Lecture notes 1

The fuzzball: initial ideas

1.1 The scale of bound states

We have seen that the information paradox arises from pair creation near the horizon of a black hole. In computing this pair creation, we had to use the fact that the quantum state around the horizon was the standard vacuum. This fact was guaranteed by the no-hair theorem, which tells us that there are no perturbative deformations of the horizon, either at the classical or the quantum level.

But could some large *non*-perturbative effect present in quantum gravity alter the structure of the black hole completely, and thereby resolve the information paradox? On the face of it, the answer would seem to be no. At the singularity $r = 0$ the curvature diverges, and we certainly expect quantum gravity effects to be nontrivial. It is conventionally assumed that that the scale of quantum gravity effects is set by the l_p, the planck length. With this assumption, we see that any quantum gravity effects at the singularity would not reach near the horizon, which is a macroscopic distance away. The curvature at the horizon itself is very low, so quantum gravity seems to be irrelevant there. Thus it seems that quantum gravity cannot really be relevant in providing a resolution of the information paradox.

However, we have noted that this argument is misleading. The natural length scale set by $G, \hbar, c$ is indeed planck length $l_p = (\frac{\hbar G}{c^2})^{\frac{1}{2}}$. But to make a large black hole we need to put together a large number of particles N, so it in in principle possible that the length scale over which quantum gravity operates gets modified to a new length scale $N^\alpha l_p$ for some $\alpha > 0$. In that case we would have to reexamine the question of whether quantum gravity effects can alter the physics at the horizon.

But what effect would generate this larger length scale? In usual treatments of quantum gravity, we just consider the metric and allow it to fluctuate, weighting each geometry by some action. In such a treatment l_p remains the only relevant length scale; until curvature scales become of order planck length, there is no reason for the semiclassical treatment of gravity to break down.

In string theory, however, we have extended objects. By themselves these objects have a tension of order planck scale, and so their extended nature should not be evident till we probe them at planck scales. But we have seen that when a large number N of branes are put together then we get an effect called fractionation, where the effective tension of extended objects becomes $\sim 1/N$

times the planck scale. One may then expect that these fractional tension objects can stretch to distances that are correspondingly larger than l_p, and that this would give the new length scale that we seek.

In this section we will give a heuristic derivation of this new length scale from fractionation. We will find that the size of a bound state of branes indeed becomes order horizon size, so that we cannot trust the standard picture of the classical black hole. It should be borne in mind that this estimate of size can be at most suggestive of new physics; to actually prove that the horizon is altered in string theory we will have to look at the actual construction of specific states of the black hole, which we will do in later sections.

1.2 The size of brane bound states

We have seen that black holes in string theory are made by making bound states of branes. In particular, we have made 2-charge, 3-charge and 4-charge holes. Let us begin by summarizing the general pattern of these constructions:

(i) We start with 10-dimensional string theory. We compactify d of these directions. We have looked at the cases $d = 5$ which gives a black hole in 4+1 noncompact dimensions, and the case $d = 6$ which gives a black hole in $3 + 1$ noncompact dimensions.

(ii) We wrap an appropriate set of branes on the compact dimensions. Let the mass of each brane of type i be m_i. For an extremal hole we take n_i branes of type i. The index i ranges over $1, \dots k$, where $k = 3$ for the 3-charge hole in 4+1 noncompact dimensions and $k = 4$ for the 4-charge hole in 3+1 noncompact dimensions. In each case the mass of the brane bound state is

$$M = n_1 m_1 + n_2 m_2 + \cdots + n_k m_k \tag{1.1}$$

The entropy of this extremal brane bound state is

$$S_{ex} = A_k \sqrt{n_1 n_2 \dots n_k} \tag{1.2}$$

where A_k is a constant of order unity.

