1. The Energy-Momentum tensor

We now wish to describe the contribution of matter to Einstein equations: Matter will be a source of curvature, and the final equations should read, symbolically, 'curvature=matter'. We begin our study by considering a simple case: dust. Here the term 'dust' signifies a collection of particles that have mass, but which do not have random velocities. Thus a small cube filled with dust will have energy, and if it is moving, also momentum, but no 'pressure'.

1.1. Currents and conservation laws

Consider a collection of particles that behave like 'dust'. For simplicity, let each of the particles have the same mass m, charge e etc. Assume that these particles can move around, but that they cannot be created or destroyed. Let us first discuss their description in flat spacetime with Cartesian coordinates.

The conservation of these particles is captured by the continuity equation

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \tag{1.1}$$

where n is the number density, and

$$\vec{J} = n\vec{v} \tag{1.2}$$

is the 'flux' vector. We can combine these quantities into the 4-vector

$$\{J^0, J^i\} = \{n, J^i\} \tag{1.3}$$

so that the continuity equation takes the form

$$J^a{}_{,a} = 0 \tag{1.4}$$

Let us focus on a small volume element containing dust. Since the dust particles have no random velocities, we can go to a reference frame where all the dust particles are at rest. Let the number desnity in this rest frame be $\bar{n}(x)$. We can write in this frame

$$J^a = \bar{n}U^a \tag{1.5}$$

where $U^a(x)$ is the 4-velocity of each the dust particles in the volume. This is evident since in this frame we have $U^a = \{1, 0, 0, 0\}$. Now change to a frame moving with velocity $-\vec{v}$, so that the dust in the chosen volume is moving with velocity $\vec{v}$. Since J^a, U^a are both vectors, and $\bar{n}$ is just a scalar, we expect in this moving frame that we will still have

$$J^a(x) = \bar{n}(x) U^a(x) \tag{1.6}$$

where U^a is now $\{\gamma, \gamma\vec{v}\}$. To see that this is correct, note that the volume occupied by the dust particles in the 'contracts' by the factor $\gamma = \frac{1}{\sqrt{1-v^2}}$ when we go to the frame where the dust is moving with velocity v. Thus

$$n = \bar{n}\gamma = \bar{n}U^0 \tag{1.7}$$

$$J^i = nv^i = \bar{n}\gamma v^i = \bar{n}U^i \tag{1.8}$$

1.2. What is the source for gravity

The vector J^a obtained above is the 'particle number' vector, since it described the number density and number flux of the dust particles. If the dust particles were charged, they would be a source for the electromagnetic potential A^a. The electromagnetic current vector is

$$J^a_{el} = eJ^a \tag{1.9}$$

We might wonder if the source for gravity would be obtained by writing

$$J^a_{grav} = mJ^a \quad ?? \tag{1.10}$$

At this point we face a question of physics: is it rest mass that lead to gravitational effects or is it energy? If it were rest mass, then (1.10) would appear to be a correct prescription, since this vector described the mass density and mass flux of the particles. But if it is energy density that is relevant, then the gravitational effect of dust will be sensitive to both the rest mass as well as the kinetic energy of the dust.

Let us consider the latter possibility in more detail. The energy of each dust particle is $m\gamma$, and the number density is $\bar{n}\gamma$. Thus the energy density is $m\bar{n}\gamma^2 = m\bar{n}(U^0)^2$. This suggests that the correct analogue of (1.6) is

$$T^{ab} = m\bar{n}(x)U^aU^b \tag{1.11}$$

Then T^{00} is the energy density.

Let us see why we have two indices on our tensor T^{ab} instead of just one on currents like J^a_{el}. The index on J^a_{el} arose from the index on U^a, so it can be traced to the 4-velocity direction of the dust particles. One of the indices on T^{ab} can be traced to the same source. The other index arises from the fact that the relevant quantity carried by the dust is not a scalar like charge but a 4-vector, whose time component is energy but which also has space components – momentum. Thus T^{ab} gives the 'current' component 'b' of the quantity which is the component 'a' of the energy-momentum 4-vector. This tesnor is called the energy-momentum tensor or the stress-energy tensor.

1.3. Curved space

We can easily see the form that (1.4) must take in curvlinear coordinates, or on curved spacetime more generally

$$J^a{}_{;a} = 0 \tag{1.12}$$

The corresponding conservation law for T^{ab} would be

$$T^{ab});b = 0 \tag{1.13}$$

Let us see if the T^{ab} for dust written in (1.11) satisfies this. We have

$$T^{ab}{}_{;b} = mU^a(\bar{n}U^b)_{;b} + m\bar{n}U^a_{;b}U^b \tag{1.14}$$

Now $(\bar{n}U^b)_{;b} = 0$ from the conservation of the particle number, eq. (1.12), and $U^a_{;b}U^b = 0$ by the geodesic equation, so we see that (1.13) is satisfied.

It may appear strange that we had to use the geodesic equation to see conservation of T^{ab}; after all there could have been forces acting on the dust particles which would have made them follow tracks that were not geodesics. But if there were such additional forces present, they would create additional components of the Stress-Energy tensor, and the total tensor would again be conserved.