

The electrostatic force

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1 The electrostatic force

Take an electric charge Q and place it at the origin $\vec{r} = 0$. Place another charge q at \vec{r} . The charge q feels a force

$$\vec{F} = \frac{1}{4\pi\epsilon_0} Qq \frac{\hat{r}}{r^2} \quad (1)$$

Note that \vec{r} is the vector from the location of Q to the location of q . Since $\hat{r} = \frac{\vec{r}}{r}$, we can also write

$$\vec{F} = \frac{1}{4\pi\epsilon_0} Qq \frac{\vec{r}}{r^3} \quad (2)$$

This form is easier to use when computing components of \vec{F} .

1.1 Many charges

Let there be several charges Q_1, Q_2, \dots at positions $\vec{r}_1, \vec{r}_2, \dots$. Place a charge q at position \vec{r} . the force on q will be the sum of the forces caused by each charge Q_i

$$\vec{F} = \frac{1}{4\pi\epsilon_0} Q_1 q \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|^3} + \frac{1}{4\pi\epsilon_0} Q_2 q \frac{\vec{r} - \vec{r}_2}{|\vec{r} - \vec{r}_2|^3} + \dots \quad (3)$$

This is a vector equation, with each side being a vector. To rewrite it in components we take the x component of the LHS and equate it to the x component of the RHS, and so on

$$\begin{aligned} F_x &= \frac{Q_1 q}{4\pi\epsilon_0} \frac{x - x_1}{[(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]^{\frac{3}{2}}} + \frac{Q_2 q}{4\pi\epsilon_0} \frac{x - x_2}{[(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2]^{\frac{3}{2}}} + \dots \\ F_y &= \frac{Q_1 q}{4\pi\epsilon_0} \frac{y - y_1}{[(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]^{\frac{3}{2}}} + \frac{Q_2 q}{4\pi\epsilon_0} \frac{y - y_2}{[(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2]^{\frac{3}{2}}} + \dots \\ F_z &= \frac{Q_1 q}{4\pi\epsilon_0} \frac{z - z_1}{[(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]^{\frac{3}{2}}} + \frac{Q_2 q}{4\pi\epsilon_0} \frac{z - z_2}{[(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2]^{\frac{3}{2}}} + \dots \end{aligned} \quad (4)$$

Example: A charge Q is placed at $(-a, 0, 0)$ and a charge $-Q$ is placed at $(a, 0, 0)$. Find the force on a charge q placed at $(x, 0, z)$.

Solution: We have

$$\vec{r} - \vec{r}_1 = (x + a, 0, z), \quad \vec{r} - \vec{r}_2 = (x - a, 0, z) \quad (5)$$

We find

$$\begin{aligned} F_x &= \frac{Qq}{4\pi\epsilon_0} \frac{x+a}{[(x+a)^2 + z^2]^{\frac{3}{2}}} - \frac{Qq}{4\pi\epsilon_0} \frac{x-a}{[(x-a)^2 + z^2]^{\frac{3}{2}}} \\ F_y &= 0 \\ F_z &= \frac{Qq}{4\pi\epsilon_0} \frac{z}{[(x+a)^2 + z^2]^{\frac{3}{2}}} - \frac{Qq}{4\pi\epsilon_0} \frac{z}{[(x-a)^2 + z^2]^{\frac{3}{2}}} \end{aligned} \quad (6)$$

1.2 The electric field

Suppose several charges Q_i are placed at positions \vec{r}_i . We want to describe the electrostatic influence of all these charges at some point \vec{r} .

If we place a charge q at \vec{r} , it will feel a force proportional to q . Let us define the *electric field* \vec{E} to be the force felt by a *unit* charge placed at \vec{r} . The force felt by a charge q at \vec{r} will be

$$\vec{F} = q\vec{E} \quad (7)$$

From (3) we see that if charges Q_1, Q_2, \dots are placed at $\vec{r}_1, \vec{r}_2, \dots$ then

$$\vec{E}(\vec{r}) = \frac{Q_1}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|^3} + \frac{Q_2}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_2}{|\vec{r} - \vec{r}_2|^3} + \dots \quad (8)$$

1.2.1 The goal of defining the electric field

It may seem that nothing has been gained by writing (7),(8) in place of the original equation (3). After all if all we wanted was to find the force on a charge, (3) gave the answer more directly. But introducing the electric field \vec{E} will help us understand the underlying physics. Suppose we have just the charge Q at $\vec{r} = 0$, and no other charges. Then there are two possibilities:

(a) Since there is no other charge to feel a force, there are no electrostatic effects operating in this situation.