(iii) Near-extremal holes have similar relations, but involving both branes and antibranes. Let the bound state be made of branes of type $i = 1, \dots k$. Let the lightest brane be of type k. Then the mass of the near-extremal state is

$$M = n_1 m_1 + \cdots + n_{k-1} m_{k-1} + (n_k + \bar{n}_k) m_k \tag{1.3}$$

The entropy of this near-extremal state is

$$S_{near-ex} = A_k \sqrt{n_1 n_2 \dots n_{k-1} n_k} + A_k \sqrt{n_1 n_2 \dots n_{k-1} \bar{n}_k} \tag{1.4}$$

(iii) The above expressions for entropy can be understood in terms of fractionation. Suppose we have a bound state of branes of type $i = 1, 2, \ldots k-1$. Then a brane of type k, when bound to this brane set, will fractionate into units that are

$$\frac{1}{n_1 n_2 \ldots n_{k-1}} \tag{1.5}$$

times a whole brane of type k. Thus n_k units of this brane charge will generate

$$N_k^{bits} = n_1 \ldots n_{k-1} n_k \tag{1.6}$$

fractional bits. The entropies above can then be understood as the number of possible ways to group these fractional bits into different sets. The number of ways to group N bits is

$$\mathcal{N} \sim e^{A_k \sqrt{N}} \tag{1.7}$$

where A_k a constant of order unity which depends on how many spin states the brane of type k has. This generates the entropies (1.2),(1.4).

1.2.1 The set-up

Let us now set up the computation that we wish to do. We wish to estimate the size of the 3-charge brane bound state in 4+1 noncompact dimensions. This size should describe some measure of how far this brane state extends in the 4 noncompact space directions. In particular, we can look at virtual fluctuations of branes emanating from the brane bound state, and ask how far they extend. We expect that nonextremal states have more energy available to generate brane fluctuations than extremal states, since all the energy in extremal states is locked up in the charge they carry. Thus to get the most stringent limits on the size of our bound state, we will ask for the size of an extremal 3-charge bound state.

We proceed in the following steps:

(i) Suppose the bound state has a transverse size D. Then we should be able to probe this size using a probe of wavelength $\lesssim D$. Such a probe will have energy

$$E_{probe} \gtrsim \frac{1}{D} \tag{1.8}$$

Note that this is a very small energy, since we are looking for the size D to be a macroscopic one.

(ii) To explore the consequences of adding the small energy (1.8) to our brane bound state, we set up the problem as follows. We put our bound state in a box of size D. We then add energy $1/D$ to the bound state. This energy can in principle create extra branes, which can wrap around the walls of the box. If we do in fact create such branes, then we will say that the size of the brane bound state is $\gtrsim D$. Note that if we did not have any novel phenomena like fractionation, then the tension of the brane would be planck scale, and the energy of the brane would be

$$E_{brane} \sim m_p^2 D \tag{1.9}$$

If we set $E_{brane} \sim E_{probe}$, then we will get

$$D \sim l_p \tag{1.10}$$

so that we would conclude that the brane bound state has size of order the planck length. With fractionation, we will see that the situation becomes very different.

(iii) We have added an energy $1/D$ to the extremal bound state. This energy can be used the by system in two ways:

(a) The extremal bound state remains unchanged, and the probe quantum p in the box carries energy $1/D$. In this case the entropy in the box is

$$S_{ex+p} = S_{ex} + s_p = A_{k-1}\sqrt{n_1 \dots n_{k-1}} + s_p \tag{1.11}$$

where

$$s_p \sim 1 \tag{1.12}$$

is the entropy of the probe quantum. If this quantum was a particle with two possible spin states, the we would have $s_p = \ln 2$.

(b) The extremal bound state absorbs the probe quantum, and becomes a near-extremal state. We will compute the entropy of this near-extremal state below; for now we write it as a small increment over the extremal entropy

$$S_{near-ex} = S_{ex} + \Delta S \tag{1.13}$$

(iv) The entropy of a set of configurations is the log of the phase space available for such configurations. Thus it will be more likely to have the situation (b) above rather than the situation (a) if

$$\Delta S > s_p \tag{1.14}$$

Let us summarize our set up before performing the computation of $S_{near-ex}$. We have placed our extremal 3-charge bound state in a box of size D, and probed it with the lowest energy quantum possible; one with energy $1/D$. We are then asking if this small energy $1/D$ will be sufficient to generate fractional branes that wrap around the walls of the box of size D. We are requiring that not only that it should be possible to excite such fractional branes, but that it should be more probable that they are excited than not excited; this is the condition (1.14). The largest value of D for which (1.14) is true will be called the size of our 3-charge extremal bound state. If this size turns out to be planck size (eq. (1.10), then we would get the traditional expectation: the size of the bound state would not grow with the number of branes in the bound state. But as we will see now, fractionation will yield a much larger size for D.

