1. Dimensional reduction (with $A_a = 0$)

Let the total spacetime have dimension D, and let the indices $A, B\ldots$ run over $0, 1, \ldots D-1$. Let p of the directions be compactified on circles; the coordinates in these directions are $y^i, i = 1, \ldots p$. Let the coordinates in the remaining $D-p$ dimensions be $x^a, a = 0, 1, \ldots D-p-1$. We assume that the metric g_{AB} is independent of the y^i, and further assume for the moment that

$$g_{ai} = 0 \tag{1.1}$$

We look for solutions where the metric of the compact directions has the form

$$g_{ij} = e^{C(x)}\delta_{ij} \tag{1.2}$$

Then we get

$$\Gamma^i_{ja} = \frac{1}{2}\delta_{ij}C_{,a}, \quad \Gamma^a_{ij} = -\frac{1}{2}e^C g^{ab}C_{,b} \tag{1.3}$$

We now wish to compute the curvature tensor in D dimensions and express it in terms of the curvature tensor in the $D-p$ dimensional space spanned by the x^a. Working around a point x we choose a coordinate system where

$$g_{ab} = \eta_{ab}, \quad g_{ab,c} = 0 \tag{1.4}$$

We now express different components of the D dimensional Riemann tensor in terms of quantities in the $D-p$ dimensional spacetime spanned by the x^a. We find

$$[R^c{}_{adb}]_D = [R^c{}_{adb}]_{D-p} \tag{1.5}$$

Here the components $[R^c{}_{adb}]_D$ are computed using the metric g_{AB} in the entire D dimensional spacetime while $[R^c{}_{adb}]_{D-p}$ is computed by assuming that we are in a $D-p$ dimensional spacetime with the metric g_{ab}. We also get

$$\begin{aligned}
[R^i{}_{ajb}]_D &= \Gamma^i_{ab,j} - \Gamma^i_{aj,b} + \Gamma^i_{Aj}\Gamma^A_{ab} - \Gamma^i_{Ab}\Gamma^A_{aj} \\
&= -\Gamma^i_{aj,b} + \Gamma^i_{cj}\Gamma^c_{ab} - \Gamma^i_{kb}\Gamma^k_{aj} \\
&= -\Gamma^i_{aj,b} - \Gamma^i_{kb}\Gamma^k_{aj} \\
&= -\frac{1}{2}\delta^i_j C_{;ab} - \frac{1}{4}\delta^i_j C_{,a}C_{,b}
\end{aligned} \tag{1.6}$$

Here we have converted ordinary partial derivatives of C to covariant derivatives: we have vanishing connection on the $D-p$ dimensional spacetime due to the coordinate choice

(1.4), and when we go to general coordinates on this space the term C_{ab} becomes $C_{;ab}$. Similarly we get

$$[R^a{}_{ibj}]_D = -e^C[\frac{1}{2}\delta_{ij}C^{;a}{}_b + \frac{1}{4}\delta_{ij}C^{;a}C_{,b}] \tag{1.7}$$

$$\begin{aligned}[R^i_{kjl}]_D &= \Gamma^i_{Aj}\Gamma^A_{kl} - \Gamma^i_{Al}\Gamma^A_{kj} \\ &= \Gamma^i_{cj}\Gamma^c_{kl} - \Gamma^i_{cl}\Gamma^c_{kj} \\ &= -\frac{1}{4}e^C C_{,c}C^{;c}[\delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj}]\end{aligned} \tag{1.8}$$

