1. Magnetic charges in electromagnetism

In usual electromagnetic theory (in flat 3+1 dimensional spacetime) we have electric charges but no magnetic charges

$$\vec{\nabla}\cdot\vec{E}=\rho_e \tag{1.1}$$

$$\vec{\nabla}\cdot\vec{B}=0 \tag{1.2}$$

The latter equation can be solved by setting

$$\vec{B}=\nabla\times\vec{A} \tag{1.3}$$

since $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V})=0$ for any $\vec{V}$. But what if we *did* want to include magnetic charges in the theory? Suppose we had a magnetic monopole at the origin O. In any simply connected region that excludes O we still have (1.2) and we can write $\vec{B}=\nabla\times\vec{A}$. This $\vec{A}$ is not unique though, since it can be changed by a gauge transformation

$$\vec{A}'=\vec{A}+\vec{\nabla}\Lambda, \quad \vec{\nabla}\times\vec{A}'=\vec{B} \tag{1.4}$$

We can thus find a potential $\vec{A}$ in each simply connected patch of space excluding the origin, but we will not be able to ensure that $\vec{A}$ agrees over the overlap region of neighboring patches; for some pairs of patches the potential will agree only upto a gauge transformation. The physical field $\vec{B}$ will be well defined and continuous, however. In this sense the magnetic monopole shows up as a 'topological' object in the electromagnetic theory.

More specifically, let us write 3-D space in polar coordinates

$$ds^2=dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{1.5}$$

Consider the potential

$$A_\phi=Q(1-\cos\theta), \quad A_r=A_\theta=0 \tag{1.6}$$

It is not immediately clear if this potential is ill-defined anywhere, since the coordinate ϕ itself breaks down at $\theta=0,\pi$. We can instead refer $\vec{A}$ to the local orthonormal frame at each point defined by the $\hat{r},\hat{\theta},\hat{\phi}$ directions. This gives

$$A_{\hat{\phi}}=A_\phi(g^{\phi\phi})^{\frac{1}{2}}=\frac{Q(1-\cos\theta)}{r\sin\theta}=\frac{Q}{r}\tan\frac{\theta}{2} \tag{1.7}$$

This is singular at $\theta=\pi$. A gauge transformation however gives a potential that is good at $\theta=\pi$

$$A'_\phi=A_\phi-2Q=-Q(1+\cos\theta), \quad A'_{\hat{\phi}}=-\frac{Q}{r}\cot\frac{\theta}{2} \tag{1.8}$$

though now we have a singularity at $\theta=0$. The field strength

$$F_{\theta\phi}=\partial_\theta A_\phi-\partial_\phi A_\theta=Q\sin\theta, \quad F_{\hat{\theta}\hat{\phi}}=\frac{Q}{r^2} \tag{1.9}$$

is well defined everywhere, and gives the radial magnetic field $B_{\hat{r}}=F_{\hat{\theta}\hat{\phi}}$ expected for a magnetic monopole.

2. The Kaluza-Klein monopole in gravity

We have seen that if we have a compact circle in spacetime then we get a gauge field A_a in the reduced spacetime. Can we have a metric which gives the A_a appropriate to a monopole? Let the compact direction be y with

$$0 \leq y < L \tag{2.1}$$

Take 3 other spatial directions from the spacetime, and for the moment ignore the remaining directions (including time). In the 4 directions thus chosen consider the metric

$$ds^2 = H[dy + Q(1 - \cos\theta)d\phi]^2 + H^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{2.2}$$

$$H^{-1} = 1 + \frac{Q}{r} \tag{2.3}$$

At large r this metric goes to flat space, with A_a at leading order given by

$$A_\phi = Q(1 - \cos\theta) \tag{2.4}$$

which is what we had in (1.6) above for a magnetic monopole. It may be checked that the metric (2.2) satisfies

$$R_{ab} = 0 \tag{2.5}$$

so that we get a solution of the vacuum Einstein equations in any spacetime if we just add on the extra dimensions (including time) in a trivial way

$$ds^2_{total} = -dt^2 + dx^i dx^i + ds^2, \quad i = 1 \ldots D-5 \tag{2.6}$$

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Problem 1: Show that for the metric (2.2) we have $R_{ab} = 0$.

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It may seem that we have found the desired monopole solution, but we have to be careful that there are no singularities anywhere, since here the curvature computation would become invalid. At $\theta = \pi$ there is a coordinate singularity, which can be removed by just going to new coordinates which implement the gauge transformation (1.8). In addition there is an apparent singularity at $r = 0$, and we must check if this too can be removed by some choice of coordinates.

