1 The story of fermions

Consider a 1-d chain of lattice sites. At each site there is a fermion, represented by a Grassman number

$$\psi_k, \quad \psi_k\psi_l + \psi_l\psi_k = 0 \tag{1}$$

so that these ψ_k are anticommuting objects. The path integral is performed with an action

$$S = i\alpha \int \psi\partial\psi \to i\alpha \sum_k \psi_k(\psi_{k+1} - \psi_k) = i\alpha \sum_k \psi_k\psi_{k+1} \tag{2}$$

where we have used the anticommuting nature of the ψ_k. Note that we get the same action if we define the derivative differently

$$S = i\alpha \int \psi\partial\psi \to i\alpha \sum_k \psi_k(\psi_k - \psi_{k-1}) = -i \sum_k \alpha\psi_k\psi_{k-1} = i\alpha \sum_k \psi_{k-1}\psi_k = i\alpha \sum_k \psi_k\psi_{k+1} \tag{3}$$

where again we had to have the anticommuting nature of the ψ_k.

To perform the path integral, we note some mathematical identities. Suppose that A_i, B_i are vectors of Grassman numbers, and M_{ij} is a matrix of commuting numbers. Consider

$$\int d[A_i]d[B_i]e^{-A_iM_{ij}B_j} \tag{4}$$

For a single variable we have

$$\int dA = 0, \quad \int dAA = 1 \tag{5}$$

Thus

$$\int dAdBe^{-AMB} = \int dAdB[1 - AMB] = -M \int dAdBAB = M \int dAA \int dBB = M \tag{6}$$

If M was a diagonal matrix we would get

$$\int d[A_i]d[B_i]e^{-A_iM_{ii}B_i} = \prod_i M_{ii} = \det M \tag{7}$$

More generally, we will get

$$\int d[A_i]d[B_i]e^{-A_iM_{ij}B_j} = \det M \tag{8}$$

Now suppose that we have just one kind of anticommuting vector

$$\int d[C_i]e^{-C_iM_{ij}C_j} \tag{9}$$

Now we must have

$$M_{ij} = -M_{ji} \tag{10}$$

As an example, let us take a 2×2 matrix

$$M_{12} = -M_{21} = 1 \tag{11}$$

Then

$$\int dC_1 dC_2 e^{-C_i M_{ij} C_j} = \int dC_1 dC_2 [1 - 2C_1 C_2] = 2 \tag{12}$$

More generally we get

$$\int d[C_i] e^{-C_i M_{ij} C_j} = Phaff[M] \tag{13}$$

where $Phaff$ can be defined for any antisymmetric matrix of even dimension N, and is given by summing terms like

$$\frac{1}{N!} \epsilon_{i_1 \dots i_N} M_{i_1 i_2} M_{i_3 i_4} \dots M_{i_{N-1} i_N} \tag{14}$$

We have

$$(Phaff[M])^2 = \det M \tag{15}$$

but we lose the information of the sign if we write the result as $(\det M)^{\frac{1}{2}}$.

Now let us compute the path integral with sources. For the case of two fields, we write

$$\int d[A_i] d[B_i] e^{-A_i M_{ij} B_j + A_i J_i + K_i B_i} \tag{16}$$

We solve this by shifting the fields. We write

$$\int d[A_i] d[B_i] e^{-A_i M_{ij} B_j + A_i J_i + K_i B_i} = \int d[A_i] d[B_i] e^{-(A_i + \tilde{K}_i) M_{ij} (B_j + \tilde{J}_i) + C} \tag{17}$$

Then

$$J = -M\tilde{J}, \quad \tilde{J} = -M^{-1} J \tag{18}$$

$$K = -\tilde{K} M, \quad \tilde{K} = -K M^{-1} \tag{19}$$

$$C = \tilde{K} M \tilde{J} = K M^{-1} M M^{-1} J = K M^{-1} J \tag{20}$$

We then get

$$\int d[A_i] d[B_i] e^{-A_i M_{ij} B_j + A_i J_i + K_i B_i} = \int d[A_i] d[B_i] e^{-A_i M_{ij} B_j} e^{K_i M^{-1}_{ij} J_j} \tag{21}$$

