1. The free scalar field

We consider the free scalar field in one space and one time dimension. Let the action be

$$S = \int dt\ L \tag{1.1}$$

$$L = \int dx\ \mathcal{L} \tag{1.2}$$

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - \frac{1}{2}m^2\phi^2 \tag{1.3}$$

Here L is the Lagrangian and $\mathcal{L}$ is the Lagrangian *density.* We should be careful not to confuse L with the length of the box in which the field theory lives; we will also call that length L.

We expand ϕ in fourier modes

$$\phi(t,x) = \sum_{n=-\infty}^{\infty} \phi_n(t) e^{ik_n x} \tag{1.4}$$

We place the system in a periodic box of length L, so

$$k_n = \frac{2\pi n}{L} \tag{1.5}$$

We have

$$\int dx\ e^{ik_n x} e^{ik_m x} = L\delta_{m+n,0} \tag{1.6}$$

We also have

$$\partial_t \phi = \sum_n \partial_t \phi_n(t) e^{ik_n x} \tag{1.7}$$

Since ϕ is real we have

$$\phi_n = \phi^*_{-n} \tag{1.8}$$

Writing

$$\phi_n = \phi^R_n + i\phi^I_n \tag{1.9}$$

we get

$$\phi^R_n = \phi^R_{-n}, \quad \phi^I_n = -\phi^I_{-n} \tag{1.10}$$

Let us now compute the Lagrangian L. We have

$$\int dx (\partial_t \phi)^2 = \int dx (\sum_n \partial_t \phi_n e^{ik_n x})(\sum_m \partial_t \phi_m e^{ik_m x}) = L \sum_n \partial_t \phi_n \partial_t \phi_{-n} \tag{1.11}$$

$$\int dx(\partial_x\phi)^2 = \int dx(\sum_n(ik_n\phi_n e^{ik_nx})(\sum_m ik_m\phi_m e^{ik_mx}) = L\sum_n k_n^2\phi_n\phi_{-n} \tag{1.12}$$

$$\int dx(\phi)^2 = \int dx(\sum_n(\phi_n e^{ik_nx})(\sum_m \phi_m e^{ik_mx}) = L\sum_n \phi_n\phi_{-n} \tag{1.13}$$

So we get

$$L = \sum_{n=1}^{\infty}[L(\dot\phi_n^R)^2 - L\omega_n^2(\phi_n^R)^2] + \sum_{n=1}^{\infty}[L(\dot\phi_n^I)^2 - L\omega_n^2(\phi_n^I)^2] + \frac{L}{2}[(\dot\phi_0)^2 - \omega_0^2(\phi_0)^2] \tag{1.14}$$

where

$$\omega_n^2 = k_n^2 + m^2 \tag{1.15}$$

We thus see that the free scalar field is equivalent to a collection of harmonic oscillators. We can thus quantize the scalar field by just quantizing these harmonic oscillators.

2. The harmonic oscillator

The Lagrangian is

$$L = \frac{1}{2}m\dot q^2 - \frac{1}{2}kq^2 \tag{2.1}$$

This gives

$$p = m\dot q, \quad H = \frac{p^2}{2m} + \frac{k}{2}q^2 \tag{2.2}$$

We write

$$\omega = \sqrt{\frac{k}{m}} \tag{2.3}$$

Upon quantization the Hamiltonian becomes

$$\hat H = \frac{\hat p^2}{2m} + \frac{k}{2}\hat q^2 \tag{2.4}$$

We can expand this as

$$\hat H = (\sqrt{\frac{k}{2}}\hat q - i\frac{\hat p}{\sqrt{2m}})(\sqrt{\frac{k}{2}}\hat q + i\frac{\hat p}{\sqrt{2m}}) + \frac{1}{2}\omega \equiv \hat A^\dagger\hat A + \frac{1}{2}\omega \tag{2.5}$$

