of our gravity theory.

Clearly, one must put only some combination of the entries of the curvature tensor to be equal to the matter variables; then in the absence of matter we would still have the freedom of some deformations of spacetime. Let us see what combinations we can choose.

At this point we are greatly helped by the fact that the objects that we are looking for have to be tensors. There are not too many possibilities of other tensors that we can make from the tensors that we have on hand. We have $R^a{}_{bcd}$, and the metric tensor g_{ab} and its inverse g^{ab}. We can try to 'contract' two indices of $R^a{}_{bcd}$ in order to obtain a tensor with only two indices. What are the possibilities? To investigate this, let us look at the symmetries of the Riemann tensor.

1. Symmetries of the Riemann tensor

From the very definition of the Rienamm tensor, it is clear that

$$R^a_{bcd} = -R^a_{bdc} \tag{1.1}$$

since we took the difference between the effects of parallel transport in the c, d directions and d, c directions to define the tensor.

Further, we have already seen that the length of a vector does not change under parallel transport. Take a vector at a point x and transport it around a closed loop made of infinitesimal segments along the coordinate directions c, d. Then we should get

$$0 = \delta(V^a V_a) = 2V_a \delta V^a = 2V_a R^a_{bcd} V^b = 2V^a V^b R_{abcd} \tag{1.2}$$

Thus for each choice of c, d, we find that the matrix $M_{ab} \equiv R_{abcd}$ is such that $V^T M V = 0$ for all V. It is easy to show that such a matrix must be antisymmetric. For consider

$$\begin{aligned} 0 &= (V+W)^a (V+W)^b [M_{ab} + M_{ba}] = V^a V^b M_{ab} + W^a W^b M_{ab} + (V^a W^b + V^b W^a) M_{ab} \\ &= V^a W^b [M_{ab} - M_{ba}] \end{aligned} \tag{1.3}$$

Since the components of V, W are arbitrary, we find that all components of $M_{ab} + M_{ba}$ vanish, and M is antisymmetric. Thus we have found

$$R_{abcd} = -R_{bacd} \tag{1.4}$$

Thus the Riemann tensor is antisymmetric in the first two indices, and in the last two indices. But there is a further symmetry, which relates the first pair of indices to the last pair of indices. this symmetry is certainly not manifest from the definition of the Riemann tensor, and to find it we turn to an explicit examination of the expression for ${R^a}_{bcd}$. But in doing that, we will find the following trick very useful.

2. Locally flat coordinates

We have observed before that any 'very small patch' of a manifold 'looks flat'. We now wish to quantify this statement.

Let the metric on a curved space be $g_{ab}(x)$, and we choose a point x_0 around which to examine the manifold. First we ask if we can choose coordinates such that at x_0 the metric becomes

$$g_{ab} = \delta_{ab} \tag{2.1}$$

This is not hard to do. First note that we can change coordinates $x^a \to x^a - x^a(x_0)$ so that the point x_0 has ccordinates $x^a = 0$. Now choose coordinates ξ^a such that

$$\xi^a = Q^a_b x^b + O(x^2) \tag{2.2}$$

where Q^a_b is a set of constants. To get a nonsingular transformation at x_0 we need $\det Q \neq 0$. Then

$$g'_{ab} = \frac{\partial x^m}{\partial \xi^a}\frac{\partial x^n}{\partial \xi^b} g_{mn} \tag{2.3}$$

which we may write as

$$g' = Q^{-1T} g Q^{-1} \tag{2.4}$$

Bur g is a symmetric matrix, and so we can diagonalize it by an orthogonal transformation. thus we can choose Q^{-1} so that $g' = diag\{c_1, c_2, \ldots c_d\}$. Since Q^{-1} is orthogonal, Q will be nonsingular (in fact the determinant will be unity), so there is certainly such an allowed transformation. Since we are working in a space with Euclidean signature, all the c_a will be positive.