Thus

$$[R_{ab}]_D = [R^c{}_{acb}]_D + [R^i{}_{aib}]_D = [R_{ab}]_{D-p} - \frac{p}{2}C_{;ab} - \frac{p}{4}C_{,a}C_{,b} \tag{1.9}$$

$$\begin{aligned}[R_{ij}]_D = [R^k{}_{ikj}]_D + [R^c{}_{icj}]_D &= -\delta_{ij}\frac{(p-1)}{4}e^C C_{,c}C^{;c} - e^C[\frac{1}{2}\delta_{ij}C^{;c}{}_c + \frac{1}{4}\delta_{ij}C^{;c}C_{,c}] \\ &= -\delta_{ij}[\frac{p}{4}e^C C_{,c}C^{;c} + \frac{1}{2}e^C C^{;c}{}_c]\end{aligned} \tag{1.10}$$

$$[R]_D = [R]_{D-p} - \frac{p}{2}C_{;c}{}^c - \frac{p}{4}C_{,c}C^{;c} - \frac{p^2}{4}C_{,c}C^{;c} - \frac{p}{2}C^{;c}{}_c = [R]_{D-p} - pC_{;c}{}^c - \frac{p(p+1)}{4}C_{,c}C^{;c} \tag{1.11}$$

The action of the D dimensional gravity theory is

$$S_D = \frac{1}{16\pi G}\int d^D\xi\sqrt{-g_D}[R]_D \tag{1.12}$$

From (1.2) we see that

$$\sqrt{-g_D} = \sqrt{-g_{D-p}}e^{\frac{p}{2}C} \tag{1.13}$$

We then get

$$S_D = \frac{V}{16\pi G}\int d^{D-p}x\sqrt{-g_{D-p}}e^{\frac{p}{2}C}([R]_{D-p} - pC_{;c}{}^c - \frac{p(p+1)}{4}C_{,c}C^{;c}) \tag{1.14}$$

where

$$V = \int dy^1 \ldots dy^p \tag{1.15}$$

is the 'coordinate volume' of the compact directions. V is a constant, and the variation in the scale of the compact directions is given through the function $C(x)$. The middle term in the bracket in (1.14) can be integrated by parts, and we get

$$S_D = \frac{V}{16\pi G}\int d^{D-p}x\sqrt{-g_{D-p}}e^{\frac{p}{2}C}([R]_{D-p} + \frac{p(p-1)}{4}C_{,c}C^{;c}) \tag{1.16}$$

The part $\int d^{D-p}x\sqrt{-g_{D-p}}e^{\frac{p}{2}C}[R]_{D-p}$ in (1.16) would look like the action for gravity in $D-p$ dimensions, if the factor $e^{\frac{p}{2}C}$ were absent. We can get rid of this factor by absorbing it in the metric g_{ab}, but this needs some care since derivatives of g_{ab} occur in $[R]_{D-p}$. We next study the steps involved in this process in some generality, since a similar computation will appear in several different contexts later on.

2. Scaling the metric

Suppose we have a D dimensional spacetime $x^0, x^1, \ldots x^{D-1}$ with a metric g_{ab}. Suppose we also have a scalar function $\mu(x)$ on this spacetime. We can then define another 'metric tensor' q_{ab}

$$q_{ab} = e^{\mu} g_{ab} \tag{2.1}$$

Quantities computed using the metric q will be labeled with the subscript q, and quantities computed using the metric g will be labeled with a subscript g.

Using the new metric q we find

$$[\Gamma^a_{\ bc}]_q = [\Gamma^a_{\ bc}]_g + \frac{1}{2}[\delta^a_b \mu_{,c} + \delta^a_c \mu_{,b} - q_{bc}\mu^{,a}] \tag{2.2}$$

where in the last term the metric is raised by q^{ab}.