Note that

$$[\hat A, \hat A^\dagger] = \omega \tag{2.6}$$

So we define

$$\hat a^\dagger = \frac{1}{\sqrt\omega}(\sqrt{\frac{k}{2}}\hat q - i\frac{\hat p}{\sqrt{2m}}), \quad \hat a = \frac{1}{\sqrt\omega}(\sqrt{\frac{k}{2}}\hat q + i\frac{\hat p}{\sqrt{2m}}) \tag{2.7}$$

getting

$$[\hat a, \hat a^\dagger] = 1 \tag{2.8}$$

Note that

$$\hat q = \sqrt{\frac{\omega}{2k}}(\hat a + \hat a^\dagger) \tag{2.9}$$

3. The quantized scalar field

We now wish to write the quantum field $\hat{\phi}$. Note that each fourier component $\phi^R_n, \phi^I_n, (n > 0)$ is the 'position' coordinate of a harmonic oscillator. This oscillator has

$$m = 2L \tag{3.1}$$

$$k = 2L\omega_n^2 \tag{3.2}$$

$$\omega = \sqrt{\frac{k}{m}} = \omega_n \tag{3.3}$$

Thus we have

$$\hat{\phi}^R_n = \frac{1}{2\sqrt{L\omega_n}}(\hat{a}^R_n + (\hat{a}^R_n)^\dagger) \tag{3.4}$$

$$\hat{\phi}^I_n = \frac{1}{2\sqrt{L\omega_n}}(\hat{a}^I_n + (\hat{a}^I_n)^\dagger) \tag{3.5}$$

For the zero mode we have

$$m = L, \quad k = L\omega_0^2, \quad \omega = \omega_0 \tag{3.6}$$

$$\hat{\phi}_0 = \sqrt{\frac{1}{2L\omega_0}}(\hat{a}_0 + \hat{a}_0^\dagger) \tag{3.7}$$

Using (1.10) we can write the classical field as

$$\begin{aligned}\phi(t) &= \sum_n (\phi^R_n(t) + i\phi^I_n(t))e^{ik_n x} \\ &= \sum_{n>0} 2\phi^R_n(t)\cos(k_n x) - \sum_{n>0} 2\phi^I_n(t)\sin(k_n x) + \phi_0(t)\end{aligned} \tag{3.8}$$

Upon quantizing we should just convert the 'position variables' $q(t)$ of the harmonic oscillators to 'position operators' $\hat{q}$

$$\hat{\phi} = \sum_{n>0} 2\hat{\phi}^R_n \cos(k_n x) - \sum_{n>0} 2\hat{\phi}^I_n \sin(k_n x) + \hat{\phi}_0 \tag{3.9}$$

In terms of creation and annihilation operators we get

$$\begin{aligned}\hat{\phi}(t) = &\sum_{n>0} \frac{1}{\sqrt{L\omega_n}}(\hat{a}^R_n + (\hat{a}^R_n)^\dagger)\cos(k_n x) - \sum_{n>0} \frac{1}{\sqrt{L\omega_n}}(\hat{a}^I_n + (\hat{a}^I_n)^\dagger)\sin(k_n x) \\ &+ \sqrt{\frac{1}{2L\omega_0}}(\hat{a}_0 + (\hat{a}_0)^\dagger)\end{aligned} \tag{3.10}$$

We thus see that the operator $(\hat{a}_n^R)^\dagger$ is a creation operator associated to the wavefunction $\cos(k_n x)$ and $(\hat{a}_n^I)^\dagger$ is associated to $\sin(k_n x)$. We would like to create and annihilate particles of definite momentum $e^{ik_n x}$. For $n > 0$ define

$$\hat{a}_n \equiv \frac{1}{\sqrt{2}}[\hat{a}_n^R + i\hat{a}_n^I], \quad n > 0 \tag{3.11}$$

