FIXING THE CONSTANTS IN STRING THEORY

We have obtained the duality structure of string theory, using by and large only classical gravity and a few additional assumptions stemming from quantum theory. In the process however we have not yet fixed many constants, which we need to give the theory its complete structure. We will fix these constants now.

1. The T-duality scale

The classical gravity theory of IIA had a symmetry which changed the length of a compact diretion to its reciprocal

$$L' = \frac{A}{L} \tag{1.1}$$

Here A is a constant with units $(length)^2$, but there is no way to fix this constant in the classical theory; any value of A will give an equally good symmetry. In the classical theory we found the freedom

$$\tilde{C}' = -\tilde{C}, \quad \phi' = \phi - \frac{1}{2}\tilde{C} \tag{1.2}$$

where $e^{\tilde{C}} = g_{99}$ is the metric in the compact direction x^9. But by a change of coordinates we can rescale the metric $\bar{g}_{99} = e^{\alpha} g_{99}$; this gives $\tilde{\bar{C}} = \tilde{C} + \alpha$. If we let the symmetry be $\tilde{\bar{C}}' = -\tilde{\bar{C}}$ then instead of (1.2) we have $\tilde{C}' = -\tilde{C} - 2\alpha$. This change is equivalent to a change in A in (1.1).

Quantum mechanics however did fix the constant A. The momentum modes P_9 have the spectrum $\frac{2\pi n}{L}$, while the winding modes (after T-duality) have spectrum $nT_{NS1}L'$ where we have written T_{NS1} for the string tension. Then T-duality is a symmetry only if

$$A = \frac{2\pi}{T_{NS1}} \tag{1.3}$$

It is conventional to write

$$T_{NS1} = \frac{1}{2\pi\alpha'} \tag{1.4}$$

thus defining a constant α' with units $(length)^2$. Note that there is a special length L which is invariant under T-duality

$$l_s = A^{\frac{1}{2}} = 2\pi\sqrt{\alpha'} \tag{1.5}$$

The coupling also changes under T-duality. The classical analysis tells us that $g' \sim \frac{g}{L}$, where L is the length of the circle before T-duality. If $L = l_s$ then the length of the circle does not change, and the coupling should not change either. (This follows because if we perform dimensional reduction to the noncompact directions then the effective coupling of the reduced theory should not change under T-duality.) Thus we get

$$g' = g\frac{l_s}{L} = g\frac{2\pi\sqrt{\alpha'}}{L} \tag{1.6}$$

2. The tension of D-branes

We have found using classical arguments that the tension of D-branes, measured in the string metric, is $\sim e^{-\phi}$. We would now like to fix the proportionality constant in this relation.

2.1. The tension of the D-string

The variable $\phi = \frac{3}{4}C$ describes the size of the circle x^{11}, with $g_{11,11} = e^C$. But by a change of coordinates we can shift $C \to C + \alpha$, so we see again that the classical theory alone does not help us to fix the proportionality constant. This time we use S-duality, and demand that it be an exact symmetry of the theory. Classical arguments allow a 1-parameter family of S-duality type symmetries

$$\phi' = -\phi + \beta \tag{2.1}$$

where β can be changed by the above mentioned shifts in ϕ. Let us choose ϕ to be such that under the correct quantum S-duality map we get

$$\phi' = -\phi \tag{2.2}$$

In the Einstein metric S-duality symmetry between the two strings then implies

$$\begin{aligned} T_{NS1} &= e^{\frac{\phi}{2}} T_1 \\ T_{D1} &= e^{-\frac{\phi}{2}} T_1 \end{aligned} \tag{2.3}$$

for some T_1. Defining

$$g = e^{\phi} \tag{2.4}$$

we see that

$$T_{D1} = \frac{T_{NS1}}{g} \tag{2.5}$$

Note that S-duality will take $g \to \frac{1}{g}$. This defines a special value of the coupling $g = 1$ which will remain unchanged under S-duality.