We can now make a further transformation

$$\tilde{\xi}^a = c_a^{1/2} \xi^a \tag{2.5}$$

The new metric will be

$$\tilde{g}_{ab} = \frac{\partial \xi^m}{\partial \tilde{\xi}^a}\frac{\partial \xi^n}{\partial \tilde{\xi}^b} g'_{mn} = \delta_{ab} \tag{2.6}$$

and we have achieved our goal.

Now we ask: can we further make the first derivatives $g_{ab,c}$ all vanish at the point x_0, while keeping the metric at x_0 to be (2.1)? Suppose the metric is already of the form (2.1) in the coordinates x. We make a coordinate change of the form

$$\xi^a = x^a + \frac{1}{2} P^a_{bc} x^b x^c + O(x^3) \tag{2.7}$$

Then the inverse map is

$$x^a = \xi^a - \frac{1}{2} P^a_{bc} \xi^b \xi^c + O(\xi^3) \tag{2.8}$$

In particular we get

$$\frac{\partial x^a}{\partial \xi^b} = \delta^a_b - P^a_{bc}\xi^c + O(\xi^2) \tag{2.9}$$

Then the new metric is

$$g'_{ab} \approx (\delta^m_a - P^m_{af}\xi^f)(\delta^n_b - P^n_{bh}\xi^h) g_{mn} \tag{2.10}$$

$$g'_{ab,c} = -g_{an}P^n_{bc} - g_{mb}P^m_{ac} + g_{ab,c} \tag{2.11}$$

Thus if we want all the first derivatives of g to vanish in the new coordinate system we need

$$g_{an}P^n_{bc} + g_{mb}P^m_{ac} = g_{ab,c} \tag{2.12}$$

Let us write

$$P_{abc} \equiv g_{am}P^m_{bc} \tag{2.13}$$

Then we need

$$P_{abc} + P_{bac} = g_{ab,c} \tag{2.14}$$

Now by construction we have $P_{abc} = P_{acb}$. We wish to extract the values of P_{abc} implied by (2.14). We find

$$g_{ab,c} + g_{ac,b} - g_{bc,a} = P_{abc} + P_{bac} + P_{acb} + P_{cab} - P_{bca} - P_{cba} = 2P_{abc} \tag{2.15}$$

which is a computation of the kind that we have seen before in deriving the expression for the connection. Thus if we choose in the transformation (2.7)

$$P^m_{bc} = g^{ma}P_{abc} = \frac{1}{2}g^{ma}[g_{ab,c} + g_{ac,b} - g_{bc,a}] \tag{2.16}$$

then all the first derivatives of the metric will vanish. In fact we observe that we must take $P^m_{bc} = \Gamma^m_{bc}$, the connection!

Can we go further and set to zero all the second derivatives of the metric at the point x_0? This is in fact not possible, since if we could do that the curvature tensor would vanish in this new frame, and since this is a tensor it would vanish in all frames. But this is not possible if the manifold is not flat.

Let us use this coordinate system to study the properties of the curvature tensor. Now we get

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} = \frac{1}{2}[g_{ad,bc} - g_{bd,ac} - g_{ac,bd} + g_{bc,ad}] \tag{2.17}$$

We now observe the properties

$$R_{abcd} = R_{cdab} \tag{2.18}$$

$$R_{abcd} + R_{adbc} + R_{abcd} = 0 \tag{2.19}$$

3. Gravitational waves

Let us return to our discussion of gravitational waves. Write

$$g_{ab} = \eta_{ab} + h_{ab}, \quad |h_{ab}| << 1 \tag{3.1}$$

We contract two indices of the Riemann tensor to get

$$R^c{}_{bcd} = \frac{1}{2}[h^c_{d,bc} - h_{bd,c}{}^c - h_{,bd} + h^c_{b,cd}] = 0 \tag{3.2}$$

Choose a solution of the form of a plane wave

$$h_{ab}(x) = h_{ab}e^{ikx_1 - i\omega t} \tag{3.3}$$

There are two free indices in (3.3). Let us first choose them to be both transverse to x_1, t. Then we must set to zero all terms that involve a b or a d derivative, and we get

$$-h_{bd}[-k^2 + \omega^2] = 0 \tag{3.4}$$

which says that if we want a nontrivial solution then we must have

$$\omega^2 = k^2 \tag{3.5}$$

which means that the wavefront will travel with the speed of light for all frequencies; further, a superposition of different wavenumbers k all in the x_1 direction will give a waveform that will not distort under propagation.