Working around a point x, choose local coordinates such that

$$q_{ab}(x) = \eta_{ab}, \quad q_{ab,c}(x) = 0 \tag{2.3}$$

We then have

$$[R^c{}_{adb}]_q = [\Gamma^c_{ab,d} - \Gamma^c_{ad,b}]_q \tag{2.4}$$

$$\begin{aligned} [R^c{}_{adb}]_g = & [\Gamma^c_{ab,d} - \Gamma^c_{ad,b}]_g + [\Gamma^c_{fd}\Gamma^f_{ab} - \Gamma^c_{fb}\Gamma^f_{ad}]_g \\ = & [\Gamma^c_{ab,d} - \Gamma^c_{ad,b}]_q - \frac{1}{2}[\delta^c_a\mu_{,b} + \delta^c_b\mu_{,a} - q_{ab}\mu^{,c}]_{,d} + \frac{1}{2}[\delta^c_a\mu_{,d} + \delta^c_d\mu_{,a} - q_{ad}\mu^{,c}]_{,b} \\ & + \frac{1}{4}[\delta^c_f\mu_{,d} + \delta^c_d\mu_{,f} - q_{fd}\mu^{,c}][\delta^f_a\mu_{,b} + \delta^f_b\mu_{,a} - q_{ab}\mu^{,f}] \\ & - \frac{1}{4}[\delta^c_f\mu_{,b} + \delta^c_b\mu_{,f} - q_{fb}\mu^{,c}][\delta^f_a\mu_{,d} + \delta^f_d\mu_{,a} - q_{ad}\mu^{,f}] \end{aligned} \tag{2.5}$$

(Indices in the second, third, fourth lines are raised by q^{ab}.)

We now assume that we are in a coordinate system where

$$q_{ab} = \eta_{ab}, \quad q_{ab,c} = 0 \tag{2.6}$$

We then find that

$$\begin{aligned} [R^c{}_{adb}]_g = & [R^c{}_{adb}]_q - \frac{1}{2}[\delta^c_b\mu_{,ad} - q_{ab}\mu^{,c}{}_d] + \frac{1}{2}[\delta^c_d\mu_{,ab} - q_{ad}\mu^{,c}{}_b] \\ & - \frac{1}{4}[\delta^c_d q_{ab} - \delta^c_b q_{ad}]\mu_{,f}\mu^{,f} + \frac{1}{4}[\delta^c_d\mu_{,a}\mu_{,b} - \delta^c_b\mu_{,a}\mu_{,d}] \\ & - \frac{1}{4}[q_{ad}\mu^{,c}\mu_{,b} - q_{ab}\mu^{,c}\mu_{,d}] \end{aligned} \tag{2.7}$$

This yields

$$[R_{ab}]_g = [R_{ab}]_q + \frac{1}{2}(D-2)\mu_{;ab} + \frac{1}{2}q_{ab}\mu_{;c}{}^{c} - \frac{1}{4}(D-2)q_{ab}\mu_{,c}\mu^{,c} + \frac{1}{4}(D-2)\mu_{,a}\mu_{,b} \quad (2.8)$$

$$[R]_g = e^{\mu}[R]_q + e^{\mu}[(D-1)\mu_{;c}{}^{c} - \frac{1}{4}(D-1)(D-2)\mu_{,c}\mu^{,c}] \quad (2.9)$$

where in the RHS of (2.8), (2.9) index raising operations and covariant derivatives are all computed using the metric q.

3. 11-D M-theory to 10-D string theory

We have noted before that M theory exists in 11 spacetime dimensions $\xi^0, \xi^1, \ldots \xi^{10}$. The metric is governed by the Einstein action

$$S_M = \frac{1}{16\pi G}\int d^{11}\xi\sqrt{-g_{11}}R_{11} \quad (3.1)$$

Let the direction ξ^{10} be compactified on a circle of coordinate length L. We write

$$g_{10,10} = e^{C(x)} \quad (3.2)$$

Then the theory in the remaining 10 dimensions is given by (1.16) with $D = 10, p = 1$

$$S = \frac{L}{16\pi G}\int d^{10}x\sqrt{-g_{10}}e^{\frac{C}{2}}R_{10} \quad (3.3)$$

We can obtain an Einstein type action in 10-D by absorbing the factor $e^{\frac{C}{2}}$ into the metric. We will do this a little later, but first we note that a somewhat differently scaled metric would be more useful for several purposes. We have seen that there is a 2-brane in string theory, with action given by

$$S_{DBI} = T_{M2}\int d^3\zeta\sqrt{-\det(\bar{g})} \quad (3.4)$$

where T_{M2} is the tension of the M theory 2-brane and $\bar{g}$ is the metric induced on the 2+1 dimensional worldvolume of the M2 brane

$$\bar{g}_{ij} = \frac{\partial x^a}{\partial\zeta^i}\frac{\partial x^b}{\partial\zeta^j}g_{ij}, \quad i, j = 0, 1, 2 \quad (3.5)$$

Here $x^a(\zeta)$ gives the embedding of the M2 brane in the 11-D spacetime, and g_{ab} is the 11-D metric.

