1. The idea of extra dimenions

We have seen that gravity is described by a metric tensor g_{ij}. The indices $i, j \ldots$ range over $0, 1, \ldots D-1$, where D is the dimension of the spacetime. But in nature we also find the electromagnetic field, which is described by a vector potential A_i. It would be nice to have a 'unified theory' where g_{ij} and A_i are both included together.

Suppose we start with a spacetime that is $D+1$ dimensional, instead of D dimensional. The metric on this spacetime is g_{ab}, with $a, b \ldots$ ranging over $0, 1, \ldots D$. Assume further this $D+1$ dimensional spacetime has the topology of a cylinder: While the coordinates $\xi^0, \xi^1, \ldots \xi^{D-1}$ range over $(-\infty, \infty)$, the last coordinate ξ^D ranges over a circle $0 \leq \xi^D < L$ with $\xi^D = L$ being equivalent to $\xi^D = 0$.

Let the metric on this $D+1$ dimensional spacetime be

$$g_{ab} = \eta_{ab} + h_{ab}, \quad h_{ab} << 1 \tag{1.1}$$

The perturbation h_{ab} describes a graviton on the total spacetime, as we have seen before. But suppose the length L is very small compared to the length scales that we are able to observe. Then an observer will see, in a sense to be made more precise below, only the directions $\xi^0, \xi^1 \ldots \xi^{D-1}$ so the spacetime will look effectively $D-1$ dimensional. We will use indices $i, j \ldots$ to run over the coordinates $\xi^0, \xi^1 \ldots \xi^{D-1}$. The components g_{ij} of g_{ab} look like a metric tensor for a $D-1$ dimensional spacetime. But we also have the components

$$g_{Di} = g_{iD} \equiv A_i \tag{1.2}$$

which are correct in number to be a vector potential in the $D-1$ dimensional spacetime. We also have the component g_{DD}, which looks like a scalar in the $D-1$ dimensional spacetime, since it carries no index along the directions $\xi^0, \xi^1 \ldots \xi^{D-1}$.

But do these different components of g_{ab} actually describe the dynamics of a graviton g_{ij}, an electromagnetic potential A_i, and a scalar? This will follow only after we check the dynamical equations for these quantities. To get these equations we start with the fact that the perturbation h_{ab} in (1.1) satisfies the equation of the graviton in $D+1$ dimensions

$$h_{ab,c}{}^{c} + h_{c}{}^{c}{}_{,ab} - h_{ac,b}{}^{c} - h_{bc,a}{}^{c} = 0 \tag{1.3}$$

We add one more fact: Since the length L is small, we will have

$$\frac{\partial h_{ab}}{\partial \xi^D} = 0 \tag{1.4}$$

This follows from quantum mechanics. Since the coordinate ξ^D is periodic, the waveform of h_{ab} must execute an integer number of oscillations in the interval $0 \le \xi^D < L$. If this number is not zero, then we have oscillations of very short wavelength (since L is small) and the corresponding quantum will have a very high energy. But when we said that L is too small a scale for us to observe we implied that we cannot observe quanta of this high energy, since the way we see a given lengthscale is by using quanta (for example, photons) that resolve that lengthscale. Thus we are forced to restrict to waveforms that have no oscillations in the direction ξ^D, and we get (1.4).

Let us now return to (1.3). First consider the case where $a = D, b = D$. This gives

$$h_{DD,k}{}^{k} = 0 \tag{1.5}$$

where (1.4) caused the vanishing of all other terms. Note that (1.5) is indeed the equation for a massless scalar field in the $D-1$ dimensional spacetime described by the coordinates $\xi^0, \xi^1 \ldots \xi^{D-1}$.

Now look at the case $a = D, b = i$. We get from (1.3) (using $h_{Di} = h_{iD} = A_i$)

$$A_{i,k}{}^{k} - A_{k,i}{}^{k} = 0 \tag{1.6}$$

This can be rewritten as

$$F_{ik}{}^{,k} = 0, \quad F_{ik} = A_{k,i} - A_{i,k} \tag{1.7}$$

and is thus seen to describe an electromagnetic vector potential in the $D-1$ dimensional spacetime described by $\xi^0, \xi^1 \ldots \xi^{D-1}$.

Finally we consider $a = i, b = j$. This should give the graviton in the $D-1$ dimensional spacetime, but things are a little tricky this time. From (1.3) we get (recall that indices have to be raised and lowered with η^{ab}, η_{ab} since we are only looking at linear order in the perturbation)

$$h_{ij,k}{}^{k} + h_k{}^{k}{}_{,ij} + h_{DD,ij} - h_{ik,j}{}^{k} - h_{jk,i}{}^{k} = 0 \tag{1.8}$$

This does not look like the equation of the graviton in the spacetime described by $\xi^0, \xi^1 \ldots \xi^{D-1}$ due to the presence of the third term on the LHS. To deal with this term let us define

$$\tilde{h}_{ij} = h_{ij} + \alpha \, h_{DD} \, \eta_{ij} \tag{1.9}$$

which gives

$$h_k{}^{k} = \tilde{h}_k{}^{k} - \alpha \, D h_{DD} \tag{1.10}$$

Here α is a constant that we will fix presently. Then (1.8) becomes

$$\tilde{h}_{ij,k}{}^{k} - \alpha h_{DD,k}{}^{k}\eta_{ij} + \tilde{h}_{k}{}^{k}{}_{,ij} - \alpha D h_{DD,ij} + h_{DD,ij} - \tilde{h}_{ik,j}{}^{k} + \alpha h_{DD,ji} - \tilde{h}_{jk,i}{}^{k} + \alpha h_{DD,ij} = 0 \tag{1.11}$$

The second term on the LHS vanishes by (1.5). We are seeking the equation

$$\tilde{h}_{ij,k}{}^{k} + \tilde{h}_{k}{}^{k}{}_{,ij} - \tilde{h}_{ik,j}{}^{k} - \tilde{h}_{jk,i}{}^{k} = 0 \tag{1.12}$$

which would describe the graviton in the spacetime $\xi^0, \xi^1 \dots \xi^{D-1}$. We see that this is achieved by setting

$$\alpha = \frac{1}{D-2} \tag{1.13}$$

We thus see that the graviton in the D dimensional spacetime is given, not directly by the components g_{ij} of the total metric g_{ab} but by a modified expression that mixes h_{ij} with the scalar h_{DD}

$$\tilde{h}_{ij} = h_{ij} + \frac{1}{D-2} h_{DD}\ \eta_{ij} \tag{1.14}$$

We have thus far worked with only linear perturbations around flat space (eq. (1.1)), but for later use we recast the above results into the notation that we will encounter in the nonlinear case. Let us set $g_{Di} = 0$ for the moment. We recover the linear results by the following nonlinear definitions

$$g_{DD} = e^{-2\phi} \approx 1 - 2\phi \ \rightarrow \ h_{DD} = -2\phi \tag{1.15}$$

$$\tilde{g}_{ij} = e^{-\frac{2}{D-2}\phi} g_{ij} \ \rightarrow \ \tilde{h}_{ij} = h_{ij} - \frac{2}{D-2}\phi\ \eta_{ij} = h_{ij} + \frac{1}{D-2} h_{DD}\ \eta_{ij} \tag{1.16}$$

From (1.16) we see that the metric in $D-1$ dimensions is g_{ij} scaled by a factor which depends in general on the coordinates $\xi^0, \xi^1 \dots \xi^{D-1}$.