(b) The charge Q causes an electrostatic influence all around it (measured by the \vec{E} it produces). If a charge q is brought to position \vec{r} , it feels a force because of the electric field at \vec{r} , but the electric field is ‘really there’ even if we do not have the charge q around.

It will turn out that (b) is the correct way to think. We will learn later that the electric field carries *energy*. If all charges are static, then this energy does not change with time, and we do not need to think about it. But if we move the charges then this energy can flow around, and ultimately will lead to *electromagnetic waves*. Thus even though it makes no difference in electrostatics whether we introduce the electric field or not, for the full theory of electromagnetism we do need the idea of electric (and magnetic) fields.

2 Analyzing the electric field

We have a value for the vector \vec{E} at each point \vec{r} . Thus we have a *vector field*. We can look at a vector field over spaces of various dimensions. Let us see what physics we get in each case.

2.0.2 A 0-dimensional point \vec{r}_0

Let us take a point \vec{r}_0 . From the vector field $\vec{E}(\vec{r})$ we get the *value* of the vector field at this point: $\vec{E}(\vec{r}_0)$. What does this signify?

This value $\vec{E}(\vec{r}_0)$ gives the force on a unit charge at \vec{r}_0 , and if we have a charge q at \vec{r}_0 , the force on this charge is

$$\vec{F} = q\vec{E}(\vec{r}_0) \quad (9)$$

2.0.3 A 1-dimensional curve C

We can perform a line integral of \vec{E}

$$I = \int_C \vec{E} \cdot d\vec{l} \quad (10)$$

What does this signify? \vec{E} is the force felt by a unit charge. Thus I is the *work done on a unit charge by the electric field as the charge moves from the start of the curve to the end of the curve*. If we have a charge q instead, then the work done is

$$W = q \int_C \vec{E} \cdot d\vec{l} \quad (11)$$

2.0.4 A 2-dimensional surface \mathcal{S}

Let us place a charge Q at $\vec{r} = 0$. Then the electric field is

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \quad (12)$$

Consider the surface \mathcal{S} given by a sphere of radius R , with center at the origin. Let us compute the integral

$$\int_{\mathcal{S}} (\vec{E} \cdot \hat{n}) dA \quad (13)$$

Note that $\hat{n} = \hat{r}$, so

$$\vec{E} \cdot \hat{n} = \frac{Q}{4\pi\epsilon_0} \frac{\hat{r} \cdot \hat{r}}{r^2} = \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2} \quad (14)$$

Since this quantity is constant for all points of the surface, we can take it out of the integral getting

$$\int_{\mathcal{S}} (\vec{E} \cdot \hat{n}) dA = \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2} \int_{\mathcal{S}} dA \quad (15)$$

But $\int_{\mathcal{S}} dA$ is just the area of the surface \mathcal{S} which is $4\pi R^2$. Thus we get

$$\int_{\mathcal{S}} (\vec{E} \cdot \hat{n}) dA = \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2} (4\pi R^2) = \frac{Q}{\epsilon_0} \quad (16)$$

The remarkable thing is that R has cancelled out. Thus whatever the size of the sphere, we get the same value for I . We will analyze this in more detail below.

2.0.5 A 3-dimensional volume \mathcal{V}

The only scalar we can make from \vec{E} is $\vec{E} \cdot \vec{E} = |\vec{E}|^2$, which we can then integrate over the volume \mathcal{V} . We will see later that the *energy density* of the electric field is

$$U = \frac{\epsilon_0}{2} |\vec{E}|^2 \quad (17)$$

Thus the energy of the electric field in the volume \mathcal{V} is

$$\mathcal{E} = \frac{\epsilon_0}{2} \int_{\mathcal{V}} d^3r |\vec{E}|^2 \quad (18)$$

3 The surface integral in more detail: Gauss law

In (19) we had seen that if we draw a sphere \mathcal{S}_r around a point charge Q , then the electric field of Q has a surface integral

$$\int_{\mathcal{S}_r} (\vec{E} \cdot \hat{n}) dA = \frac{Q}{\epsilon_0} \quad (19)$$

It is easy to see why the integral did not depend on what radius we took for the sphere. If we increase the radius from r_1 to r_2 then the area of the surface element dA increases to $\frac{r_2^2}{r_1^2} dA$. But because the electric field decreases with distance precisely as the inverse square, the value of $\vec{E} \cdot \hat{n}$ drops by a factor $\frac{r_1^2}{r_2^2}$. The net contribution to the flux $\vec{E} \cdot \hat{n} dA$ stays unchanged.