For $n < 0$ we can similarly define

$$\hat{a}_n \equiv \frac{1}{\sqrt{2}}[\hat{a}_n^R + i\hat{a}_n^I], \quad n < 0 \tag{3.12}$$

where the operators $\hat{a}_n^R, \hat{a}_n^I$ for negative n are given through (1.10). The two sets (3.11), (3.12) encode all the $\hat{a}_n^R, \hat{a}_n^I$. We then find

$$\hat{a}_n^\dagger = \frac{1}{\sqrt{2}}[(\hat{a}_n^R)^\dagger - i(\hat{a}_n^I)^\dagger], \quad n \neq 0 \tag{3.13}$$

The factor $\frac{1}{\sqrt{2}}$ in (3.11)-(3.13) ensures that

$$[\hat{a}_m, \hat{a}_n^\dagger] = \delta_{m,n} \tag{3.14}$$

The zero mode operators stays as before. Writing

$$\cos(k_n x) = \frac{1}{2}[e^{ik_n x} + e^{-ik_n x}], \quad \sin(k_n x) = \frac{1}{2i}[e^{ik_n x} - e^{-ik_n x}] \tag{3.15}$$

we get

$$\hat{\phi} = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{L}} \frac{1}{\sqrt{2\omega_n}}[\hat{a}_n e^{ik_n x} + \hat{a}_n^\dagger e^{-ik_n x}] \tag{3.16}$$

This form is not very symmetrical between x, t since the operators depend on x but not on t. We move to the Heisenberg picture where we have

$$\hat{O}(t) = e^{i\hat{H}t}\hat{O}e^{-i\hat{H}t} \tag{3.17}$$

For a harmonic oscillator

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} \tag{3.18}$$

Note that

$$[\hat{a}, \hat{H}] = \omega \hat{a}, \quad [\hat{a}^\dagger, \hat{H}] = -\omega \hat{a}^\dagger \tag{3.19}$$

We can thus move $\hat{a}$ across $e^{-i\hat{H}t}$ in

$$\hat{a}(t) = e^{i\hat{H}t}\hat{a}e^{-i\hat{H}t} \tag{3.20}$$

getting

$$\hat{a}(t) = \hat{a}e^{-i\omega t} \tag{3.21}$$

$$\hat{a}^\dagger(t) = \hat{a}^\dagger e^{i\omega t} \tag{3.22}$$

Then in the Heisenberg picture the operator $\hat{\phi}$ becomes

$$\hat{\phi}(x,t) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_n}} [\hat{a}_n e^{ik_n x - i\omega_n t} + \hat{a}_n^\dagger e^{-ik_n x + i\omega_n t}] \tag{3.23}$$

We now see a covariant dot product between the position vector (t, x) and the momentum (ω_n, k_n).

4. Cross section

Imagine that we have two particles in a box. There is some probability per unit time that they collide, and thus scatter into new states. We want to describe this process and compute its relevant numbers.

By rotating our frame of reference appropriately we can let the particles come at each other along the x-direction. We assume that if the two particles come within a certain distance of each other then they will interact and scatter. To describe this distance we imagine a disc of area σ drawn around one of the particles (say particle 2) in the directions perpendicular to the particle's velocity $\vec{v}_2$. If particle 1 passes tries to pass through this disc then we assume that it scatters.

The probability density for each particle is $\frac{1}{V}$. Then the probability that particle 1 comes at particle 2 per unit time within a disc of area σ is

$$\frac{1}{V}|v_{1x} - v_{2x}|\sigma \tag{4.1}$$

In time T the probability for scattering is

$$\frac{1}{V}|v_{1x} - v_{2x}|\sigma \; T \tag{4.2}$$

Now look at the same process from a quantum viewpoint. Suppose the amplitude for the initial state to go to some scattered state over time T is $\mathcal{A}$. Then the probability is $|\mathcal{A}|^2$. If we sum over all possible final states then we get the total probability to scatter in time T.