1.2.2 Computing $S_{near-ex}$

We are starting with the 3-charge extremal state in 4+1 noncompact dimensions. Thus 5 dimensions have been compact; let us call these directions $X^1, \ldots X^5$. Let each of these directions be compactified to a circle; thus we have the D1D5P system on the spacetime

$$M_{9,1} \to M_{4,1} \times T^4 \times S^1 \tag{1.15}$$

Let the box of size D be a circle in the direction X^6. We leave the other 3 space directions uncompactified. Thus our question is: will branes wrap around the direction X^6?

We see immediately that we already have the tools to address this question. The 3-charge extremal state with 5 compact directions was characterized by charges n_1, n_2, n_3. But with 6 compact directions, the most entropic states are those with 4 kinds of charges. The branes of this fourth charge will indeed wrap the new compact circle X^6. We have not added any net charge however; all we have is an extra energy $1/D$. Thus branes of the fourth type will have no net charge; thus we will have equal numbers of branes and antibranes for this brane type

$$n_4 = \bar{n}_4 \tag{1.16}$$

Each brane of type 4 has mass m_4; thus the added energy $1/D$ gives

$$n_4 = \frac{1}{2Dm_4} \tag{1.17}$$

Note that this will be a fractional number in general; thus we are looking to excite fractional branes.

Since we have added a very small amount of energy $1/D$, we do not expect the brane bound state to change its nature completely from a 3-charge state with entropy (1.2) to a 4-charge state with entropy (1.4). Rather, we expect a state with properties partway between these two types of structures. We proceed as follows:

(i) First consider the extremal state. Here we have brane charges n_1, n_2, n_3, Fractionation generates

$$N = n_1 n_2 n_3 \tag{1.18}$$

fractional bits, whose different possible groupings give the entropy

$$S_{ex} = A_3 \sqrt{N} = 2\pi\sqrt{N} \tag{1.19}$$

where we have recalled that $A_3 = 2\pi$ when the compactification is on a torus $T^4 \times S^1$.

(ii) Let us assume that a fraction f of these N bits will be involved in generating a 4-charge near-extremal state, and the remaining fraction $(1 - f)$

will continue to generate the groupings that generate 3-charge extremal entropy. Then the 3-charge extremal entropy will be

$$S_3 = 2\pi\sqrt{(1-f)N} \tag{1.20}$$

(iii) The fraction fN of three charge bits are going to 'fractionate' branes of the fourth charge. Thus the branes of this fourth type will arise in units that are $1/(Nf)$ times a full brane of this type. The number of bits generated by n_4 such branes is then

$$N_4 = fNn_4 \tag{1.21}$$

An equal number of fractional bits arise from the $\bar{n}_4$ antibranes

$$\bar{N}_4 = fN\bar{n}_4 = fNn_4 \tag{1.22}$$

The entropies from these grouping these two sets of bits give the 4-charge near-extremal entropy of type (1.4)

$$S_4 = A_4\sqrt{N_4} + A_4\sqrt{\bar{N}_4} = 2A_4\sqrt{N_4} = 4\pi\sqrt{fNn_4} \tag{1.23}$$

where we have noted that $A_4 = 2\pi$ when the compactification is on a torus $T^4 \times S^1 \times \tilde{S}^1$.