But more is true. We can take a surface of *any shape*, and still get the same flux as long as the surface encloses Q . Imagine a surface element that is tilted through an angle θ . The area of this

element increases from $\delta a \delta b$ to $\delta a \frac{\delta b}{\cos \theta}$. But the normal to this surface element is no longer along the direction of \vec{E} , and we get $\vec{E} \cdot \hat{n} = |\vec{E}| \cos \theta$. Thus again $\vec{E} \cdot \hat{n} dA$ remains unchanged.

Thus we find that if we draw *any* surface \mathcal{S} enclosing Q , then the flux through this surface equals $\frac{Q}{\epsilon_0}$

$$\int_{\mathcal{S}} (\vec{E} \cdot \hat{n}) dA = \frac{Q}{\epsilon_0} \quad (20)$$

Problem: Show that if there are several charges Q_i enclosed by a surface \mathcal{S} , then

$$\int_{\mathcal{S}} (\vec{E} \cdot \hat{n}) dA = \frac{Q_{encl}}{\epsilon_0} \quad (21)$$

where

$$Q_{encl} = \sum_i Q_i \quad (22)$$

is the total charge enclosed by \mathcal{S} .

Eq. (21) is called the Gauss law.

4 Applications of the Gauss law

What can we do with the Gauss law? Suppose we know the charges enclosed in a surface \mathcal{S} . Does this mean that we can find the value of \vec{E} at any point on \mathcal{S} ? In general we cannot learn much from the law, since knowing Q_{encl} only gives us the value of an *integral* of \vec{E} , and this does not give \vec{E} at any one point.

But in special cases there may be a *symmetry*, which may us a lot about the structure of \vec{E} . Suppose this symmetry determines \vec{E} upto an overall constant which needs to be determined. If we only want this one number, then the Gauss law can give it to us, and thus determine \vec{E} .

Example: Consider a spherical shell of radius R carrying a uniform surface charge density. Let the total charge carried be Q . Find \vec{E} inside and outside the shell.

Solution: Let the shell be a sphere of radius R centered at the origin. By spherical symmetry, we expect that

(i) \vec{E} points radially outwards

(ii) The magnitude of \vec{E} depends on r but not on the angular coordinates θ, ϕ

Thus

$$\vec{E} = E(r)\hat{r} \quad (23)$$

To find \vec{E} at any point at radius r , let \mathcal{S} be a sphere of radius r centered at the origin. Then

$$\int_{\mathcal{S}} (\vec{E} \cdot \hat{n}) dA = \int_{\mathcal{S}} E(r)(\hat{r} \cdot \hat{r}) dA = E(r) \int_{\mathcal{S}} dA = E(r)4\pi r^2 \quad (24)$$

(a) If $r > R$, then $Q_{encl} = Q$. Thus

$$E(r)4\pi r^2 = \frac{Q}{\epsilon_0} \quad (25)$$

which gives

$$E(r) = \frac{Q}{r^2\epsilon_0}, \quad \vec{E} = \frac{Q}{\epsilon_0 r^2} \hat{r} \quad (26)$$

(b) If $r < R$, then $Q_{encl} = 0$, and we get

$$E(r) = 0, \quad \vec{E} = 0 \quad (27)$$

Example: Find \vec{E} produced by an infinite line charge λ per unit length.

Solution: Let the line charge be along the z axis, and let z, ρ, ϕ be cylindrical coordinates set up around this axis. By symmetry, we expect

(i) \vec{E} will point along $\hat{\rho}$

(ii) $|\vec{E}|$ will depend only on ρ , but not on z or ϕ

Thus

$$\vec{E} = E(\rho)\hat{\rho} \quad (28)$$

Let \mathcal{S} be a cylinder centered on the z axis, radius ρ and length L . We compute $\int_{\mathcal{S}} (\vec{E} \cdot \hat{n}) dA$ by adding the contributions from various sides of the cylinder:

(a) The top has normal \hat{z} , so $\vec{E} \cdot \hat{n} = 0$. So there is no contribution from the top, and similarly no contribution from the bottom of the cylinder.