So we get the basic equation

$$\frac{1}{V}|v_{1x} - v_{2x}|\sigma T = \Sigma_{final\ states}|\mathcal{A}|^2 \tag{4.3}$$

or

$$\sigma = \frac{V}{T|v_{1x} - v_{2x}|}\Sigma_{final\ states}|\mathcal{A}|^2 \tag{4.4}$$

Note that

$$v_{1x} = \frac{p_1^x}{p_1^0} \tag{4.5}$$

The scattering will be produced by the interaction Hamiltonian

$$H_{int} = \int d\vec{x}\ \lambda\hat{\phi}^4 \tag{4.6}$$

We have

$$\hat{\phi} = \sum_{\vec{n}} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_{\vec{n}}}}[\hat{a}_{\vec{n}} e^{-i\omega_{\vec{n}} t + i\vec{k}_{\vec{n}}\cdot\vec{x}} + \hat{a}_{\vec{n}}^\dagger e^{i\omega_{\vec{n}} t - i\vec{k}_{\vec{n}}\cdot\vec{x}}] \tag{4.7}$$

The initial state is made as

$$|\psi_i\rangle = a_{\vec{n}_1}^\dagger a_{\vec{n}_2}^\dagger|0\rangle \tag{4.8}$$

The final state is

$$|\psi_f\rangle = a_{\vec{n}_3}^\dagger a_{\vec{n}_4}^\dagger|0\rangle \tag{4.9}$$

These states are normalized. We then have

$$\mathcal{A} = \langle\psi_f|(-i)\int dt_1 H_{int}(t_1)|\psi_i\rangle \tag{4.10}$$

The combinatoric factor is $4! = 24$. This is seen as follows. The interaction Hamiltonian has the schematic structure

$$(a + a^\dagger)(a + a^\dagger)(a + a^\dagger)(a + a^\dagger) \tag{4.11}$$

The amplitude (4.10) then has the schematic form

$$\langle 0|a_3 a_4 (a + a^\dagger)(a + a^\dagger)(a + a^\dagger)(a + a^\dagger) a_1^\dagger a_2^\dagger|0\rangle \tag{4.12}$$

So the operators chosen from the brackets occurring in the interaction Hamiltonian must be $a_3^\dagger, a_4^\dagger, a_1, a_2$, but any of these can be picked from any of the 4 brackets. We thus get $4! = 24$. Apart from this factor we see that the contribution from (4.12) is unity since all the operators involved are in different states and we just have the algebra of 4 different harmonic oscillators. (We assume that the final state is different from the initial state.)

We thus get

$$\mathcal{A} = 24\lambda(-i)\frac{1}{V^2}\frac{1}{\sqrt{2\omega_{\vec{n}_1}}}\frac{1}{\sqrt{2\omega_{\vec{n}_2}}}\frac{1}{\sqrt{2\omega_{\vec{n}_3}}}\frac{1}{\sqrt{2\omega_{\vec{n}_4}}}$$
$$\int_{t_1=0}^{T} dt_1 \int d\vec{x} e^{i(\omega_{\vec{n}_3}+\omega_{\vec{n}_4}-\omega_{\vec{n}_1}-\omega_{\vec{n}_2})t_1} e^{-i(\vec{k}_3+\vec{k}_4-\vec{k}_1-\vec{k}_2)\cdot\vec{x}} \tag{4.13}$$

We have

$$\int d\vec{x} e^{-i(\vec{k}_3+\vec{k}_4-\vec{k}_1-\vec{k}_2)\cdot\vec{x}} = V\delta_{\vec{n}_3+\vec{n}_4-\vec{n}_1-\vec{n}_2,0} \tag{4.14}$$

The time integral gives

$$\int_0^T dt_1 e^{i\Delta\omega t_1} = \frac{1}{(i\Delta\omega)}[e^{i\Delta\omega T} - 1] \tag{4.15}$$

where

$$\Delta\omega \equiv \omega_{\vec{n}_3} + \omega_{\vec{n}_4} - \omega_{\vec{n}_1} - \omega_{\vec{n}_2} \tag{4.16}$$

is the energy difference between the final and initial states. The mod squared of (4.15) gives

$$\frac{1}{(\Delta\omega)^2}[2 - e^{i\Delta\omega T} - e^{-i\Delta\omega T}] \tag{4.17}$$

In the limit of large T this turns out to equal

$$2\pi T\delta(\Delta\omega) \tag{4.18}$$

(To find the coefficient of the delta function integrate (4.17) over $\Delta\omega$. We use the method of contour integrals and shift the pole to $\Delta\omega - i\epsilon$. Only one term from the square bracket gives a nonzero contribution.)