Before examining the other possible equations arising from other choices of b, d let us note that some of the solutions of the equations are actually 'fake' waves: they do not represent any real deformation of the spacetime. We have seen that if we choose curvilinear coordinates then the metric will look different from η_{ab} in general. Since we are looking for small perturbations to the metric, consider the effect of a small change in the coordinates:

$$\xi^a = x^a + \epsilon^a(x), \quad |\epsilon^a| << 1 \tag{3.6}$$

Then we have

$$x^a = \xi^a - \epsilon^a(\xi) + O(\xi^2) \tag{3.7}$$

and the new metric is

$$g'_{ab} = (\delta^m_a - \epsilon^a_{,m})(\delta^n_b - \epsilon^b_{,n})\eta_{mn} = \eta_{ab} - \epsilon_{a,b} - \epsilon_{b,a} \tag{3.8}$$

Thus we can get a deformation of the form

$$h_{ab} = -\epsilon_{a,b} - \epsilon_{b,a} \tag{3.9}$$

from just a change of coordinates. Since these are just curvilinear coordinates on flat space, the curvature tensor computed for such h_{ab} will vanish, and we must exclude such solutions from our count of possible gravitational waves. Note that if we are looking for solutions that have the profile $e^{ikx_1-i\omega t}$, then we must choose

$$\epsilon^a(x) = \epsilon^a e^{ikx_1-i\omega t} \tag{3.10}$$

Under such a change of coordinates any metric that we had will change to

$$h'_{ab} = h_{ab} - i\epsilon_a k_b - i\epsilon_b k_a \tag{3.11}$$

Thus in particular let $a = 0$. Then we get

$$h'_{00} = h_{00} + 2i\epsilon_0\omega \tag{3.12}$$

$$h'_{01} = h_{01} - i\epsilon_0 k_1 + i\epsilon_1\omega \tag{3.13}$$

$$h'_{0i} = h_{0i} + i\epsilon_i\omega \tag{3.14}$$

From (3.12) we see that we can choose ϵ_0 to make $h'_{00} = 0$. We can then choose ϵ_1 in (3.13) to make $h_{01} = 0$, and ϵ_i in (3.14) to make $h_{0i} = 0$. thus we get $h_{0b} = 0$, for all b, with our choice of change of coordinates. Since any coordinates are as good as any other, we might as well assume that we are in these latter coordinates since a lot of the 'fake' deformations of the metric have been made to vanish.

Let us now return to the equation (3.2), and choose $b = 0, d = 0$. Then we get

$$h^c_{0,0c} - h_{00,c}{}^c - h_{,00} + h^c_{0,c0} = 0 \tag{3.15}$$

Noting that a time derivative is $(-i\omega)$ and that $h_{0b} = 0$, we get

$$\omega^2 h = 0 \tag{3.16}$$

which gives

$$h = 0 \tag{3.17}$$

for a nontrivial solution. Let us now choose $b = 0, d = 1$. Then we get

$$h^c_{1,0c} - h_{01,c}{}^c - h_{,01} + h^c_{0,c1} = h^1_{1,01} = \omega^2 h_{11} = 0 \tag{3.18}$$

Thus we find that h_{00}, h_{01}, h_{11} all vanish. Now we try $b = 1, d = i$. Then we get

$$h^c_{i,1c} - h_{1i,c}{}^c - h_{,1i} + h^c_{1,ci} = h^1_{i,11} - h_{1i,c}{}^c = \omega^2 h_{1i} = 0 \tag{3.19}$$

Thus $h_{1b} = 0$ for all b. Thus we see that the wave has to be transverse and traceless.