(b) On the curved side of the cylinder we have $\hat{n} = \hat{\rho}$. Thus $\vec{E} \cdot \hat{n} = E(\rho)$, and

$$\int_{\mathcal{S}} (\vec{E} \cdot \hat{n}) dA = E(\rho) \int dA = E(\rho)2\pi\rho L \quad (29)$$

Now we relate this surface integral to the charge enclosed. The charge enclosed is $Q_{encl} = \lambda L$. Thus

$$E(\rho)2\pi\rho L = \frac{\lambda L}{\epsilon_0} \quad (30)$$

which gives

$$E(\rho) = \frac{\lambda}{2\pi\epsilon_0\rho}, \quad \vec{E} = \frac{\lambda}{2\pi\epsilon_0\rho}\hat{\rho} \quad (31)$$

Example: Find the electric field produced by a plane carrying a uniform surface charge density σ .

Solution: Let the plane be $x = 0$. Based on the symmetry of the problem we expect the following:

(i) \vec{E} points normal to the plane. (If there was any tangential component then it would lie along a particular direction along the plane; this does not make sense since all directions in the plane look the same and no particular one should get picked out.)

(ii) If \vec{E} points to the right on the right side of the plane, then it should point to the left on the left side of the plane

For $x > 0$ we can write $x = |x|$, and we write

$$\vec{E} = E(|x|)\hat{x} \quad \text{for } x > 0 \quad (32)$$

For negative x we expect the same magnitude for \vec{E} as on the positive x side, but an opposite sign. Thus we write

$$\vec{E} = -E(|x|)\hat{x} \quad \text{for } x < 0 \quad (33)$$

Draw a surface S in the shape of a cylinder, with axis normal to the plane, and half the cylinder on each side of the plane. Let the cross sectional area be A , and the length extend from $-x$ to x .

(a) On the curved side of the cylinder, we have $\vec{E} \cdot \hat{n} = 0$

(b) From the right end, we have $\int \vec{E} \cdot \hat{n} dA = E(|x|)A$.

(c) From the left end we also get $\int \vec{E} \cdot \hat{n} dA = E(|x|)A$.

Thus

$$\int_S (\vec{E} \cdot \hat{n}) dA = 2E(|x|)A \quad (34)$$

The charge enclosed is σA . Thus

$$2E(|x|)A = \frac{\sigma A}{\epsilon_0}, \quad E(|x|) = \frac{\sigma}{2\epsilon_0} \quad (35)$$

and

$$\begin{aligned} \vec{E} &= \frac{\sigma}{2\epsilon_0} \quad \text{for } x > 0 \\ \vec{E} &= -\frac{\sigma}{2\epsilon_0} \quad \text{for } x < 0 \end{aligned} \quad (36)$$

5 The differential form of the Gauss law

Suppose we have a volume charge distribution $\rho(x, y, z)$. Take a small cube of sides dx, dy, dz around the point (x, y, z) . The volume of this cube is $dV = dxdydz$. The charge enclosed by this cube is

$$dQ_{encl} = \rho(x, y, z)dV \quad (37)$$

The flux through the surface of this cube will satisfy the Gauss law

$$\int_S (\vec{E} \cdot \hat{n})dA = \frac{\rho dV}{\epsilon_0} \quad (38)$$

But by a general theorem on vector fields we can replace the surface integral by a volume integral over \mathcal{R} , the region inside the cube:

$$\int_S (\vec{E} \cdot \hat{n})dA = \int_{\mathcal{R}} (\vec{\nabla} \cdot \vec{E})dV \quad (39)$$

Since the cube is small, the value of the integrand on the RHS is approximately the same at all points of the cube, and we can write

$$\int_{\mathcal{R}} (\vec{\nabla} \cdot \vec{E})dV = (\vec{\nabla} \cdot \vec{E})(x, y, z) \int_{\mathcal{R}} dV = (\vec{\nabla} \cdot \vec{E})(x, y, z)V \quad (40)$$

Then the Gauss law gives

$$(\vec{\nabla} \cdot \vec{E})(x, y, z)V = \frac{dQ_{encl}}{\epsilon_0} = \frac{\rho(x, y, z)dV}{\epsilon_0} \quad (41)$$

which gives

$$(\vec{\nabla} \cdot \vec{E}) = \frac{\rho}{\epsilon_0} \quad (42)$$

This is the differential form of the Gauss law. It relates the density at each point to the divergence of \vec{E} at that point. Note that this does not mean that we can find \vec{E} at each point immediately; we get $\vec{\nabla} \cdot \vec{E}$ at each point, but then we have to integrate to get \vec{E} itself.