Collecting all these terms we find

$$|\mathcal{A}|^2 = (24)^2\lambda^2(2\pi T)\delta(\omega_{\vec{n}_3}+\omega_{\vec{n}_4}-\omega_{\vec{n}_1}-\omega_{\vec{n}_2})\frac{1}{V^2}\frac{1}{16}\frac{1}{\omega_{\vec{n}_1}}\frac{1}{\omega_{\vec{n}_2}}\frac{1}{\omega_{\vec{n}_3}}\frac{1}{\omega_{\vec{n}_4}}\delta_{\vec{n}_3+\vec{n}_4-\vec{n}_1-\vec{n}_2,0} \tag{4.19}$$

In doing the sum over final states, we note that

$$\sum_{\vec{n}} = \int d\vec{n} = \frac{V}{(2\pi)^3}\int d\vec{p} \tag{4.20}$$

$$\delta_{\vec{n}_3+\vec{n}_4-\vec{n}_1-\vec{n}_2,0} = \frac{(2\pi)^3}{V}\delta(\vec{p}_4+\vec{p}_3-\vec{p}_1-\vec{p}_2) \tag{4.21}$$

$$\sum_{\vec{n}_3}\sum_{\vec{n}_4} = \frac{V^2}{(2\pi)^6}\int d\vec{p}_3 d\vec{p}_4 \tag{4.22}$$

Let us now take a specific case. We will be in the CM frame, and so

$$\vec{p}_1 = -\vec{p}_2, \quad |\vec{p}_1| = |\vec{p}_2| \equiv p_i \tag{4.23}$$

We can now eliminate one delta function

$$\int d\vec{p}_3 \delta(\vec{p}_4 + \vec{p}_3 - \vec{p}_1 - \vec{p}_2) = 1 \tag{4.24}$$

which sets

$$\vec{p}_3 = -\vec{p}_4, \quad |\vec{p}_3| = |\vec{p}_4| \equiv p_f \tag{4.25}$$

The energy delta function is

$$\delta(2\sqrt{p_f^2 + m^2} - 2\sqrt{p_i^2 + m^2}) \tag{4.26}$$

We can write

$$d\vec{p}_4 = p_f^2 dp_f d\Omega_f \tag{4.27}$$

Then we see that

$$\int dp_f \delta(2\sqrt{p_f^2 + m^2} - 2\sqrt{p_i^2 + m^2}) = [\frac{d(2\sqrt{p_f^2 + m^2})}{dp_f}]^{-1} = \frac{\omega_f}{2p_f} \tag{4.28}$$

The δ-function also sets

$$p_i = p_f, \quad \omega_i = \omega_f \equiv \omega \tag{4.29}$$

We can now collect all terms.

$$\Sigma_{final\ states}|\mathcal{A}|^2 = (24)^2\lambda^2(2\pi T)(\frac{\omega_f}{2p_f})(\frac{1}{V^2})\frac{1}{16}\frac{1}{\omega_f^4}(\frac{V}{(2\pi)^3})p_f^2 d\Omega_f \tag{4.30}$$

Note that

$$|v_{1x} - v_{2x}| = 2|v_{1x}| = 2\frac{p_i}{\omega_i} \tag{4.31}$$

We get

$$\sigma = \frac{V}{T}\frac{\omega_i}{2p_i}(24)^2\lambda^2(2\pi T)(\frac{\omega_f}{2p_f})(\frac{1}{V^2})\frac{1}{16}\frac{1}{\omega_f^4}(\frac{V}{(2\pi)^3})p_f^2 d\Omega_f \tag{4.32}$$

This simplifies to

$$\sigma = \frac{9\lambda^2}{(2\pi)^2}\frac{1}{\omega^2}d\Omega_f \tag{4.33}$$

Since the two final state particles are identical, we see that we need to sum over only half the final solid angle to account for all final states. Thus the total cross section is

$$\sigma_T = \frac{9\lambda^2}{2\pi}\frac{1}{\omega^2} \tag{4.34}$$

5. Decay

We now consider the physical process where we have a single particle to start with, which then decays into two other particles. Clearly the three particles cannot all have the the same mass, since then energy would not be conserved. We assume that all three particles are of different species and have different masses.