If the direction ξ^{10} is compactified on a circle, then we have seen that we can wrap the M2 brane on this circle, getting a string in 10-D. We can choose the M2 worldvolume coordinate ζ^2 as

$$\zeta^2 = \xi^{10} \tag{3.6}$$

The string configuration is then given by just the two coordinates ζ^0, ζ^1, and the action (3.2) is

$$S_{DBI} = T_{M2} L \int d^2\zeta e^{\frac{C}{2}} \sqrt{-\det(\bar{q})} \tag{3.7}$$

where

$$\bar{q}_{ij} = \frac{\partial x^a}{\partial \zeta^i} \frac{\partial x^b}{\partial \zeta^j} g_{ij}, \quad i, j = 0, 1 \tag{3.8}$$

The action (3.7) is similar to the simple 'area action' (3.2), but the tension is position dependent due to the factor $e^{\frac{C}{2}}$. If one is going to work with extensively with the string (as we do in string theory) then it is useful to rewrite the string DBI action so that the tension is constant. Note that the determinant in (3.7) is taken for a 2×2 matrix. Thus we see that if we define the 'string metric'

$$g^S_{ab} = e^{\frac{C}{2}} g_{ab} \tag{3.9}$$

and the corresponding induced metric

$$\bar{q}^S_{ij} = \frac{\partial x^a}{\partial \zeta^i} \frac{\partial x^b}{\partial \zeta^j} g_{ij}, \quad i, j = 0, 1 \tag{3.10}$$

then the action (3.7) becomes

$$S_{DBI} = T_{M2} L \int d^2\zeta \sqrt{-\det(\bar{q}^S)} \tag{3.11}$$

This *does* look like a simple 'area action' for the string, with a tension for the string that does not depend on x

$$T_S = T_{M2} L \tag{3.12}$$

What are we to make of the new 10-D metric g^S_{ab} defined in (3.9)? We should first find the gravity action in terms of this metric. We take the action S in (3.3) and convert $g = g_{10}$ to g^S. We have

$$\sqrt{-g_{10}} = e^{-\frac{5}{2}C} \sqrt{-g^S} \tag{3.13}$$

Using (2.9) with $D=10$, $\mu=\frac{C}{2}$ we get

$$[R]_g \equiv R_{10} = e^{\frac{C}{2}}[R]_S + e^{\frac{C}{2}}[\frac{9}{2}C_{;c}{}^{c} - \frac{9}{2}C_{;c}C^{;c}] \tag{3.14}$$

and the action S in (3.3) becomes

$$\begin{aligned} S &= \frac{L}{16\pi G}\int d^{10}x e^{-\frac{3}{2}C}\sqrt{-g_S}([R]_S + \frac{9}{2}C_{;c}{}^{c} - \frac{9}{2}C_{;c}C^{;c}) \\ &= \frac{L}{16\pi G}\int d^{10}x e^{-\frac{3}{2}C}\sqrt{-g_S}([R]_S + \frac{27}{4}C_{;c}C^{;c} - \frac{9}{2}C_{;c}C^{;c}) \\ &= \frac{L}{16\pi G}\int d^{10}x e^{-\frac{3}{2}C}\sqrt{-g_S}([R]_S + \frac{9}{4}C_{;c}C^{;c}) \end{aligned} \tag{3.15}$$

In field theory the coupling constant appears outside the action as $S=\frac{1}{g^2}\int S[\chi]$, where χ are the dynamical fields. We wish to regard the factor $e^{-\frac{3}{2}C}$ as the effect of a position dependent coupling. Writing $g=e^{\phi}$, we would like to define

$$e^{-\frac{3}{2}C} = \frac{1}{g^2} = e^{-2\phi} \tag{3.16}$$

so that $C=\frac{4}{3}\phi$. We then get from (3.15)

$$S = \frac{L}{16\pi G}\int d^{10}x\sqrt{-g_S}e^{-2\phi}([R]_S + 4\phi_{;c}\phi^{;c}) \tag{3.17}$$

This is the well known string action. The scalar ϕ is called the dilaton, and as will be proved later, gives the string coupling through $g=e^{\phi}$. It is related to the metric in the compact direction through (3.16).