Problem: Suppose the electric field is

$$\vec{E} = (-8x + 2y)\hat{x} + 2y\hat{y} + 3z^2\hat{z} \quad (43)$$

Find the charge density.

Solution: We have

$$\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E} \quad (44)$$

But

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial}{\partial x}(-8x + 2y) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z^2) = -8 + 2 + 6z = -6 + 6z \quad (45)$$

Thus

$$\rho = (-6 + 6z)\epsilon_0 \quad (46)$$

6 The electric potential Φ

The electric field of a point charge Q is

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0} \frac{\hat{r}}{r^3} \quad (47)$$

A vector field is described by *three* numbers for each point of space. But if the vector field is special, these different components might be related, and there would be a smaller number of independent numbers describing the vector field. We will now see that this is the case for \vec{E} in electrostatics, which can be written as a *gradient* of a function. Since the function is described by just *one* number per point, this gives a great simplification in the description of \vec{E} .

Problem: Show that

$$\vec{\nabla} \frac{1}{r} = -\frac{\vec{r}}{r^3} \quad (48)$$

Solution: $\vec{\nabla} \frac{1}{r}$ has three components:

$$\begin{aligned} (\vec{\nabla} \frac{1}{r})_x &= \partial_x \frac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = -\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \\ (\vec{\nabla} \frac{1}{r})_y &= \partial_y \frac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = -\frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \\ (\vec{\nabla} \frac{1}{r})_z &= \partial_z \frac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = -\frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \end{aligned} \quad (49)$$

which gives (48).

We can thus write the field of a point charge as

$$\vec{E} = -\vec{\nabla}\Phi \quad (50)$$

with

$$\Phi(\vec{r}) = \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \quad (51)$$

Problem: Write the electric field of several charges in terms of the potential of those charges.

Solution: Let the charges Q_1, Q_2, \dots be placed at $\vec{r}_1, \vec{r}_2, \dots$. Then

$$\vec{E}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}(\vec{r}_2) + \dots \quad (52)$$

But

$$\vec{E}_1(\vec{r}) = -\vec{\nabla}\Phi_1(\vec{r}), \quad \Phi_1(\vec{r}) = \frac{Q_1}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_1|} \quad (53)$$

and similarly for the other charges. Thus

$$\vec{E}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}) \quad (54)$$

with

$$\Phi(\vec{r}) = \frac{Q_1}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_1|} + \frac{Q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_2|} + \dots \quad (55)$$

6.1 The curl of \vec{E}

Recall that the curl of a gradient is zero. Thus if $\vec{E} = -\vec{\nabla}\Phi$, then we must have

$$\vec{\nabla} \times \vec{E} = 0 \quad (56)$$

Problem: Verify (56) for the field of a point charge Q .

Solution: Assume the charge Q is placed at $\vec{r} = 0$. Then

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3} \quad (57)$$

$$(\vec{\nabla} \times \vec{E})_z = \partial_x E_y - \partial_y E_x \quad (58)$$

But

$$\partial_x E_y = \partial_x \left[\frac{Q}{4\pi\epsilon_0} \frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right] = -3 \frac{Q}{4\pi\epsilon_0} \frac{xy}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \quad (59)$$

This is symmetric under the interchange $x \leftrightarrow y$, so

$$(\vec{\nabla} \times \vec{E})_z = \partial_x E_y - \partial_y E_x = 0 \quad (60)$$

Similarly, the other components of the curl vanish.

6.2 The line integral of \vec{E}

Consider a curve C which begins at \vec{r}_i and ends at \vec{r}_f . We can compute the line integral

$$I = \int_{\vec{r}_i}^{\vec{r}_f} \vec{E} \cdot d\vec{l} \quad (61)$$

Since we know that $\vec{E} = -\vec{\nabla}\Phi$, we can evaluate the line integral

$$I = - \int_{\vec{r}_i}^{\vec{r}_f} \vec{\nabla}\Phi \cdot d\vec{l} = -[\Phi(\vec{r}_f) - \Phi(\vec{r}_i)] \quad (62)$$

Problem: Suppose a charge q moves under the influence of an electric field \vec{E} from point \vec{r}_i to \vec{r}_f . Find the work done by the electric field on the charge.