Thus the 2-point function will be given by

$$< A_k B_l >= \frac{1}{Z} \frac{\delta}{\delta J_k} \frac{\delta}{\delta K_l} Z = M^{-1}_{lk} \tag{22}$$

For the case of a single field we have

$$\int d[C_i] e^{-C_i M_{ij} C_j + C_i J_i} = \int d[C_i] e^{-(C_i + \frac{\tilde{J}_i}{2}) M_{ij} (C_j + \frac{\tilde{J}_j}{2}) + \frac{1}{4} \tilde{J}_i M_{ij} \tilde{J}_j} \tag{23}$$

where we have used the antisymmetry of M. We have

$$J = -M\tilde{J}, \quad \tilde{J} = -M^{-1} J \tag{24}$$

Thus

$$\int d[C_i] e^{-C_i M_{ij} C_j + C_i J_i} = \int d[C_i] e^{-C_i M_{ij} C_j} e^{\frac{1}{4} \tilde{J}_i M_{ij} \tilde{J}_j} = \int d[C_i] e^{-C_i M_{ij} C_j} e^{-\frac{1}{4} J_i M^{-1}_{ij} J_j} \tag{25}$$

where we have used that

$$M^{-1T} M M^{-1} = -M^{-1} M M^{-1} = -M^{-1} \tag{26}$$

We have

$$< C_k C_l >= \frac{1}{Z} \frac{\delta}{\delta J_k} \frac{\delta}{\delta J_l} Z = \frac{1}{2} M^{-1}_{kl} \tag{27}$$

2 Fermions in 2-d

In 2-d the fermions will be 2-component spinors. The Γ matrices will be

$$\Gamma^\tau = \sigma_1, \quad \Gamma^\sigma = \sigma_2 \tag{28}$$

We write

$$\Gamma^z = \Gamma^\tau + i\Gamma^\sigma = 2\sigma^+ = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{29}$$

$$\Gamma^{\bar{z}} = \Gamma^\tau - i\Gamma^\sigma = 2\sigma^- = 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{30}$$

The chirality operator is

$$\Gamma^5 = \Gamma^\tau\Gamma^\sigma = i\sigma^3 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{31}$$

The action is

$$S = \int d^2z \frac{1}{2} i\alpha\psi^\dagger\Gamma^a\partial_a\psi = \int d^2z \frac{1}{2} i\alpha\psi^\dagger(\Gamma^z\partial_z + \Gamma^{\bar{z}}\partial_{\bar{z}})\psi \tag{32}$$

where we will choose the constant α later. Let us write

$$\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \tag{33}$$

Then we have

$$S = \int d^2z i\alpha(\psi^+)^*\partial_z\psi^- + \int d^2z i\alpha(\psi^-)^*\partial_{\bar{z}}\psi^+ \tag{34}$$

Thus the right and left moving parts of S split up, and we can consider them one at a time.

3 2-point function of fermions

Consider the action

$$S = i\alpha \int d^2z\psi(z)\partial_{\bar{z}}\psi(z) + \int d^2z\psi(z)J(z) \tag{35}$$

The correlation function will be

$$< \psi(z_1)\psi(z_2) >= \frac{1}{Z} < e^{-S}\psi(z_1)\psi(z_2) >= \frac{1}{Z}\frac{\delta}{\delta J(z_1)}\frac{\delta}{\delta J(z_2)}Z \tag{36}$$

The matrix M in this case is

$$M = i\alpha\partial_{\bar{z}} \tag{37}$$

Thus the inverse will satisfy

$$\int d^2z' \; [i\alpha\partial_{\bar{z}}](z, z')M^{-1}(z', z'') = \delta^2(z - z'') \tag{38}$$

