1. The gauge field from dimensional reduction

Consider a $D+1$ dimensional spacetime with coordinates ξ^A, where $A, B \ldots = 0, \ldots D$. Let the direction ξ^D be compact, with

$$0 \leq \xi^D < 2\pi \tag{1.1}$$

We write

$$x^a = \xi^a, \ a = 0, 1, \ldots D-1; \qquad y = \xi^D \tag{1.2}$$

We have seen that at linear order around flat spacetime the components $g_{aD} = g_{Da}$ behave as a gauge field A_a in the D dimensional reduced spacetime. Our first impulse might be to extend this identification also to the nonlinear level where A_a is not small

$$g_{AB} = \begin{pmatrix} \begin{pmatrix} \; g_{ab} \; \end{pmatrix} & \begin{pmatrix} A_a \end{pmatrix} \\ \begin{pmatrix} \; A_b \; \end{pmatrix} & e^C \end{pmatrix} \tag{1.3}$$

In other words the line element would be

$$ds^2 = g_{ab} dx^a dx^b + 2A_a dx^a dy + e^C dy^2 \tag{1.4}$$

But note that a basic feature of gauge fields is their change under gauge transformations

$$A'_a = A_a + \Lambda_{,a}, \quad F'_{ab} = \partial_a A'_b - \partial_b A'_a = F_{ab} \tag{1.5}$$

It is easy to see where such a freedom lies in the metric description. Over each point x of the D dimensional spacetime we have a circle but there is no canonical way to choose the point on this circle where $\xi^D = 0$. Suppose we change this choice

$$x'^a = x^a, \quad y' = y - \lambda(x) \tag{1.6}$$

Then

$$g'_{ay} = g_{ay} + e^C \lambda_{,a} \tag{1.7}$$

(Note that $\frac{\partial}{\partial x^a}\lambda = \frac{\partial}{\partial x'^a}\lambda$.) Eq. (1.7) with $g_{ay} = A_a$ does look a little like the equation for a gauge transformation (1.5) except for the factor e^C. But also note that

$$g'_{ab} = g_{ab} = g_{yb}\lambda_{,a} + g_{ay}\lambda_{,b} + e^C \lambda_{,a}\lambda_{,b} \tag{1.8}$$

This change in g_{ab} is a little difficult to interpret physically. From the point of view of D dimensions, we have a metric g_{ab} satisfying some equations arising from the 'matter' fields A_a, C. If we just make a gauge transformation on A_a then the physical field strength F_{ab} should not change and the metric g_{ab} should not change either. If we want this invariance under gauge transformations then we should write the $D+1$ dimensional metric as

$$ds^2 = g_{ab}dx^a dx^b + e^C(dy + A_a dx^a)^2 \tag{1.9}$$

Under the change (1.6) we have

$$dy' = dy - \lambda_{,a}dx^a \tag{1.10}$$

If A_a transformed as a gauge field

$$A'_a = A_a + \lambda_{,a} \tag{1.11}$$

then we would get

$$dy' + A'_a dx'^a = dy + A_a dx^a \tag{1.12}$$

and the line element (1.9) would be invariant without any change in g_{ab}. This looks more promising, so let us check that the coordinate change (1.6) indeed gives (1.11). The metric from (1.9) is

$$g_{AB} = \begin{pmatrix} \begin{pmatrix} g_{ab} + e^C A_a A_b \end{pmatrix} & \begin{pmatrix} e^C A_a \end{pmatrix} \\ \begin{pmatrix} e^C A_b \end{pmatrix} & e^C \end{pmatrix} \tag{1.13}$$

The coordinate change (1.6) gives

$$g'_{AB} = \begin{pmatrix} \begin{pmatrix} g_{ab} + e^C A'_a A'_b \end{pmatrix} & \begin{pmatrix} e^C A'_a \end{pmatrix} \\ \begin{pmatrix} e^C A'_b \end{pmatrix} & e^C \end{pmatrix} \tag{1.14}$$

with A'_a gives by (1.11).

**■

Problem 1: Verify that the transformation (1.6) indeed gives (1.14).

**■

The definition (1.13) gives a somewhat curious way of obtaining the D dimensional metric g_{ab} from g_{AB}, but algebraically this definition pays immediate dividends. We have from (1.13)

$$\det g_{AB} = (\det g_{ab})(e^{C}) \tag{1.15}$$

By contrast, the determinant of g_{AB} using the definition (1.3) is more messy. Further, with the definition (1.13)

$$g^{AB} = \begin{pmatrix} \begin{pmatrix} & g^{ab} & \end{pmatrix} & \begin{pmatrix} -A^{a} \end{pmatrix} \\ (\quad -A^{b} \quad) & e^{-C} + A_a A^a \end{pmatrix} \tag{1.16}$$

where all indices are raised and lowered by the D dimensional metric g_{ab} obtained through (1.13).