Close to $r = 0$ we can approximate $H^{-1} = \frac{Q}{r}$, and then (2.2) becomes

$$ds^2 = \frac{r}{Q}[dy + Q(1 - \cos\theta)d\phi]^2 + \frac{Q}{r}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{2.7}$$

We rewrite different terms in (2.7) as follows, in trying to remove the singularity ar $r = 0$

$$Q\frac{dr^2}{r} = Q(\frac{dr}{\sqrt{r}})^2 = 4Qd\tilde{r}^2, \quad \tilde{r} = \sqrt{r} \tag{2.8}$$

$$\frac{Q}{r}r^2(d\theta^2 + \sin^2\theta d\phi^2) = Q\tilde{r}^2(4(\frac{d\theta}{2})^2 + 4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}d\phi^2) \tag{2.9}$$

$$\frac{r}{Q}dy^2 = 4Q\tilde{r}^2\sin^2\frac{\theta}{2}\;(\frac{dy}{2Q})^2 + 4Q\tilde{r}^2\cos^2\frac{\theta}{2}\;(\frac{dy}{2Q})^2 \tag{2.10}$$

$$\frac{r}{Q}[Q(1 - \cos\theta)d\phi]^2 = 4Q\tilde{r}^2\sin^4\frac{\theta}{2}\;d\phi^2 \tag{2.11}$$

$$2\frac{r}{Q}Q(1 - \cos\theta)dyd\phi = 8Q\tilde{r}^2\sin^2\frac{\theta}{2}\;(\frac{dy}{2Q})d\phi \tag{2.12}$$

We can combine the terms in $d\phi^2$

$$4Q\tilde{r}^2\sin^2\frac{\theta}{2}(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2})d\phi^2 = 4Q\tilde{r}^2\sin^2\frac{\theta}{2}d\phi^2 \tag{2.13}$$

Some terms combine to a perfect square

$$4Q\tilde{r}^2\sin^2\frac{\theta}{2}[(\frac{dy}{2Q})^2 + 2(\frac{dy}{2Q})d\phi + d\phi^2] = 4Q\tilde{r}^2\sin^2\frac{\theta}{2}[d(\phi - \frac{y}{2Q})]^2 \tag{2.14}$$

Defining new variables

$$\tilde{\theta} = \frac{\theta}{2}, \quad \tilde{y} = \frac{y}{2Q}, \quad \tilde{\phi} = \phi - \frac{y}{2Q}, \quad 0 \le \tilde{\theta} < \frac{\pi}{2}, \quad 0 \le \tilde{y} < \frac{L}{2Q}, \quad 0 \le \tilde{\phi} < 2\pi \tag{2.15}$$

we get

$$ds^2 = 4Q[d\tilde{r}^2 + \tilde{r}^2(d\tilde{\theta}^2 + \cos^2\tilde{\theta}d\tilde{y}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2)] \tag{2.16}$$

2.1. The metric on S^3

We wish to interpret the metric $(d\tilde{\theta}^2 + \cos^2\tilde{\theta} d\tilde{y}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2)$ appearing in (2.16) as the metric on a unit S^3. To see this start with the metric of flat R^4

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = dr^2 + r^2 d\Omega_3^2 \tag{2.17}$$

The S^3 of radius r is given by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2 \tag{2.18}$$

We can write

$$x_1^2 + x_2^2 = r^2\cos^2\theta, \quad x_3^2 + x_4^2 = \sin^2\theta \tag{2.19}$$

with

$$0 \leq \theta < \frac{\pi}{2} \tag{2.20}$$

We can further write

$$\begin{aligned} x_1 &= r\cos\theta\cos\phi, \ \ x_2 = r\cos\theta\sin\phi \\ x_3 &= r\sin\theta\cos\chi, \ \ x_4 = r\sin\theta\sin\chi \end{aligned} \tag{2.21}$$

with

$$0 \leq \phi < 2\pi, \quad 0 \leq \chi < 2\pi \tag{2.22}$$

A little computation using (2.21) gives

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = dr^2 + r^2(d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta d\chi^2) \tag{2.23}$$

so that we can identify the metric on S^3 as

$$d\Omega_3^2 = d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta d\chi^2, \quad 0 \leq \theta < \frac{\pi}{2}, \ \ 0 \leq \phi < 2\pi, \ \ 0 \leq \chi < 2\pi \tag{2.24}$$

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Problem 2: Show that (2.23) follows from the definitions (2.21).