Solution: The force on the charge is $\vec{F} = \vec{E}$. The work done on the charge by the field is

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{l} = q \int_{\vec{r}_i}^{\vec{r}_f} \vec{E} \cdot d\vec{l} = -q[\Phi(\vec{r}_f) - \Phi(\vec{r}_i)] \equiv -q\Delta\Phi \quad (63)$$

Problem: A shell of radius R carries a uniform surface charge with total charge Q . Find the potential due to this shell, both outside and inside the shell.

Solution: Let us set $\Phi = 0$ at infinity. Let the shell be placed with center at the origin. Let us compute Φ on the positive x axis; i.e., at points at $(x, 0, 0)$ for $x > 0$. By spherical symmetry, we will then have

$$\Phi(r = x) = \Phi(x, 0, 0) \quad (64)$$

To compute $\Phi(x, 0, 0)$ we have to find how much work needs to be done to move a unit charge from infinity to the point $(x, 0, 0)$. We can choose any path to move the charge, and we choose to move along the x -axis in the negative x direction from $x' = \infty$ to $x' = x$. Then

$$\Phi(x) - \Phi(\infty) = - \int_{x'=\infty}^x \vec{E} \cdot d\vec{l} \quad (65)$$

(a) First let $x > R$, so that the entire path is outside the shell. Then

$$\vec{E}(x', 0, 0) = \frac{Q}{4\pi\epsilon_0} \frac{1}{x'^2} \hat{x} \quad (66)$$

We have $d\vec{l} = dx' \hat{x}$. Thus

$$\vec{E} \cdot d\vec{l} = \frac{Q}{4\pi\epsilon_0} \frac{1}{x'^2} dx' \quad (67)$$

and

$$\Phi(x, 0, 0) = - \int_{x'=\infty}^x \frac{Q}{4\pi\epsilon_0} \frac{1}{x'^2} dx' = \int_{x'=x}^{\infty} \frac{Q}{4\pi\epsilon_0} \frac{1}{x'^2} dx' = \frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{x'}\right]_x^{\infty} = \frac{Q}{4\pi\epsilon_0} \frac{1}{x} \quad (68)$$

Thus using spherical symmetry we write

$$\Phi(r) = \frac{Q}{4\pi\epsilon_0} \frac{1}{r}, \quad \text{for } r > R \quad (69)$$

(b) Now let $r < R$. We break the integral from infinity to x into two parts: one from infinity to $x' = R$, and one from $x' = R$ to $x' = x$. Thus

$$\Phi(x, 0, 0) = - \left[\int_{x'=\infty}^R \vec{E} \cdot d\vec{l} + \int_{x'=R}^x \vec{E} \cdot d\vec{l} \right] \quad (70)$$

But $\vec{E} = 0$ inside the shell, so the second integral vanishes. The first integral is the result (68) at $x = R$, so we get

$$\Phi(x, 0, 0) = \frac{Q}{4\pi\epsilon_0} \frac{1}{R} \quad (71)$$

Thus we have

$$\Phi(r) = \frac{Q}{4\pi\epsilon_0} \frac{1}{R} \quad \text{for } r < R \quad (72)$$

Problem: Find the potential due to a slab $|x| < a$ carrying a uniform volume charge density ρ . Set $\Phi = 0$ at $x = 0$.

Problem: Find the potential due to two parallel plates separated by a distance a , if one carries a surface charge density σ and the other $-\sigma$.

Problem: A spherical shell of radius R_1 carrying a uniform surface charge density σ_1 is placed with center at $(-a, 0, 0)$ and another shell of radius R_2 carrying a uniform surface charge density σ_2 is placed with center at $(b, 0, 0)$. Suppose a test charge q is moved slowly from the point $(-(a + R_1))$ to the point $(b + R_2)$. How much work will have to be done on this charge?

7 Energy in electrostatics

Let us ask a physical question: where is the energy of an electron?

The electron has charge, and thus produces an electric field. If this field is something ‘real’, then it must have an energy. How do we find the energy of an electric field?