4. T-duality

We obtained a 10-D geometry by compactifying the coordinate ξ^{10} in 11-D M theory. Let us now compactify another direction $\xi^9 = x^9$, on a circle of coordinate length $\tilde{L}$. It will be helpful to use the string metric g_S defined in (3.9) to describe the geometry. We write

$$g_{S,99} = e^{\tilde{C}} \tag{4.1}$$

Next we compute the action for the 9-D theory obtained in the remaining directions $x^0, x^1, \ldots x^8$. We start with the action (3.17) which is written in terms of the metric

g_S, and use (1.11) to get (we have $p = 1$ since we are compactifying one dimension in going from $R_{S,10}$ to $R_{S,9}$)

$$\begin{aligned} S &= \frac{L\tilde{L}}{16\pi G} \int d^9x \sqrt{-g_{S,9}} e^{-2\phi} e^{\frac{\tilde{C}}{2}} ([R]_{S,9} - \tilde{C}_{;c}{}^c - \frac{1}{2}\tilde{C}_{,c}\tilde{C}^{;c} + 4\phi_{;c}\phi^{;c}) \\ &= \frac{L\tilde{L}}{16\pi G} \int d^9x \sqrt{-g_{S,9}} e^{-2\phi} e^{\frac{\tilde{C}}{2}} ([R]_{S,9} - 2\phi_{;c}\tilde{C}^{;c} + \frac{1}{2}\tilde{C}_{;c}\tilde{C}^{;c} - \frac{1}{2}\tilde{C}_{,c}\tilde{C}^{;c} + 4\phi_{;c}\phi^{;c}) \\ &= \frac{L\tilde{L}}{16\pi G} \int d^9x \sqrt{-g_{S,9}} e^{-2\phi} e^{\frac{\tilde{C}}{2}} ([R]_{S,9} - 2\phi_{;c}\tilde{C}^{;c} + 4\phi_{;c}\phi^{;c}) \end{aligned} \tag{4.2}$$

The geometry is thus described by the 9-D metric and two scalars, $\phi, \tilde{C}$. Suppose we make another geometry which has the *same* 9-D metric, but with the opposite sign of $\tilde{C}$

$$\tilde{C}' = -\tilde{C} \tag{4.3}$$

From (4.1) this implies that we have made the change

$$g'_{S,99} = \frac{1}{g_{S,99}} \tag{4.4}$$

so that longer lengths of the compact direction x^9 change to *shorter* lengths. We can keep the prefactor of $[R]_{S,9}$ in (4.2) invariant if we also make the change

$$\phi' = \phi - \frac{1}{2}\tilde{C} \tag{4.5}$$

Interestingly though, we also find that

$$-2\phi'_{;c}\tilde{C}'^{;c} + 4\phi'_{;c}\phi'^{;c} = -2\phi_{;c}\tilde{C}^{;c} + 4\phi_{;c}\phi^{;c} \tag{4.6}$$

so that the entire action S in (4.2) is left invariat

$$S[g_{S,9}, \phi, \tilde{C}] = S[g_{S,9}, \phi', \tilde{C}'], \quad \phi' = \phi - \frac{1}{2}\tilde{C},\ \tilde{C}' = -\tilde{C} \tag{4.7}$$

This is an amazing symmetry, since it suggests that we can, roughly speaking, make the length of the compact direction change to its reciprocal, if we also scale the coupling $g = e^{\phi}$ by a suitable amount. Is this just a curiosity, or a real symmetry of the physics? If it is to be a real symmetry then there should be no way to tell the difference between the two systems related by the symmetry.