(iv) The total entropy of our nonextremal brane state is the sum of (1.20) and (1.23)

$$S_{non-ex} = 2\pi\sqrt{(1-f)N} + 4\pi\sqrt{fNn_4} \tag{1.24}$$

We must maximize this function over the possible choices of f. The maximum occurs at

$$f = \frac{4n_4}{1+n_4} \tag{1.25}$$

Substituting this in (1.24) we get

$$S_{non-ex} = 2\pi\sqrt{N(1+4n_4)} \approx 2\pi\sqrt{N} + 4\pi\sqrt{N}n_4 = S_{ex} + 4\pi\sqrt{N}n_4 \tag{1.26}$$

where we have anticipated that at the end that n_4 will be a small fractional value; i.e., $n_4 \ll 1$.

(v) From the above relation we find that the quantity ΔS in (1.13) is

$$\Delta S = 4\pi\sqrt{N}n_4 \tag{1.27}$$

Using (1.17) and (1.18), this is

$$\Delta S = \frac{2\pi\sqrt{n_1n_2n_3}}{Dm_4} \tag{1.28}$$

The condition (1.14) says that we get the size of the bound state D by setting ΔS equal to s_p. This gives

$$D = \frac{2\pi\sqrt{n_1 n_2 n_3}}{s_p m_4} \qquad (1.29)$$

We have taken a compactification where our first three charges are D1, D5, P. In the compactification (1.15), the volume of T^4 is $(2\pi)^4 V$ and the length of S^1 is $2\pi R$. The fourth charge in this situation is the KK monopole, with the direction X^6 as the nontrivially fibred circle. The length of this circle $\tilde{S}^1$ is now D, the size of the box we have used to confine the 3-charge state. The mass of the KK-monopole is

$$m_4 = \frac{D^2 V_4 R}{4\pi^2 g^2 \alpha'^4} \qquad (1.30)$$

Substituting in (1.29) gives

$$D = \frac{8\pi^3 g^2 \alpha'^4 \sqrt{n_1 n_2 n_3}}{D^2 V_4 R s_p} \qquad (1.31)$$

which gives

$$D = \left(\frac{8\pi^3 g^2 \alpha'^4 \sqrt{n_1 n_2 n_3}}{V_4 R s_p}\right)^{\frac{1}{3}} \qquad (1.32)$$

Recall that $s_p \sim 1$ (eq.(1.12). Thus we find for the transverse size of our 3-charge extremal bound state

$$D \sim \left(\frac{8\pi^3 g^2 \alpha'^4 \sqrt{n_1 n_2 n_3}}{V_4 R}\right)^{\frac{1}{3}} \qquad (1.33)$$

Let us compare this scale D to the horizon radius of the 3-charge extremal hole. We know that the microscopic entropy of this hole reproduces the Bekenstein entropy, so

$$2\pi\sqrt{n_1 n_2 n_3} = \frac{A}{4G^{(5)}} \qquad (1.34)$$

We have

$$A = 2\pi^2 r_h^3, \quad G^{(5)} = \frac{G^{(10)}}{(2\pi R)((2\pi)^4 V_4)} = \frac{\pi g^2 \alpha'^4}{4RV_4} \qquad (1.35)$$

This gives

$$r_h = \left(\frac{\pi g^2 \sqrt{n_1 n_2 n_3}\,\alpha'^4}{RV_4}\right)^{\frac{1}{3}} \qquad (1.36)$$

Thus we find

$$D \sim r_h \qquad (1.37)$$

1.2.3 Summary

The relation (1.37) is a remarkable one, so let us pause for a moment to consider its significance.

Eq. (1.37) is a crude estimate, obtained through a heuristic picture of fractionated branes. It should be noted however that computations with this heuristic picture have yielded satisfactory results in other situations. In [] it was found that the classically observed transition between black holes and black strings can be reproduced by extremizing functions like (1.24). Similar computations also predict correctly the transition point where string degrees of freedom start behaving like black hole degrees of freedom. Further, the technical inputs going into the estimate (1.37) were the entropy expressions (1.2) and (1.4), and these are believed to be robust expressions in string theory.

The derivation of (1.37) also does not tell us anything about the structure of brane bound state. To understand this structure must clearly be our next task, and we will do this in the next few sections. We will take specific instances of the brane bound state, and work out their explicit structure. We will indeed find that none of these states have the traditionally expected structure of a black hole: in no case do we get either a horizon or a singularity.