****

Problem 2: Verify that (1.16) is indeed the inverse of (1.13).

****

2. Dimensional reduction of the action with $A_a \neq 0$

The action in $D+1$ dimensions is

$$\frac{1}{16\pi G}\int d^{D+1}\xi\sqrt{-g_{D+1}}[R]_{D+1} \tag{2.1}$$

From (1.1) we have

$$\int d^{D+1}\xi = 2\pi\int d^{D}x \tag{2.2}$$

From (1.15) we have

$$\sqrt{-g_{D+1}} = \sqrt{-g_D}\,e^{\frac{C}{2}} \tag{2.3}$$

We now need to compute $[R]_{D+1}$ in terms of D dimensional quantities.

2.1. Coordinate choices

To convert $[R]_{D+1}$ to D dimensional variables all we have to do in principle is to take g_{AB} given in (1.13) and use it to compute $[R]_{D+1}$. But as before it will be more convenient to note first that $[R]_{D+1}$ is a scalar, and so can be computed at a point $\bar{\xi} = (\bar{x}, \bar{y})$ by choosing convenient coordinates around this point. All quantities will be independent of y. We choose the coordinates x around $\bar{x}$ so that

$$g_{ab}(\bar{x}) = \eta_{ab}, \quad g_{ab,c}(\bar{x}) = 0 \tag{2.4}$$

Note that $g_{ab,cd}(\bar{x}) \neq 0$ in general. We can also choose coordinates to make

$$A_a(\bar{x}) = 0 \tag{2.5}$$

though $A_{a,b}(\bar{x}) \neq 0$ in general.

****

Problem 3: Show how to construct coordinates that will achieve (2.5). Show that the change of coordinates used has the form of a gauge transformation.

****

2.2. Computing the connection

We have

$$\begin{aligned}
[\Gamma^a_{bc}]_{D+1} =& \frac{1}{2}[g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d})]_{D+1} + \frac{1}{2}[g^{ay}(g_{yb,c} + g_{yc,b} - g_{bc,y})]_{D+1} \\
=& \frac{1}{2} g^{ad}[(g_{db} + e^C A_d A_b)_{,c} + (g_{dc} + e^C A_d A_c)_{,b} - (g_{bc} + e^C A_b A_c)_{,d}] \\
& - \frac{1}{2} A^a [(e^C A_b)_{,c} + (e^C A_c)_{,b}]
\end{aligned} \tag{2.6}$$

Note that in computing the curvature we will have at most one derivative on Γ, and then we will set $x = \bar{x}$ which by (2.5) sets $A_a = 0$. Thus Γ can have at most one A which is not differentiated. So in a term like $(e^C A_d A_b)_{,c}$ appearing in (2.6) we must let the derivative fall on one of the two A factors and not on C. With such simplifications we get

$$[\Gamma^a_{bc}]_{D+1} = \Gamma^a_{bc} + \frac{1}{2} e^C g^{ad}[(A_{d,c} - A_{c,d})A_b + (A_{d,b} - A_{b,d})A_c] \tag{2.7}$$

Finally, note that at the end of the computation that all quantities should be expressible in terms of the gauge invariant field strength $F_{ab} = A_{b,a} - A_{a,b}$. Thus we write

$$[\Gamma^a_{bc}]_{D+1} = \Gamma^a_{bc} - \frac{1}{2}e^C g^{ad}[F_{db}A_c + F_{dc}A_b] \tag{2.8}$$