2.2. The mass of D0

Start with IIB string theory, and compactify x^9 to the length l_s. Wrap a D-string on this circle; this has mass $T_{D1}l_s$. The operation T_9 will leave this compactification length and coupling g unchanged, but change the D1 to a D0 brane. If T-duality is to be a symmetry we should get the same mass as before the duality, and we have

$$m_{D0} \equiv T_{D0} = \frac{T_{NS1}}{g} l_s = \frac{1}{g\sqrt{\alpha'}} \tag{2.6}$$

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Problem 1: Using similar arguments, prove that the tension of a D-p-brane is

$$T_{Dp} = m_{D0}(l_s)^{-p} = \frac{1}{g}\frac{1}{(2\pi)^p}\frac{1}{(\alpha')^{\frac{p+1}{2}}} \tag{2.7}$$

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3. The size of the 11th dimension

The D0 brane is just a graviton moving in its lowest harmonic along x^{11}. Thus we must have

$$m_{D0} = \frac{2\pi}{L_{11}} \tag{3.1}$$

where L_{11} is the length of the length of the circle x^{11}. Equating (3.1), (2.6) we get the exact relation between the coupling g and the size of x^{11}

$$g = e^{\phi} = \frac{L_{11}}{2\pi\sqrt{\alpha'}} = \frac{L_{11}}{l_s} \tag{3.2}$$

The tension of the string T_{NS1} is related to the tension of the M2 brane by

$$T_{NS1} = L_{11}T_{M2} \tag{3.3}$$

which gives using (3.2)

$$T_{M2} = \frac{1}{g}\frac{1}{(2\pi)^2}\frac{1}{(\alpha')^{\frac{3}{2}}} \tag{3.4}$$

From (2.7) we see that this is the same as the tension of the D2 brane, which is expected since the D2 of IIA is just the M2 brane not wrapped on x^{11}.

4. The tension of the NS5 brane

The tension of the D5 brane is

$$T_{D5} = \frac{1}{g}\frac{1}{(2\pi)^5}\frac{1}{(\alpha')^3} \tag{4.1}$$

In the Einstein frame using our classical results and demanding and demanding S-duality we get

$$\begin{aligned} T_{D5} &= e^{\frac{\phi}{2}} T_5 \\ T_{NS5} &= e^{-\frac{\phi}{2}} T_5 \end{aligned} \tag{4.2}$$

for some T_5. Thus we see that

$$T_{NS5} = e^{-\phi} T_{D5} = \frac{T_{D5}}{g} = \frac{1}{g^2}\frac{1}{(2\pi)^5}\frac{1}{(\alpha')^3} \tag{4.3}$$

By applying T-duality on a circle on length l_s along the NS5, we find that we get the same tension for the NS5 of IIA.

5. Fixing the Newton constant G_N^{10}

5.1. *The constant G_N^{10}*

Consider the gravity action

$$S = \frac{1}{16\pi G_N}\int d^D x\sqrt{-g}R \tag{5.1}$$

With $\hbar = 1$ the action has no units (we need to take e^{iS} for example in a path integral). The metric g_{ab} is dimensionless, while the coordinates x^a have units of length. Then we see that Γ^a_{bc} has units $(length)^{-1}$, and R has units $(length)^{-2}$. Thus G_N has units

$$G_N \sim (length)^{D-2} \tag{5.2}$$

Now consider the action of IIA string theory in 10-d

$$S_{IIA} = \frac{1}{16\pi G_N^{10}}\int d^{10}x\sqrt{-g_S}e^{-2\phi}[[R]_S + 4\phi_{,a}\phi^{,a} + \ldots] \tag{5.3}$$

where we have not written all the terms in the action. G_N^{10} has units $(length)^8$. What is the value of this constant? We already have quantities with units of length in the theory, for example $\sqrt{\alpha'}$, and we have expressed all other quantities in terms of this unit. We expect that

$$G_N^{10} = k(\alpha')^4 \tag{5.4}$$

for some constant k, and our goal is to find k.