The weaknesses noted above should not detract from the interesting nature of the estimate (1.37). The horizon radius r_h is a complicated expression (1.36) of seven parameters: $R, V, g, n_1, n_2, n_3, \alpha'$. Our crude estimate for the size D of the bound state turned out to have the same dependence on all these parameters. It is hard to see how a result like this could be obtained in a theory which did not have extended objects like strings and branes. In most approaches to quantum gravity, the semiclassical approximation is violated only where the curvature radius becomes planck scale in the classical metric. But in string theory the estimate (1.37) shows that the size of a string bound state grows with the number of branes n_1, n_2, n_3 in the bound state. It also grows with the coupling g. These growths are such that the brane bound state always has a size of order the horizon radius r_h predicted for a hole with the same charges.

This circumstance opens up a whole new direction of thought in the context of the information paradox. If the structure of the black hole can be altered at the horizon, then we cannot trust any of the steps that led to the creation of entanglement in radiation from this horizon. We will in fact be able to take specific instances of states of nonextremal holes, and observe that radiation from these states does not in generate the problematic entanglement that led to the information paradox.

Lecture notes 2

The extremal 2-charge hole

The computations of this chapter will play a central role in our analysis of the quantum physics of black hole. The simplest black hole in string theory is the extremal hole made with 2-charges. We have seen that the microscopic count of states of this system gives an entropy $S_{micro} = C\sqrt{n_1 n_2}$; this is similar to the form of the entropy for all other black holes in string theory. We can assume the traditional spherically symmetric ansatz and find a metric for the 2-charge system. If the compactification is chosen to be $K3 \times S^1$, then we have noted that this metric has a 'small' horizon, and the Bekenstein-Wald entropy S_{bek} computed from this horizon agrees with S_{micro}.

But for the 2-charge extremal system we can do something more: we can construct all the $Exp[S_{micro}]$ quantum states of the system. It turns out that none of these states fall in the spherically symmetric ansatz which had a horizon. Instead all states are 'fuzzballs', with no horizon or singularity.

A fuzzball structure resolves all the puzzles associated with black holes; if there is no vacuum region around a horizon, then there is no radiation by pair creation from this vacuum. The black hole behaves like any other normal body, radiating energy and information from its surface. One may argue that other black holes need not share the features of the 2-charge extremal hole, but as we will see shortly, extensive work with 3 and 4 charge holes has supported the fuzzball picture for all holes; no microstates have been found that possess a traditional horizon.

Let us now see how the microstates for the 2-charge extremal hole are constructed.

2.0.1 The NS1-P system

Let us recall the NS1-P bound state discussed in section 4.1. We start with IIB string theory. We compactify $x^6 \dots x^9$ to a torus T^4, and an additional direction x^5 to a circle S^1. In what follows, we will write

$$x^6 = z_1, \quad x^7 = z_2, \quad x^8 = z_3, \quad x^9 = z_4 \tag{2.1}$$

and

$$x^5 = y \tag{2.2}$$

We will take n_w fundamental strings (i.e., NS1 branes) wrapping the S^1, and n_p units of momentum along this S^1. The bound state of these charges gives our

2-charge extremal NS1-P system. Our goal is to find the nature of this bound state, in particular whether it has a horizon.

Let us place our bound state at the location $\vec{x} = 0$ in the transverse directions $x^1, \dots x^4$. Since the strings and momentum modes all run along the compact direction y, one might think that we will get a point mass at the location $\vec{x} = 0$ upon dimensional reduction. Such a point mass would generate a spherically symmetric metric. Why should this expectation be violated?

We have seen in section 4.1 that the string NS1 carries the momentum P in the form of traveling waves. The key point is that the string does not have a longitudinal vibration mode. Thus all the momentum must be carried by *transverse* vibration modes. In a transverse vibration, the string must distort away from the direction y towards one of the other spatial directions. This breaks the spherical symmetry, and also gives the string state a nonzero transverse size. Taking a larger value for n_p will give a larger entropy, but will also give larger transverse vibrations and thus a larger size for the bound state. In the end, we will find that no state of this NS1-P system generates a traditional horizon.