With a similar prescription for the terms to be retained we find

$$[\Gamma^y_{ab}]_{D+1} = \frac{1}{2}e^{-C}[(e^C A_a)_{,b} + (e^C A_b)_{,a}] \tag{2.9}$$

$$[\Gamma^a_{yb}]_{D+1} = -\frac{1}{2}e^C g^{ad}[F_{db} + C_{,d}A_b] \tag{2.10}$$

$$[\Gamma^y_{ya}]_{D+1} = \frac{1}{2}C_{,a} - \frac{1}{2}e^C F_{ab}A^b \tag{2.11}$$

$$[\Gamma^a_{yy}]_{D+1} = -\frac{1}{2}e^C g^{ad}C_{,d} \tag{2.12}$$

$$[\Gamma^y_{yy}]_{D+1} = \frac{1}{2}e^C A^a C_{,a} \tag{2.13}$$

2.3. Computing the curvature

Recall that after computing the derivatives involved in the curvature we set $g_{ab,c} = 0, A_a = 0$. We then find

$$\begin{aligned}
[R^a{}_{bcd}]_{D+1} =&[\Gamma^a_{bd,c} - \Gamma^a_{bc,d}]_{D+1} + [\Gamma^a_{Ac}\Gamma^A_{bd} - \Gamma^a_{Ad}\Gamma^A_{bc}]_{D+1}\\
=&[R^a{}_{bcd}]_D - \frac{1}{2}e^C F^a{}_b F_{cd} - \frac{1}{2}e^C(F^a{}_d A_{b,c} - F^a{}_c A_{b,d}) + \Gamma^a_{yc}\Gamma^y_{bd} - \Gamma^a_{yd}\Gamma^y_{bc}\\
=&[R^a{}_{bcd}]_D - \frac{1}{2}e^C F^a{}_b F_{cd} + \frac{1}{4}e^C(F^a{}_d F_{bc} - F^a{}_c F_{bd})
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
[R^a{}_{ycy}]_{D+1} =&[\Gamma^a_{yy,c}]_{D+1} - [\Gamma^a_{fy}\Gamma^f_{yc} + \Gamma^a_{yy}\Gamma^y_{yc}]_{D+1}\\
=& -\frac{1}{2}e^C C^a{}_{,c} - \frac{1}{4}e^C C^{,a}C_{,c} - \frac{1}{4}e^{2C}F^a{}_f F^f{}_c
\end{aligned} \tag{2.15}$$

where indices are raised with $g^{ab}(\bar{x}) = \eta^{ab}$.

$$[R^y{}_{byd}]_{D+1} = -\frac{1}{2}C_{,bd} - \frac{1}{4}C_{,b}C_{,d} - \frac{1}{4}e^C F_{bf}F^f{}_d \tag{2.16}$$

We can now compute the Ricci tensor and Ricci scalar

$$[R_{bd}]_{D+1} = [R_{bd}]_D - \frac{1}{2}C_{,bd} - \frac{1}{4}C_{,b}C_{,d} - \frac{1}{2}e^C F_{bf}F_d{}^f \tag{2.17}$$

$$[R_{yy}]_{D+1} = -\frac{1}{2}e^C C^a{}_{,a} - \frac{1}{4}e^C C^{,a}C_{,a} - \frac{1}{4}e^{2C}F^a{}_f F^f{}_a \tag{2.18}$$

$$[R]_{D+1} = [R]_D - C^a{}_{,a} - \frac{1}{2}C^{,a}C_{,a} - \frac{1}{4}e^C F_{ab}F^{ab} \tag{2.19}$$

3. The dimensionally reduced action

Let us return to the computation of the action (2.1). Using (2.2), (2.3), (2.16) we get

$$\begin{aligned} S =& \frac{(2\pi)}{16\pi G}\int d^Dx e^{\frac{C}{2}}\sqrt{-g_D}([R]_D - C^a{}_{,a} - \frac{1}{2}C^{,a}C_{,a} - \frac{1}{4}e^C F_{ab}F^{ab}) \\ =& \frac{(2\pi)}{16\pi G}\int d^Dx e^{\frac{C}{2}}\sqrt{-g_D}([R]_D - \frac{1}{4}e^C F_{ab}F^{ab}) \end{aligned} \tag{3.1}$$

3.1. The string worldsheet action with $A_a \neq 0$

Let us now apply the above to the dimensional reduction of 11-D M theory to 10-D string theory. The 2-brane action was just an 'area action'. The string was obtained by wrapping the 2-brane on the compact circle, but the action was a 'weighted area action'. To convert it to a simple 'area action' we had to use a rescaled metric g^S_{ab} for the 10-D string theory. Let us now see how these steps change when $A_a \neq 0$.