7.1 The energy of a collection of point charges

Let us begin with something simple. We have point charges Q_1, Q_2, \dots at positions $\vec{r}_1, \vec{r}_2, \dots$. What is the energy of this setup?

First we must make the question a little more precise. We are not asking for *how* the point charges are made. It may cost energy to make the point charges, but that is not our concern right now. *Given* the point charges Q_1 , we can move them around. Different configurations will have different energies. Let us start with a configuration where the charges are all dispersed away from each other, so that no charge feels the force of another charge. We can do that by putting the charges at infinity, in different directions. Let us call the energy of this configuration *zero*, since we can choose our zero of energy any way we want. Now we bring the charges into their final positions \vec{r}_i . Some work will have to be done on the charges to push them against their mutual repulsion, and this will show up as the final energy of the configuration:

$$\text{Energy of configuration} = \text{Work done on charges} \quad (73)$$

Let us now compute this energy.

(a) Bring charge Q_1 to position \vec{r}_1 . Since no other charge is nearby, no force is felt on Q_1 , so no force is needed to move Q_1 to this position. So work done so far is zero.

(b) Now bring Q_2 to position \vec{r}_2 . The charge Q_1 creates a potential at position \vec{r}_2

$$\Phi(\vec{r}_2) = \frac{Q_1}{4\pi\epsilon_0} \frac{1}{|\vec{r}_2 - \vec{r}_1|} \quad (74)$$

Thus bringing Q_2 to this position will need us to do work

$$W = Q_2\Phi(\vec{R}_2) = \frac{Q_1Q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_2 - \vec{r}_1|} \quad (75)$$

(c) Now bring charge Q_3 to \vec{r}_3 . Now that both Q_1 and Q_2 are present, the potential that we must consider gets a part from each of these two charges

$$\Phi(\vec{r}_3) = \frac{Q_1}{4\pi\epsilon_0} \frac{1}{|\vec{r}_3 - \vec{r}_1|} + \frac{Q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_3 - \vec{r}_2|} \quad (76)$$

The work done to bring Q_3 to \vec{r}_3 is

$$W = Q_3\Phi(\vec{r}_3) = \frac{Q_1Q_3}{4\pi\epsilon_0} \frac{1}{|\vec{r}_3 - \vec{r}_1|} + \frac{Q_2Q_3}{4\pi\epsilon_0} \frac{1}{|\vec{r}_3 - \vec{r}_2|} \quad (77)$$

and we must add this to the work (75) that had been done before.

(d) Proceeding in this way, after all the charges have been brought to their required final positions, we get the total work done W_{total} which must equal the energy \mathcal{E} of the configuration:

$$W_{total} = \mathcal{E} = \sum_{i < j} \frac{Q_i Q_j}{4\pi\epsilon_0} \frac{1}{|\vec{r}_i - \vec{r}_j|} \quad (78)$$

We can also write this as

$$\mathcal{E} = \frac{1}{2} \sum_{i \neq j} \frac{Q_i Q_j}{4\pi\epsilon_0} \frac{1}{|\vec{r}_i - \vec{r}_j|} \quad (79)$$

7.2 Energy for a continuous charge distribution

Now suppose that instead of point charges Q_i we have a charge *density* $\rho(\vec{r})$. We will compute the energy in three different forms:

7.2.1 Charge-charge interaction:

We can break up our region into small cubes, and consider the charge in each cube as a small point charge which can interact with the charges from all the other little cubes. Then we can use relation (79).

The charge in the small cube around \vec{r} is

$$dq = \rho(\vec{r})dv \quad (80)$$

For a cube around the point \vec{r}' we have

$$dq' = \rho(\vec{r}')dv' \quad (81)$$

The energy is then

$$\mathcal{E} = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \int dv \int dv' \rho(\vec{r}) \frac{1}{|\vec{r} - \vec{r}'|} \rho(\vec{r}') \quad (82)$$

7.2.2 Charge-potential interaction:

We observe that

$$\phi(\vec{r}) = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \int dv' \frac{1}{|\vec{r} - \vec{r}'|} \rho(\vec{r}') \quad (83)$$

Thus

$$\mathcal{E} = \frac{1}{2} \int dv \rho(\vec{r}) \phi(\vec{r}) \quad (84)$$

7.2.3 Energy in the electric field:

Recall that $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$. Thus

$$\mathcal{E} = \frac{\epsilon_0}{2} \int dv (\vec{\nabla} \cdot \vec{E})(\vec{r}) \phi(\vec{r}) \quad (85)$$