Take three scalar fields, with masses m_1, m_2, m_3, and interaction

$$\mathcal{L}_{int} = -\beta\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3 \tag{5.1}$$

We wish to compute the probability of decay per unit time. The initial state is

$$|\psi_i\rangle = a^\dagger_{1,\vec{k}_1}|0\rangle \tag{5.2}$$

The final state is

$$|\psi_f\rangle = a^\dagger_{2,\vec{k}_2}a^\dagger_{3,\vec{k}_3}|0\rangle \tag{5.3}$$

The amplitude for decay in time T is

$$\mathcal{A} = \langle\psi_f|(-i)\int dt_1 H_{int}(t_1)|\psi_i\rangle \tag{5.4}$$

with

$$H_{int} = \int d^3x\beta\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3 \tag{5.5}$$

The combinatoric factor is unity. We get

$$\mathcal{A} = \beta(-i)\frac{1}{V^{\frac{3}{2}}}\frac{1}{\sqrt{2\omega_{\vec{n}_1}}}\frac{1}{\sqrt{2\omega_{\vec{n}_2}}}\frac{1}{\sqrt{2\omega_{\vec{n}_3}}}\int_{t_1=0}^{T}dt_1\int d\vec{x}e^{i(\omega_{\vec{n}_2}+\omega_{\vec{n}_3}-\omega_{\vec{n}_1})t_1}e^{-i(\vec{k}_2+\vec{k}_3-\vec{k}_1)\cdot\vec{x}} \tag{5.6}$$

We have

$$\int d\vec{x}e^{-i(\vec{k}_2+\vec{k}_3-\vec{k}_1)\cdot\vec{x}} = V\delta_{\vec{n}_2+\vec{n}_3-\vec{n}_1,0} \tag{5.7}$$

The time integral gives

$$\int_0^T dt_1 e^{i\Delta\omega t_1} = \frac{1}{(i\Delta\omega)}[e^{i\Delta\omega T}-1] \tag{5.8}$$

Collecting all these terms (and using (4.17), (4.18)) we find

$$|\mathcal{A}|^2 = \beta^2(2\pi T)\delta(\omega_2+\omega_3-\omega_1)\frac{1}{V}\frac{1}{8}\frac{1}{\omega_1}\frac{1}{\omega_2}\frac{1}{\omega_3}\delta_{\vec{n}_2+\vec{n}_3-\vec{n}_1,0} \tag{5.9}$$

We now sum over the final states. We have

$$\delta_{\vec{n}_2+\vec{n}_3-\vec{n}_1,0} = \frac{(2\pi)^3}{V}\vec{\delta}(\vec{p}_2+\vec{p}_3-\vec{p}_1) \tag{5.10}$$

$$\sum_{\vec{n}_2}\sum_{\vec{n}_3} = \frac{V^2}{(2\pi)^6}\int d\vec{p}_2 d\vec{p}_3 \tag{5.11}$$

We take the initial particle to have momentum zero. Thus

$$\vec{n}_1 = 0, \quad \vec{p}_1 = 0, \quad \omega_1 = m_1 \tag{5.12}$$

We can now eliminate one delta function

$$\int d\vec{p}_2\delta(\vec{p}_2+\vec{p}_3-\vec{p}_1) = 1 \tag{5.13}$$

which sets

$$\vec{p}_2 = -\vec{p}_3, \quad |\vec{p}_2| = |\vec{p}_3| \equiv p_f \tag{5.14}$$

