1. The implications of T-duality

Let us recall what we learnt about T-duality at the classical level. Starting with 11-d M theory we compactified one direction y with $g^{11}_{yy} = e^C$. (We took all off-diagonal components of the metric to zero for simplicity). After doing dimensional reduction to 10-d, we defined the string metric

$$g^S_{ab} = e^{\frac{C}{2}} g^{10}_{ab} = e^{\frac{2\phi}{3}} g^{10}_{ab} \tag{1.1}$$

In this 10-d theory we compactified another circle, say the direction x^9. Let the *string* metric in this direction be

$$g^S_{99} = e^{\tilde{C}} \tag{1.2}$$

Then the gravity action is invariant under the change

$$\tilde{C}' = -\tilde{C} \tag{1.3}$$

$$\phi' = \phi - \frac{1}{2}\tilde{C} \tag{1.4}$$

The 9-d string metric is left unchanged. While this is a symmetry of the classical action we wonder if it can be a symmetry for the full quantum theory. (1.3) implies that lengths L in the x^9 direction change to a new length that is proportional to the *reciprocal* $\frac{1}{L}$. Can we not tell the difference between compact directions of these two different lengths by some experiments? If we can, then we would not really have a symmetry under (1.3), (1.4). To examine this issue, let us consider a point x in the 9-direction noncompact space, and look at the x^9 direction circle at this x before and after the T-duality. We write the effect of T-duality as

$$L' = \frac{A}{L} \tag{1.5}$$

where L is the length of the x^9 circle before T-duality, L' the length after T-duality, and A is a constant with units $(length)^2$. We certainly have a massless particle in the theory – the graviton, and we can let this graviton run around the x^9 circle. Quantum mechanics tells us that the energy levels of such modes are (for a circle of length L)

$$E = \frac{2\pi n}{L} \tag{1.6}$$

where n is an integer. After the duality map we have a circle of length L', which has energy levels

$$E = \frac{2\pi m}{L'} = \frac{2\pi m}{A} L \tag{1.7}$$

For an experimenter looking at the situation from the 9-d perspective, these are different spectra; the spectrum (1.6) is *closely* spaced at points x where L is large, while the spectrum (1.7) is *widely* spaced where L is large. But suppose we had in the system a *string* which we could wrap around the compact circle. The string has zero rest length, and a tension T; the energy of the string stretched to a length l is Tl. We can wrap the string m times around the compact circle, and this gives the total energy spectrum (before duality)

$$E = \frac{2\pi n}{L} + mTL \tag{1.8}$$

After T-duality we have the circle length (1.5) and the spectrum

$$E = \frac{2\pi m}{L'} + nTL' = \frac{2\pi m}{A}L + nT\frac{A}{L} \tag{1.9}$$

(We have assumed that the string tension does not change under the duality.) To match the parts of (1.8), (1.9) linear in L we need

$$A = \frac{2\pi}{T} \tag{1.10}$$

We observe that with this choice the other terms in (1.8), (1.9) match up as well. So we see that if we do have a string in the theory, then under T-duality we interchange the roles of *gravitons* (running along x^9) with *strings* (wrapped on x^9). The gravitons carry momentum, so they are termed *momentum modes.* The strings wind around the compact circle, and so the wrapped strings are called *winding modes.* The combined spectrum of these modes is indeed invariant under a change

$$L' = \frac{2\pi}{T}\frac{1}{L} \tag{1.11}$$

and we can hope to make T-duality a symmetry of the full quantum theory.

2. IIA and IIB string theories

We have concentrated on only the bosonic fields in our gravity theory, but consistency at the quantum level needs fermionic fields as well; in fact equal in number to the bosonic fields so that we have *supersymmetry.* We will study fermionic fields later, but at this stage we need one piece of information about these fields. A supersymmetry operation Q maps a bosonic state to a fermionic one

$$|F\rangle = Q|B\rangle \tag{2.1}$$

Bosonic states have integer spin while fermionic ones have half-integer spin; thus Q has half-integer spin. The vectors and tensors $T_{a_1 \ldots a_n}$ all have integer spin. Half integer spin representations are called *spinors*. A spinor in D spacetime dimensions has $2^{[\frac{D}{2}]}$ components, where $[X]$ means the integer part of X. For D even the spinor can be decomposed into two halves, of dimension $2^{\frac{D}{2}-1}$ each, which do not mix under rotations, but go into each other under *reflections*. These two parts are called the it right handed and *left handed* representations.