2.0.2 The metric of a vibrating string

Our string that has a large number of strands n_w, and we will finally be interested in waves on such a multiwound string. But we start with the analysis of waves on a single string. We will also ignore the compactifications for now; thus we have a string in 9+1 dimensional Minkowski spacetime.

First consider the string with no wave on it. Thus we have a straight string, which we take to lie along the y direction. We had seen that the gravitational solution produced by such a string is

$$\begin{aligned} ds^2_S &= H_1^{-1}[-dt^2 + dy^2] + dx_i dx_i \\ B_{tx_1} &= H_1 \\ e^{2\phi} &= H_1^{-1} \end{aligned} \tag{2.3}$$

where

$$H_1 = 1 + \frac{Q_1}{|\vec{x}|^6} \tag{2.4}$$

The gravitational equations are nonlinear, and it may seem that there is no easy way to find the metric produced by a string carrying a general vibration profile. But it turns out that when the wave is taken to propagate only in one direction along the string, then there is a simple technique to find the solution for an arbitrary profile of the wave. This is indeed the case we are interested in, since in the extremal NS1-P system the momentum modes P travel in only one direction along the string.

To apply this technique, we begin by writing

$$u = t + y, \quad v = t - y \tag{2.5}$$

Then the solution (2.3) becomes

$$\begin{aligned} ds_S^2 &= -H_1^{-1} du dv + dx_i dx_i \\ B_{uv} &= -\frac{1}{2} H_1 \\ e^{2\phi} &= H_1^{-1} \end{aligned} \tag{2.6}$$

This solution has a null killing vectors ∂_u. That is, the metric is invariant under a coordinate shift

$$\xi^\mu \to \xi^\mu + \epsilon k^\mu \tag{2.7}$$

with

$$k^u = 1, \quad k^v = 0, \quad k^i = 0 \tag{2.8}$$

Note that as a covariant vector, k has only one nonvanishing component k_v. A general result of Garfinkle and Vachaspati says that in this situation, we can generate another solution by adding a component proportional to $k_\mu k_\nu$ as follows:

$$\begin{aligned} ds_S^2 &= -H_1^{-1}[du dv + T(v,x) dv^2] + dx_i dx_i \\ B_{uv} &= -\frac{1}{2} H_1 \\ e^{2\phi} &= H_1^{-1} \end{aligned} \tag{2.9}$$

Here $T(v,x)$ is a harmonic function of the x^i

$$\sum_{i=1}^{8} \partial_i \partial_i T(v,x) = 0 \tag{2.10}$$

The only regular solutions to this equation have the form

$$T(v,x) = \vec{f}(v) \cdot \vec{x} \tag{2.11}$$

Such a T in (2.9) generates a new solution from the original solution (2.6). But in its present form, the solution is not expressed in an optimal set of coordinates. This is because the metric does not approach flat spacetime at large distances, as the term $T(v,x)dv^2$ does not fall off at large $|\vec{x}|$.

To remedy this difficulty, we go to new coordinates as follows. We define a function

$$\vec{f}(v) = -2\ddot{\vec{F}}(v) \tag{2.12}$$

where a dot denotes $\frac{d}{dv}$. Then our new coordinates $\{u', v', \vec{x}'\}$ are defined through

$$\begin{aligned} v &= v' \\ u &= u' - 2\dot{\vec{F}} \cdot \vec{x}' + 2\dot{\vec{F}} \cdot \vec{F} - \int^{v'} \dot{F}^2 dv \\ \vec{x} &= \vec{x}' - \vec{F} \end{aligned} \tag{2.13}$$