The world sheet of the 2-brane is parametrized by coordinates $\zeta^0, \zeta^1, \zeta^2$. We choose the world sheet coordinates as

$$\begin{aligned} \zeta^0 =& \zeta^0(x^0, x^1, \dots x^9) \\ \zeta^1 =& \zeta^1(x^0, x^1, \dots x^9) \\ \zeta^2 =& x^{10} \end{aligned} \tag{3.2}$$

The 2-brane action is

$$S_{M2} = T_{M2}\int d^3\zeta\sqrt{-\bar{g}} \tag{3.3}$$

where

$$\bar{g}_{ij} = \frac{\partial x^A}{\partial \zeta^i}\frac{\partial x^B}{\partial \zeta^j} g^{11}_{AB}, \quad i,j = 0,1,2 \tag{3.4}$$

Following our experience with dimensional reduction we write the induced 3×3 metric as

$$\bar{g}_{ij} = \begin{pmatrix} \begin{pmatrix} g^{(2)}_{ij} + e^{\bar{C}}\bar{A}_i\bar{A}_j \end{pmatrix} & \begin{pmatrix} e^{\bar{C}}\bar{A}_1 \\ e^{\bar{C}}\bar{A}_2 \end{pmatrix} \\ (\, e^{\bar{C}}\bar{A}_1 \quad e^{\bar{C}}\bar{A}_2 \,) & (e^{\bar{C}}) \end{pmatrix} \tag{3.5}$$

From (3.4) and using the decomposition (1.13) for the 11-D metric we find that

$$\bar{C} = C \tag{3.6}$$

$$\bar{A}_i = \frac{\partial x^a}{\partial \zeta^i}A_a, \quad a = 0,1\dots 9, \quad i = 0,1 \tag{3.7}$$

$$\bar{g}^{(2)}_{ij} = \frac{\partial x^a}{\partial \zeta^i}\frac{\partial x^b}{\partial \zeta^j} g^{11}_{ab} - e^C(\frac{\partial x^a}{\partial \zeta^i}A_a)(\frac{\partial x^b}{\partial \zeta^j}A_b) = \frac{\partial x^a}{\partial \zeta^i}\frac{\partial x^b}{\partial \zeta^j} g^{10}_{ab} \tag{3.8}$$

We finally have, using (1.15)

$$\det \bar{g} = \det \bar{g}^{(2)} e^{\bar{C}} = \det \bar{g}^{(2)} e^{C} \tag{3.9}$$

and the action (3.3) becomes

$$S_{M2} = 2\pi T_{M2} \int d^2\zeta e^{\frac{C}{2}} \sqrt{-\bar{g}^{(2)}} \tag{3.10}$$

Note that $g^{(2)}_{ij}$ in (3.8) is obtained naturally as the induced metric when the 10-D metric is g^{10}_{ab} defined with the prescription (2.2). The last step of rescaling the metric is now the same as in the case $A_a = 0$

$$g^S_{ab} = e^{\frac{C}{2}} g^{10}_{ab} \tag{3.11}$$

This gives an 'area action'

$$S_{M2} = 2\pi T_{M2} \int d^2\zeta \sqrt{-\bar{g}} \tag{3.12}$$

with

$$\bar{g}_{ij} = \frac{\partial x^a}{\partial \zeta^i}\frac{\partial x^b}{\partial \zeta^j} g^S_{ab}, \quad i,j = 0,1 \tag{3.13}$$

3.2. The gravity action in the string metric

We write the action (3.1) in the 10-D string metric (3.13)

$$\begin{aligned} S =& \frac{(2\pi)}{16\pi G} \int d^{10}x e^{\frac{C}{2}} \sqrt{-g_{10}} [[R]_{10} - \frac{1}{4} e^C F_{ab} F_{cd} (g^{10})^{ac} (g^{10})^{bd}] \\ =& \frac{(2\pi)}{16\pi G} \int d^{10}x \sqrt{-g^S} [e^{-\frac{3C}{2}} ([R]_S + \frac{9}{4} C_{,a} C^{;a}) - \frac{1}{4} F_{ab} F^{ab})] \end{aligned} \tag{3.14}$$

where in the last step we let the indices be raised and lowered by g^S. After the redefinition $\frac{3}{2}C = 2\phi$ we get the conventional form of the action in the string metric

$$S = \frac{(2\pi)}{16\pi G} \int d^{10}x \sqrt{-g^S} [e^{-2\phi}([R]_S + 4C_{,a}C^{;a}) - \frac{1}{4} F_{ab} F^{ab})] \tag{3.15}$$

****

Problem 4: Note that powers of e^ϕ multiplying the gauge field action in (3.15) all canceled out. Does this fact depend on the particular dimensions $11 \to 10$ involved in the reduction from M theory to string theory or would we get this cancellation for any $D+1 \to D$ reduction, assuming all other steps in the reasoning were the same?

****