In 10-d space with Lorentzian signature (which is what we have for our string theory) the spinors can be taken to have real components; thus the left and right handed spinors have 16 real components each. In the theories we start with there are in fact *two* spinors Q_1, Q_2 that generate transformations (2.1). The theories are thus called type II theories. But there are two possibilities – both spinors can have opposite handedness or both can have the same handedness (the choice of left/right is then convention). The former gives Type IIA string theory, while the latter gives Type IIB string theory.

The 11-d M theory has only one spinor with 32 real components; we cannot break this into left and right parts. If we have one such spinor Q for M theory, then on dimensional reduction to 10-d we get 2 spinors with 16 components each, one right handed and one left handed. Thus from M-theory we get type IIA theory after we compactify the coordinate y. Type IIB string theory will not come as a dimensional reduction, since such reduction always generates equal numbers of left and right handed spinors.

All this is relevant for us at this stage for one reason – T duality maps the IIA theory to IIB and vice versa

$$\begin{aligned} IIA &\rightarrow IIB \\ IIB &\rightarrow IIA \end{aligned} \tag{2.2}$$

Each of the two theories has a graviton, and by the arguments above, if we wish to have T-duality as a real symmetry then the graviton of Type IIA implies that we need a string in Type IIB, and the graviton in IIB implies a string in IIA.

3. The content of Type IIA string theory

We have already seen that the string of IIA should arise from a 2-brane of M theory, called the M2 brane. If we wrap this M2 on the compact circle y then we get a string in 10-d. If we do *not* wrap it then both spatial directions of the brane lie along the directions of the 10-d theory, and we get a 2-brane that we call the D2 brane of Type IIA string theory.

In the 11-d M theory if we have a M2 brane then electric-magnetic duality suggests that we should also have a brane of dimension $11-(4+2)=5$. This is the M5 brane, and it completes the brane content of M theory. But in IIA string theory we get two kinds of branes from M5: If we wrap M5 along y then we get a 4-brane – which will be called the D4 brane. If we do not wrap the M5 along y then we get a 5-brane in 10-d which will be called the NS5 brane of the IIA theory. (The reasons for these names will be clear only later.)

We have already seen that there is a D6 brane in IIA. We can thus collect all the branes of IIA in the following diagram

$$\begin{array}{lccccccc} Type\ IIA: & & F1 & & & NS5 & & \\ & D0 & & D2 & D4 & & D6 & \end{array} \tag{3.1}$$

4. The action of T-duality on the D0 brane

We have seen that momentum modes and winding modes go into each other under T-duality. What is the action of T-duality on the D0 brane, which is also an object in IIA string theory? From our analysis of objects in the theory so far we expect that the T-dual of the D0 brane will be another brane, and we wish to find the dimension p of this brane. If the mass of the brane scales linearly with the length of a compact direction, then we will know that we have a brane that wraps that direction. Thus we wish to study the masses of objects in string theory.

4.1. Measuring masses in string theory

This issue becomes a little complex because we have more than one metric which we have defined on the same space. Start with 11-d M theory, and let all off-diagonal terms in the metric vanish for simplicity. Let the y direction be compactified with coordinate range $0 \le y < l$, and $g^{11}_{yy} = e^C$. The proper length of the y direction is then $L = le^{\frac{C}{2}}$. A single D0 brane is the lowest possible momentum mode along the y circle, and this has energy

$$E = \frac{2\pi}{L} = \frac{2\pi}{l}e^{-\frac{C}{2}} \tag{4.1}$$

We are using units where $c = \hbar = 1$, and mass has units $(length)^{-1}$. A 10-d experimenter who sees a D0 brane at location x can then mark off a coordinate range in the 10-d spacetime which also has proper length L. If he marks this length off in the time direction, he gets a coordinate interval $\Delta t = \frac{L}{(g_{tt})^{\frac{1}{2}}}$, where g_{tt} is at the location x.

But we have put another metric on this 10-d space, $g^S_{ab} = e^{\frac{C}{2}} g^{10}_{ab}$. A person using the metric g^S will find for the coordinate interval Δt a proper length

$$L^S = (g^S_{tt})^{\frac{1}{2}} \Delta t = l e^{\frac{3C}{4}} = l e^{\phi} \tag{4.2}$$

This experimenter, who uses the string metric, will then see that the mass of the D0 brane is

$$m^S_{D0} \sim (L^S)^{-1} \sim e^{-\phi} \sim \frac{1}{g} \tag{4.3}$$

where our interest is only in the dependence of this mass on the size of the y circle at the location x. We see that the mass of the D0 brane in the string metric depends on the 'coupling' g as $m \sim \frac{1}{g}$. This coupling dependence is an important result, and characterizes all D-branes.