The energy delta function is

$$\delta(\sqrt{p_f^2+m_2^2}+\sqrt{p_f^2+m_3^2}-m_1) \tag{5.15}$$

We can write

$$d\vec{p}_3 = p_f^2 dp_f d\Omega_f \tag{5.16}$$

Then we see that

$$\int dp_f\delta(\sqrt{p_f^2+m_2^2}+\sqrt{p_f^2+m_3^2}-m_1) = [\frac{d}{dp_f}(\sqrt{p_f^2+m_2^2}+\sqrt{p_f^2+m_3^2})]^{-1} = \frac{\omega_2\omega_3}{p_f(\omega_2+\omega_3)} \tag{5.17}$$

Collecting all terms, the probability for decay is

$$P = |\mathcal{A}|^2 = T\frac{\beta^2}{4\pi^2}\frac{1}{8}\frac{1}{\omega_1\omega_2\omega_3}|p_f|^2 d\Omega\frac{\omega_2\omega_3}{p_f(\omega_2+\omega_3)} \tag{5.18}$$

Thus the probability of decay per unit time per unit solid angle is

$$\Gamma' = \frac{P}{T} = \frac{\beta^2}{32\pi^2}d\Omega\frac{p_f}{m_1(\omega_2+\omega_3)} \tag{5.19}$$

Integrating over all directions,

$$\Gamma = \frac{\beta^2}{8\pi} \frac{p_f}{m_1(\omega_2 + \omega_3)} \tag{5.20}$$

Note that by the energy conservation delta function

$$\omega_2 + \omega_3 = m_1 \tag{5.21}$$

so we get

$$\Gamma = \frac{\beta^2}{8\pi} \frac{p_f}{m_1^2} \tag{5.22}$$

Also

$$\sqrt{p_f^2 + m_2^2} + \sqrt{p_f^2 + m_3^2} = m_1 \tag{5.23}$$

which gives

$$p_f^2 = \frac{\sum_i m_i^4 - 2\sum_{i<j} m_i^2 m_j^2}{4m_1^2} \tag{5.24}$$

We then get

$$\Gamma = \frac{\beta^2}{16\pi} [\frac{(\sum_i m_i^4 - 2\sum_{i<j} m_i^2 m_j^2)^{\frac{1}{2}}}{m_1^3}] \tag{5.25}$$

6. Units

We are using units where $c = \hbar = 1$. Thus the action, which has the same units as $\hbar$, is dimensionless. In the kinetic term $\int d^4x \frac{1}{2}\partial\phi\partial\phi$ the integration measure d^4x has units L^4 (L stands for length). The two derivatives give L^{-2}. Thus ϕ must have units L^{-1}. The mass term $\int d^4x \frac{1}{2}m^2\phi^2$ is then seen to also be dimensionless once we note that the mass m has units L^{-1} in quantum mechanics.

The interaction term $\int d^4x \lambda\phi^4$ then tells us that the coefficient λ must be dimensionless. For the interaction $\int d^4x \beta\phi^3$ we see that β has units L^{-1}.

The cross section has units L^2 since it is an area, and Γ has units L^{-1} since it is a rate (probability per unit time).

7. Identical particles

In the computations above we often sum over final states. The initial state is given, and we first compute the *amplitude* to reach a *given* final state. The modulus squared of this amplitude gives the *probability* to reach this final state. We then add the *probabilities*.

Suppose the final state consists of two particles of the *same* species. Let the momenta of these particles be $\vec{p}_1, \vec{p}_2$. Then we can reach the same final state in two ways: The first particle can have momentum $\vec{p}_1$ and the second $\vec{p}_2$, or the other way around. Since these are two ways to reach the *same* final state we must add the *amplitudes* to get the total amplitude to reach the final state. We then square this amplitude to get the probability.

Now we must some over final states. But note that we must just count *different* final outcomes. If the first particle goes north and the second south, then this is *not* different from the case that the first particle goes south and the second north. So if we are counting final states by looking at the momentum of the first particle (the second will go in the opposite direction) then we must count only over *half* the total solid angle.