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There is another standard way to write the metric on S^3. Let

$$\psi_+ = \phi + \chi, \quad \psi_- = \phi - \chi, \quad \theta' = 2\theta \tag{2.25}$$

Then (2.24) becomes

$$
\begin{aligned}
d\Omega_3^2 =& \frac{1}{4}[d\theta'^2 + d\psi_+^2 + d\psi_-^2 + 2\cos\theta' d\psi_+ d\psi_-] \\
=& \frac{1}{4}[d\theta'^2 + (d\psi_+ + \cos\theta' d\psi_-)^2 + \sin^2\theta' d\psi_-^2]
\end{aligned}
\tag{2.26}
$$

From (2.25) we see that

$$
\psi_+ = \psi_- + 2\chi \equiv \psi_- + \chi' \tag{2.27}
$$

With this substitution we get

$$
d\Omega_3^2 = \frac{1}{4}[d\theta'^2 + (d\chi'^2 + (1+\cos\theta')d\psi_-)^2 + \sin^2\theta' d\psi_-^2] \tag{2.28}
$$

We cover all points of S^3 once with the ranges $0 \le \chi < 2\pi,\ 0 \le \psi_- < 2\pi$, so that we get

$$
0 \le \theta' \le \pi, \quad 0 \le \psi_- < 2\pi, \quad 0 \le \chi' < 4\pi \tag{2.29}
$$

The metric (2.28) is seen to give the angular coordinates of (2.7) after we choose the radial coordinate as in (2.8), and define $\chi' = \frac{y}{Q}$.

2.2. The requirement for regularity of the monopole solution

Returning to (2.16) we see that around $r = 0$ we indeed have the metric of smooth 4-dimensional space if the ranges of the coordinates were as in (2.24). While $\tilde{\theta}, \tilde{\phi}$ are seen to have the correct ranges, the range for $\tilde{y}$ is correct only if

$$
\frac{L}{2Q} = 2\pi, \quad Q = \frac{L}{4\pi} \tag{2.30}
$$

Since Q gave the charge of the monopole (see (2.4)) the relation (2.30) fixes the charge of the monopole.

2.3. The D-6 brane of string theory

We can now add on the remaining coordinates of spacetime as in (2.6) to get a solution of the complete theory. In particular, we can start with 11-D M theory and compactify one direction to a circle of length L to get 10-D string theory. Using this compact circle and 3 other directions we make the monopole solution described above. Then in these 3 chosen directions the metric is nontrivial over a distance of order L from the origin, going over to flat space at distances much larger than L. We have 6 remaining spatial directions and one time direction in which the solution is completely homogeneous. Thus we get a 6-dimensional ‘brane’ in the 10-dimensional string theory. This brane acts like a monopole source for the gauge field A_a of the string theory. The 11-D solution is

$$ds^2 = -dt^2 + dx^i dx^i + H^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] + H[dy + Q(1-\cos\theta)d\phi]^2 \quad (2.31)$$

with $i = 1, \ldots 6$. For later use it will be helpful to write this 10-D solution in the string metric. From (2.20) we read off the parameters of the solution when we reduce from 11 to 10 dimensions (The metric (2.31) is already written in the form where we can read of the components of the 10-D metric)

$$e^C = H \quad (2.32)$$

$$A_\phi = Q(1-\cos\theta) \quad (2.33)$$

$$ds_{10}^2 = -dt^2 + dx^i dx^i + H^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \quad (2.34)$$

We obtain the string metric by scaling the 10-D components of the 11-D metric

$$g^S_{ab} = e^{\frac{C}{2}} g^{10}_{ab} \quad (2.35)$$

so we get

$$ds_S^2 = H^{\frac{1}{2}}[-dt^2 + dx^i dx^i] + H^{-\frac{1}{2}}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \quad (2.36)$$

Using Cartesian coordinates for the directions perpendicular to the brane we have

$$ds_S^2 = H^{\frac{1}{2}}[-dt^2 + dx^i dx^i] + H^{-\frac{1}{2}}[dz^\alpha dz^\alpha] \quad (2.37)$$

with $\alpha = 1 \ldots 4$. Note that this is a form that we have seen before, where the directions along and perpendicular to the brane are each described by a metric that is flat upto a conformal factor.