One might wonder if the above computations have any physical meaning if we can change the mass by using a different metric $g'_{ab} = e^{\alpha C} g_{ab}$. But in fact there is a direct physical significance to the masses measured in the string metric. The string has a tension T, which gives a unit of length $l_s \equiv T^{-\frac{1}{2}}$. In the ground state of the string we can expect that quantum fluctuations give the string a size l_s, so this is a physical length scale. We can then compare the Compton wavelength of the D0 brane $\lambda = 1/m_{D0}$ with l_s, and the ratio $\frac{\lambda}{l_s}$ is an interesting physical quantity.

The ratio $\frac{\lambda}{l_s}$ is independent of what metric we use to measure lengths. But using the string metric certainly simplifies things. Recall that the string had a position dependent tension in the metric g^{10} obtained by dimensional reduction of g^{11}, and the string metric was constructed just to get rid of this position dependence. Thus the tension T of the string is a constant in the string metric, and this makes l_s constant in the string metric. Thus measuring λ in the string metric directly compares the D0 brane Compton wavelength to the string length l_s. The conclusion (4.3) thus says that

$$m_{D0} \sim \frac{1}{g l_s} \tag{4.4}$$

and we will fix the proportionality constant later.

4.2. T-duality on D0

Let the x^9 direction be compactified with $0 \leq x^9 < \tilde{l}$, and $g^S_{99} = e^{\tilde{C}}$. The mass of the D0 brane in the string metric is

$$m^S_{D0} \sim e^{-\phi} \tag{4.5}$$

The string metric on the 9-d noncompact spacetime is not changed under T-duality, and if T-duality is a symmetry then the Compton wavelength of the D0 brane (as seen in these 9 dimensions) must be the same as the Compton wavelength of the object obtained after T-duality. We should therefore just see what the mass (4.5) looks in terms of the variables $\phi', \tilde{C}'$ obtained after T-duality. We have from (1.3), (1.4)

$$m^S_{d0} \sim e^{-\phi} = e^{-\phi'} e^{\frac{\tilde{C}'}{2}} \tag{4.6}$$

The factor $e^{\frac{\tilde{C}'}{2}}$ shows that the mass depends linearly on the length of the x^9 direction, while the factor $e^{-\phi'} \sim \frac{1}{g'}$ shows that the mass per unit length has the $(coupling)^{-1}$ dependence that characterizes D-branes. Thus we conclude that *the T-dual of the D0 brane of IIA string theory is a D1 brane of IIB string theory.*

Since $(T_9)^2 = 1$, we also have that the T-dual of the D1 brane of IIB string theory is a D0 brane of IIA string theory. We summarize these results

$$\begin{aligned} T: \quad & D0\ (IIA) \ \rightarrow \ D1\ (IIB) \\ & D1\ (IIB) \ \rightarrow \ D0\ (IIA) \end{aligned} \tag{4.7}$$

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Problem 1: Assume that any D-brane has a tension $T \sim \frac{1}{g}$. Generalize the above argument to show that T-duality takes a D-p-brane to a D-(p+1)-brane (if the brane did not span the direction of the T-duality) or to a D-(p-1)-brane (if the brane did span the direction of the T-duality).

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5. The gauge field action

The D0 brane couples to a gauge field A_a with field strength F_{ab}, and the action for this gauge field is

$$S = -\frac{l}{16\pi G} \int d^{10}x \sqrt{-g^S} \frac{1}{4} F_{ab} F^{ab} \tag{5.1}$$

where the indices are raised by g_S^{ab}. For the D0 brane at rest we have $A_a = (A_0, 0, \ldots 0)$. With x^9 compactified as above we will let the distribution of D0 brane be uniform along x^9, and then $A_0 = A_0(r)$ where r is the radial coordinate in the 8 remaining noncompact directions. We find

$$S = -\frac{l\tilde{l}}{16\pi G}\int d^9x\sqrt{-g_9^S}\,e^{\frac{\tilde{C}}{2}}\frac{1}{2}(A_{0,r})^2 g_S^{00} g_S^{rr} \tag{5.2}$$