But we know that

$$\partial_{\bar{z}}\frac{1}{z} = \pi\delta^2(z) \tag{39}$$

Thus

$$M^{-1}(z'-z'') = \frac{1}{i\pi\alpha}\frac{1}{(z'-z'')} \tag{40}$$

We then find, using the expression for the correlator in the case of a single field

$$< \psi(z_1)\psi(z_2) >= -\frac{1}{2\pi i\alpha}\frac{1}{(z_1-z_2)} \tag{41}$$

We would finally like to use a normalization where

$$< \psi(z_1)\psi(z_2) >= -\frac{1}{2\pi i\alpha}\frac{1}{(z_1-z_2)} \tag{42}$$

Thus we choose

$$\alpha = -\frac{1}{2\pi i} = \frac{i}{2\pi} \tag{43}$$

The factor of i that arises in this normalization reflects the fact that we are working in Euclidean signature, so that $t \to -i\tau$. Thus the action, which has a factor i in the Lorentzian signature, does not have such a factor in Euclidean signature.

4 Currents

Let us take a set of fermions

$$\psi^k, \quad k = 1, \dots N \tag{44}$$

These are anticommuting objects, and the 2-point functions are

$$< \psi^k(z_1)\psi^l(z_2) >= \frac{\delta^{kl}}{(z_1-z_2)} \tag{45}$$

Now consider the matrices $T^a, a = 1, \dots r$ forming a Lie algebra

$$[T^a, T^b] = f^{ab}_c T^c \tag{46}$$

We assume that these have been brought to an antisymmetric form

$$T^a_{ij} = -T^a_{ji} \tag{47}$$

We also assume that they are normalized by

$$tr(T^aT^b) = \delta^{ab} \tag{48}$$

Make the following bilinears in the fermions

$$J^a(z) = \frac{1}{2}T^a_{ij}\psi^i(z)\psi^j(z) \tag{49}$$

These are called currents. Note that the scaling dimension is

$$(\Delta, \bar{\Delta}) = (1, 0) \tag{50}$$

since the fermions had holomorphic dimension $\frac{1}{2}$ each. Thus we can define charges

$$Q^a = \int_C dz J^a(z) \tag{51}$$

where C is a contour that encircles the region to which we wish to apply the charge operator.

5 OPE of currents

Consider the OPE of two currents

$$J^a(z)J^b(z') = \frac{1}{2}T^a_{ij}\psi^i(z)\psi^j(z)\ \frac{1}{2}T^b_{kl}\psi^k(z')\psi^l(z') \tag{52}$$

The most singular term arises from contracting all fermions. This gives

$$\frac{1}{4}T^a_{ij}T^b_{kl}\ [\delta^{jk}\delta^{kl} - \delta^{ik}\delta^{jl}]\frac{1}{(z-z')^2} = \frac{1}{2}tr(T^aT^b)\frac{1}{(z-z')^2} = \frac{\frac{1}{2}\delta^{ab}}{(z-z')^2} \tag{53}$$

The next term comes when one pair of fermions is contracted. This is

$$\frac{1}{4}T^a_{ij}T^b_{kl}\ [\delta^{jk}\psi^i(z)\psi^l(z') - \delta^{jl}\psi^i(z)\psi^k(z') - \delta^{ik}\psi^j(z)\psi^l(z') + \delta^{il}\psi^j(z)\psi^k(z')]\frac{1}{(z-z')} \tag{54}$$

In this term let us put

$$\psi(z) \approx \psi(z') \tag{55}$$

since the corrections will be terms with no singularity. Then we get from the first part of the above expression

$$\frac{1}{4}T^a_{ij}T^b_{kl}\ \delta^{jk}\psi^i(z)\psi^l(z')\frac{1}{(z-z')} = \frac{1}{4}(T^aT^b)_{il}\psi^i(z')\psi^l(z')\frac{1}{(z-z')} \tag{56}$$

Doing this with all four terms, and using the antisymmetry of the T^a we find

$$\frac{1}{2}(T^aT^b - T^bT^a)_{il}\psi^i(z')\psi^l(z')\frac{1}{(z-z')} = \frac{1}{2}f^{ab}_cT^c_{il}\psi^i(z')\psi^l(z')\frac{1}{(z-z')} = \frac{f^{ab}_cJ^c(z')}{(z-z')} \tag{57}$$

Thus overall we get the OPE

$$J^a(z)J^b(z') = \frac{\frac{1}{2}\delta^{ab}}{(z-z')^2} + \frac{f^{ab}_cJ^c(z')}{(z-z')} + \ldots \tag{58}$$