5.2. Using the D6 brane solution

The IIA theory has a D6 brane which has a mass per unit area

$$T_{D6} = \frac{1}{g}\frac{1}{(2\pi)^6}\frac{1}{(\alpha')^{\frac{7}{2}}} \tag{5.5}$$

Let us compactify directions six directions $x^4 \ldots x^9$, with the volume of this 6-torus being $\bar{V}_6$. If we wrap the D6 brane on this torus, then we will have an object with mass

$$M = T_{D6}\bar{V}_6 = \frac{\bar{V}_6}{g}\frac{1}{(2\pi)^6}\frac{1}{(\alpha')^{\frac{7}{2}}} \tag{5.6}$$

But we know the full gravitational solution for a D6 brane in M theory, and we have also obtained the 10-d string metric and dilaton that result in the IIA description

$$ds_S^2 = H^{\frac{1}{2}}[-dt^2 + dx^i dx^i] + H^{-\frac{1}{2}}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{5.7}$$

$$e^{(\phi-\phi_\infty)} = H^{\frac{3}{4}} \tag{5.8}$$

where $H^{-1} = 1 + \frac{Q}{r}$ and $x^i, i = 4 \ldots 9$ are the coordinates along the D6. (In our earlier classical analysis the variable ϕ could be shifted by a constant, but we have now fixed this freedom and thus we must let ϕ go to its value $\phi_\infty = \log g$ at large r.) We can write the action (5.3) dimensionally reduced to 3+1 spacetime dimensions

$$S_4 = \frac{1}{16\pi G_N^{10}}\int d^4x\sqrt{-g_4}V_6 e^{-2\phi}[[R]_{S,4} + \ldots] \equiv \frac{\bar{V}_6}{16\pi G_N^{10}}\int d^4x\sqrt{-g_4}e^{-2\phi_4}[[R]_{S,4} + \ldots] \tag{5.9}$$

Here V_6 is the volume of the directions x^i, with $\bar{V}_6$ the value of V_6 at infinity; we have taken $\bar{V}_6$ out in the last expression in (5.9) for later convenience. For the solution (5.7)

$$V_6 = \bar{V}_6 H^{\frac{3}{2}} \tag{5.10}$$

and

$$e^{-2\phi_4} = e^{-2\phi}\frac{V_6}{\bar{V}_6} = e^{-2\phi_\infty} \tag{5.11}$$

We can convert the action (5.9) to an Einstein action for 3+1 dimensional spacetime by rescaling

$$g_{ab}^E = e^{-2\phi_4}g_{ab}^S \tag{5.12}$$

In terms of g^E_{ab} we get an action

$$S^E_4 = \frac{1}{16\pi G^4_N}\int d^4x\sqrt{-g^E_4}[[R]_{E,4} + \ldots] \tag{5.13}$$

where we have defined

$$G^4_N = \frac{G^{10}_N}{\bar{V}_6} \tag{5.14}$$

The 3+1 dimensional Einstein metric for our configuration describes a point source at $r = 0$ with spherical symmetry around $r = 0$. At large r (where all matter fields fall off) we know that the unique spherically symmetric solution for the action (5.13) behaves like the Schwarzschild metric, which has in particular

$$g^E_{tt} = -(1 - \frac{2G^4_N M}{r}) \tag{5.15}$$

for a source of mass M. For the metric (5.7) we get

$$g^E_{tt} = -e^{-2(\phi_4)_\infty} H^{\frac{1}{2}} = -e^{-2(\phi_4)_\infty}(1 + \frac{Q}{r})^{-\frac{1}{2}} \tag{5.16}$$

We can perform a uniform rescaling of coordinates to get $g^E_{ab} \to e^{2(\phi_4)_\infty} g^E_{ab}$, which makes g^E_{ab} go to η_{ab} at large r, and allows a comparison with (5.15)

$$g^E_{tt} = -(1 + \frac{Q}{r})^{-\frac{1}{2}} \approx -(1 - \frac{Q}{2r}) \tag{5.17}$$

Equating (5.15), (5.17) we get

$$G^4_N = \frac{Q}{4M} \tag{5.18}$$

But the KK-monopole solution gives

$$Q = \frac{L_{11}}{4\pi} = g\frac{\sqrt{\alpha'}}{2} \tag{5.19}$$

Collecting all these results we find from (5.14)

$$G^{10}_N = 8\pi^6 g^2 (\alpha')^4 \tag{5.20}$$