After T-duality we have a D1 brane, and we expect a potential A_{ab} to couple to this brane. The directions along the brane worldsheet are x^0, x^9, so we expect A_{09}. We thus write

$$T_9: \qquad A_0 \to e^{\alpha\phi+\beta\tilde{C}} A_{09} \tag{5.3}$$

where we have allowed for possible factors depending on the scalars $\phi, \tilde{C}$ in the 9-d theory. The action for A_{ab} will have a field strength F_{abc} with the nonzero component $F_{09r} = A_{09,r}$

$$\begin{aligned} S = &-\frac{l\tilde{l}}{16\pi G}\int d^9x\sqrt{-g_9^S}\,e^{\frac{\tilde{C}'}{2}}\frac{1}{12}F_{abc}F^{abc} = -\frac{l\tilde{l}}{16\pi G}\int d^9x\sqrt{-g_9^S}\,e^{\frac{\tilde{C}'}{2}}\frac{1}{2}(A_{09,r})^2 g_S^{00} e^{-\frac{\tilde{C}}{2}} g_S^{rr} \\ = &-\frac{l\tilde{l}}{16\pi G}\int d^9x\sqrt{-g_9^S}\frac{1}{2}(A_{09,r})^2 g_S^{00} g_S^{rr} \end{aligned} \tag{5.4}$$

We must equate the 9-d actions (5.2) and (5.4) if T-duality is a symmetry, and then we find that in (5.3) we need $\alpha = \beta = 0$

$$T_9: \quad A_0 \to A_{09} \tag{5.5}$$

Thus the gauge potential just acquires a new index along the direction of the T-duality if it did not have this index before; since $(T_9)^2 = 1$ we also see that if we started with A_{09} which had an index 9 then we *lose* this index.

6. S-duality

We now have two strings in type IIB theory – the string we obtained by T-duality on the 'momentum mode' called the fundamental string, also for reasons to be seen later, NS1) and the D string (also termed the D1 brane). These two strings are not the same, since the F1 string has a tension T that is *constant* in the string frame (i.e. independent of the coupling $g = e^{\phi}$) while the D-string has a tension that behaves as $\frac{1}{g} = e^{-\phi}$.

Since we have two different strings, one wonders if there is a symmetry that interchanges these two strings. the tensions of the two strings appear to be rather asymmetrical, until we realize that we are measuring these tensions in the *string* metric, which was designed to make the tension constant for NS1, and which is thus not symmetrical between the two strings.

One metric that would be canonically defined in 10-d spacetime (without having anything to do with any specific string) is the Einstein metric of 10-d. The action in the string metric is

$$S = \frac{L}{16\pi G} \int d^{10}x \sqrt{-g_S} e^{-2\phi}([R]_S + 4\phi_{;c}\phi^{;c}) \tag{6.1}$$

We see that to absorb the factor $e^{-2\phi}$ into the curvature term we need to define the Einstein metric as

$$g^E_{ab} = e^{-\frac{\phi}{2}} g^S_{ab} \tag{6.2}$$

Recall that if we define a new metric $q_{ab} = e^{\mu} g_{ab}$ then

$$[R]_g = e^{\mu}[R]_q + e^{\mu}[(D-1)\mu_{;c}{}^{c} - \frac{1}{4}(D-1)(D-2)\mu_{,c}\mu^{,c}] \tag{6.3}$$

Using this we find the action in the 10-d Einstein frame

$$S = \frac{L}{16\pi G} \int d^{10}x \sqrt{-g_E}([R]_E - \frac{1}{2}\phi_{;c}\phi^{;c}) \tag{6.4}$$

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Problem 2: Derive (6.4) from (6.1) using (6.3).

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Since the tension of a string has units $(length)^{-1}$ we see from arguments similar to those given earlier that the tension in the Einstein metric $T_{Einstein}$ is related to the tension in the string metric T_{string} by

$$T_{Einstein} = e^{\frac{\phi}{2}} T_{string} \tag{6.5}$$

We then find that

$$\begin{aligned} (T_{NS1})_{Einstein} &\sim e^{\frac{\phi}{2}} \\ (T_{D1})_{Einstein} &\sim e^{-\frac{\phi}{2}} \end{aligned} \tag{6.6}$$

We now see that we have a symmetry if we interchange the two strings, if we also let

$$\phi \to -\phi \tag{6.7}$$

which implies $g \to \frac{1}{g}$.

This is a very interesting symmetry, since it takes it weak coupling to *strong* coupling.