6 The current algebra

Define the operators

$$J^a_n = \int_C' dzJ^a(z)z^n = \frac{1}{2\pi i}\int_C dzJ^a(z)z^n \tag{59}$$

We wish to compute the commutator

$$[J^a_n, J^b_m] \tag{60}$$

We have

$$J^a_nJ^b_m = \int_{C_2}' dz' \int_{C_1}' dz' J^b(z')J^a(z)z'^nz^m \tag{61}$$

where C_2 is outside C_2. In the other order we will have

$$J^b_mJ^a_n = \int_{C_2}' dz' \int_{C_1}' dz' J^b(z')J^a(z)z'^nz^m \tag{62}$$

with C_2 inside C_1. Thus in the commutator we will get

$$[J^a_n, J^b_m] = \int'_C dz' \int'_{C_1} dz' J^b(z') J^a(z) z'^n z^m \tag{63}$$

where C is a circle the encircles z counterclockwise. Let us first do this z' integral. The leading term in the OPE gives

$$\int'_C dz' \frac{\frac{\delta^{ab}}{2}}{(z'-z)^2} z'^n = \frac{\delta^{ab}}{2} n z^{n-1} \tag{64}$$

The dz integral then is

$$\frac{\delta^{ab}}{2} n \int'_{C_1} dz z^{n+m-1} = \frac{n}{2} \delta^{ab} \delta_{n+m,0} \tag{65}$$

Now let us look at the term with the single pole. The dz' integral gives

$$\int'_C dz' f^{ab}_c \frac{J^c(z)}{z'-z} z'^n = f^{ab}_c J^c(z) z^n \tag{66}$$

The dz integral then gives

$$f^{ab}_c \int'_{C_1} dz J^c(z) z^{n+m} = f^{ab}_c J^c_{n+m} \tag{67}$$

Thus we find the algebra

$$[J^a_n, J^b_m] = f^{ab}_c J^c_{n+m} + \frac{1}{2} \delta^{ab} n \delta_{n+m,0} \tag{68}$$

This is called a current algebra of level 1. More generally we have

$$[J^a_n, J^b_m] = f^{ab}_c J^c_{n+m} + \frac{k}{2} \delta^{ab} n \delta_{n+m,0} \tag{69}$$

which is called the current algebra of level k.

Consider the limit $k \to \infty$. Then we can ignore the fosrt term on the RHS, and we get

$$[J^a_n, J^b_m] \approx \frac{k}{2} \delta^{ab} n \delta_{n+m,0} \tag{70}$$

This is just like the algebra of free bosons

$$[\alpha^a_n, \alpha^b_m] = n \eta^{ab} \delta_{n+m,0} \tag{71}$$

Thus we describe a flat Euclidean space of dimension r, where r is the dimension of the group. The physics here is that of strings propagating on the group manifold which corresponds to the Lie algebra that we have taken. The string has a string length $l_s = \sqrt{\alpha'}$, and we can ask how this compares to the curvature length scale of the group manifold. In the limit $k \to \infty$ the string is very small compared to the size of the group manifold, so we do not see the curvature of the group manifold and it just looks like flat space. Thus we get the oscillator algebra noted above. In the opposite limit $k = 1$, the string length is comparable to the size of the group manifold, and the entire motion is very quantum; we cannot ignore the curvature of the target space. It is remarkable that we can solve the motion of the string exactly in this situation. We will see later that the central charge contributed by such a target space is

$$c = \frac{kD}{c_v + k} \tag{72}$$

where $D = r$ is the dimension of the group manifold, and c_v is the second quadratic Casimir

$$f_c^{ab} f_b^{a'c} = -c_v \delta^{aa'} \tag{73}$$

Thus for $SU(2)$ we will have

$$f_3^{12} f_2^{13} + f_2^{13} f_3^{12} = -2 \tag{74}$$

so that

$$c_v = 2 \tag{75}$$

We see that for $k \to \infty$

$$c \to D \tag{76}$$

which agreed with the central charge of D free bosons, representing a target space that is D flat dimensions.