We now integrate by parts

$$\int dv (\vec{\nabla} \cdot \vec{E}) \phi = \int \vec{\nabla} \cdot (\phi \vec{E}) - \int \vec{E} \cdot \vec{\nabla} \phi \quad (86)$$

But

$$\int \vec{\nabla} \cdot (\phi \vec{E}) = \int_S (\vec{E} \cdot \hat{n}) dA \quad (87)$$

Since the volume of integration is all space, the boundary \mathcal{S} is a large sphere at infinity. Far away from all charges, we have

$$\vec{E} = \frac{Q_{total}}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}, \quad \Phi = \frac{Q_{total}}{4\pi\epsilon_0} \frac{1}{r} \quad (88)$$

Thus

$$\int_S (\vec{E} \cdot \hat{n}) dA = \frac{Q_{total}^2}{(4\pi\epsilon_0)^2} \int \frac{1}{r^3} dA = \frac{Q_{total}^2}{(4\pi\epsilon_0)^2} \frac{4\pi r^2}{r^3} = \frac{Q_{total}^2}{(4\pi\epsilon_0)^2} \frac{4\pi}{r} \quad (89)$$

This vanishes because $r \rightarrow 0$. The other term in (87) is

$$- \int \vec{E} \cdot \vec{\nabla} \phi = \int \vec{E} \cdot \vec{E} = - \int |\vec{E}|^2 \quad (90)$$

Thus we find

$$\mathcal{E} = \frac{\epsilon_0}{2} \int dv |\vec{E}|^2 \quad (91)$$

7.2.4 Summary of the energy of a continuous distribution

Thus we see that the energy \mathcal{E} of a continuous charge distribution can be written *either* as

(a) The interaction between charges at different points

or as

(b) The interaction between charges and the potential at their location

or as

(c) The energy of the electric field filling all space

It is important that we should not *add* together two or more of these contributions, since that would be overcounting.

Problem: Find the electrostatic energy in a sphere of radius R carrying a uniform charge density ρ .

Solution: First we find \vec{E} . By spherical symmetry

$$\vec{E} = E(r)\hat{r} \quad (92)$$

Draw a Gaussian surface of radius r . The flux is

$$\int \vec{E} \cdot \hat{n} dA = 4\pi r^2 E(r) \quad (93)$$

The charge enclosed is

$$\begin{aligned} Q_{encl} &= \frac{4\pi}{3}\rho r^3, \quad r < R \\ &= \frac{4\pi}{3}\rho R^3, \quad r > R \end{aligned} \quad (94)$$

Thus

$$\begin{aligned} E(r) &= \frac{\rho}{3\epsilon_0}r, \quad r < R \\ &= \frac{\rho}{3\epsilon_0}\frac{R^3}{r^2}, \quad r > R \end{aligned} \quad (95)$$

Thus

$$\begin{aligned} |\vec{E}|^2 &= \left(\frac{\rho}{3\epsilon_0}\right)^2 r^2, \quad r < R \\ &= \left(\frac{\rho}{3\epsilon_0}\right)^2 \frac{R^6}{r^4}, \quad r > R \end{aligned} \quad (96)$$

The energy of the field inside the sphere is

$$\mathcal{E}_{in} = \frac{\epsilon_0}{2} \left(\frac{\rho}{3\epsilon_0}\right)^2 \int_0^R dr 4\pi r^2 r^2 = \frac{4\pi}{5} \frac{\epsilon_0}{2} \left(\frac{\rho}{3\epsilon_0}\right)^2 R^5 \quad (97)$$

From outside the sphere we get

$$\mathcal{E}_{out} = \frac{\epsilon_0}{2} \left(\frac{\rho}{3\epsilon_0}\right)^2 R^6 \int dr 4\pi r^2 \frac{1}{r^4} = 4\pi \frac{\epsilon_0}{2} \left(\frac{\rho}{3\epsilon_0}\right)^2 R^5 \quad (98)$$

The total energy is thus

$$\mathcal{E} = \mathcal{E}_{in} + \mathcal{E}_{out} = \frac{24\pi}{5} \frac{\epsilon_0}{2} \left(\frac{\rho}{3\epsilon_0}\right)^2 R^5 \quad (99)$$