6.1. The D5 and NS5 branes

In the string metric, the D5 brane, like all D-branes, has a tension $\sim e^{-\phi}$. The units of this tension are $(length)^{-6}$, since tensions is now mass per unit 5-volume. In the Einstein metric we thus get

$$(T_{D5})_{Einstein} = e^{\frac{3\phi}{2}}(T_{D5})_{string} \sim e^{\frac{\phi}{2}} \tag{6.8}$$

Under S-duality (6.7) this tension is not invariant; thus if S-duality is to be a symmetry then there must be another 5-brane with tension

$$(T_{NS5})_{Einstein} \sim e^{-\frac{\phi}{2}} \tag{6.9}$$

If we go back to the string metric this latter tension will be

$$(T_{NS5})_{string} = e^{-\frac{3\phi}{2}}(T_{NS5})_{Einstein} \sim e^{-2\phi} \sim \frac{1}{g^2} \tag{6.10}$$

7. The 3-form gauge fields

We can now pause to update the set of gauge fields that are needed in the theory. First look at 11-d M theory. If we are to have the M2 brane, then we need a 3-form potential A_{MNP} to couple to this brane. (The magnetic dual M5 brane will also couple to the same potential.)

Adding the action for A_{MNP} to the M-theory action we get

$$S = \frac{1}{16\pi G}\int d^{11}\xi\sqrt{-g}[R - \frac{1}{2}\frac{1}{4!}F_{MNPQ}F^{MNPQ}] \tag{7.1}$$

When we dimensionally reduce to 10-d IIA string theory by compactifying the coordinate y, we get two fields. If all indices of A_{MNP} are in the 10 noncompact directions, then we get a 3-form potential A_{abc} which couples to the D2 brane. If one index is in the y direction then we get a 2-form potential

$$B_{ab} \equiv A_{aby} \tag{7.2}$$

which couples to the NS1 (fundamental string) which is just the M2 brane with one direction along y. the field strength is

$$H_{abc} = \partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab} \tag{7.3}$$

Let us set all off diagonal terms in the metric to zero for simplicity. The action for A_{MNP} gives two terms in 10-d

$$S_A = \frac{l}{16\pi G}\int d^{10}x\sqrt{-g_{10}}e^{\frac{C}{2}}[\frac{1}{2}\frac{1}{4!}F_{abcd}F^{abcd} + \frac{1}{2}\frac{1}{3!}e^{-C}F_{abc}F^{abc}] \tag{7.4}$$

Converting to the string metric we get

$$\begin{aligned} S_A^{string} &= \frac{l}{16\pi G}\int d^{10}x\sqrt{-g_{10}^S}e^{-2C}[\frac{1}{2}\frac{1}{4!}e^{2C}F_{abcd}F^{abcd} + \frac{1}{2}\frac{1}{3!}e^{\frac{C}{2}}H_{abc}H^{abc}] \\ &= \frac{l}{16\pi G}\int d^{10}x\sqrt{-g_{10}^S}[\frac{1}{2}\frac{1}{4!}F_{abcd}F^{abcd} + \frac{1}{2}\frac{1}{3!}e^{-\frac{3C}{2}}H_{abc}H^{abc}] \\ &= \frac{l}{16\pi G}\int d^{10}x\sqrt{-g_{10}^S}[\frac{1}{2}\frac{1}{4!}F_{abcd}F^{abcd} + \frac{1}{2}\frac{1}{3!}e^{-2\phi}H_{abc}H^{abc}] \end{aligned} \tag{7.5}$$

Collecting all the terms we have found so far in the IIA action we get (in the string metric)

$$\begin{aligned} S = \frac{l}{16\pi G}\int d^{10}x\sqrt{-g_S}[e^{-2\phi}([R]_S + 4\phi_{,a}\phi^{;a} - \frac{1}{12}H_{abc}H^{abc}) \\ + (-\frac{1}{4}F_{ab}F^{ab} - \frac{1}{48}F_{abcd}F^{abcd})] \end{aligned} \tag{7.6}$$

Note that the gauge fields for the D2, D4 branes appear with no power of e^{ϕ} while the other terms in the action all have a power $e^{-2\phi}$.

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Problem 3: Show that in the Einstein frame there is a manifest S-duality symmetry between the two 3-form field strengths in (7.6). To do this, show that the action for these fields in the Einstein frame has the form

$$S \sim \int d^{10}x\sqrt{-g_E}[-\frac{1}{12}e^{-\phi}H_{abc}H^{abc} - e^{\phi}\frac{1}{12}F_{abc}F^{abc}] \tag{7.7}$$

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