The solution (2.9) then becomes (we drop the primes on the new variables)

$$\begin{aligned} ds_S^2 &= -H^{-1}dudv - (H^{-1} - 1)[\dot{F}^2 dv^2 + 2\dot{\vec{F}} \cdot d\vec{x}dv] + d\vec{x} \cdot d\vec{x} \\ B_{uv} &= -\frac{1}{2}H^{-1}, \quad B_{vi} = (H^{-1} - 1)\dot{F}_i \\ e^{2\phi} &= H^{-1} \end{aligned} \tag{2.14}$$

where

$$H = 1 + \frac{Q_1}{|\vec{x} - \vec{F}(v)|^6} \tag{2.15}$$

We see that we have added a wave to the profile of the string source; the source location has been shifted as

$$\vec{x} \to \vec{x} + \vec{F}(v) \tag{2.16}$$

so $\vec{F}(v)$ describes the oscillation profile of the string. We have therefore obtained the gravitational solution of the string carrying a wave moving in the positive y direction.

It turns out that we can make an even larger class of solutions. In the case of a string with no waves, we had seen in eq.(4.1) that we could linearly superpose the harmonic functions produced by several different strings, and thus obtain a new solution to the nonlinear supergravity gravity equations. The solution for a string carrying a wave (2.14) is more complicated, but remarkably, it admits a similar superposition of solutions. To achieve this superposition, we first write (2.14) in the 'chiral null model' form

$$\begin{aligned} ds^2 &= H^{-1}(\vec{x})\left(-dudv + K(\vec{x}, v)dv^2 + 2A_i(\vec{x}, v)dx^i dv\right) + d\vec{x} \cdot d\vec{x} \\ B_{uv} &= g_{uv} = -\frac{1}{2}H^{-1}(\vec{x}), \quad B_{vi} = g_{vi} = H^{-1}(\vec{x})A_i(\vec{x}, v) \\ e^{2\phi} &= H^{-1}(\vec{x}) \end{aligned} \tag{2.17}$$

We find that for the solution (2.14),

$$\begin{aligned} H(\vec{x}, v) &\equiv 1 + \tilde{H} = 1 + \frac{Q_1}{|\vec{x} - \vec{F}(v)|^6} \\ K(\vec{x}, v) &= \frac{Q_1|\dot{\vec{F}}|^2}{|\vec{x} - \vec{F}(v)|^6} \\ A_i(\vec{x}, v) &= \frac{Q_1\dot{F}_i}{|\vec{x} - \vec{F}(v)|^6} \end{aligned} \tag{2.18}$$

Regarding A_i as a gauge field we can construct the field strength $F_{ij} = A_{j,i}?A_{i,j}$. The functions in the chiral null model are required satisfy the equations

$$\partial^2 H = 0, \quad \partial^2 K = 0, \quad \partial_i F^{ij} = 0 \tag{2.19}$$

where

$$\partial^2 = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i} \tag{2.20}$$

and all indices are raised and lowered by the flat metric δ_{ij} on the transverse space $\vec{x}$.

While the gravitational equations are very nonlinear, it turns out that we can superpose the functions $\tilde{H}, K, A_i$ from different solutions to obtain a new solution. Thus if we have several strings, carrying different vibration profiles $\vec{F}_m(v)$, then we can get a new solution of the form (2.17) with

$$\begin{aligned} H(\vec{x}, v) &\equiv 1 + \tilde{H} = 1 + \sum_m \frac{Q_{1,m}}{|\vec{x} - \vec{F}_m(v)|^6} \\ K(\vec{x}, v) &= \sum_m \frac{Q_{1,m}|\dot{\vec{F}}_m|^2}{|\vec{x} - \vec{F}_m(v)|^6} \\ A_i(\vec{x}, v) &= \sum_m \frac{Q_{1,m}\dot{F}_{m,i}}{|\vec{x} - \vec{F}_m(v)|^6} \end{aligned} \tag{2.21}$$

2.0.3 Compactification

The above solution was for a string in 9+1 noncompact spacetime. For our final black hole, we want a compactification $T^4 \times S^1$.

Let us first consider the effect of a T^4 compactification. As we had seen in section 4.1, compactifying a direction like $x^6 = z_1$ is equivalent to taking an array of strings, with separation equal to the length of the z_1 circle. From (2.21) we see that we can superpose the harmonic functions for each string in the array, in exactly the same way that we did in section 4.1 for strings carrying no vibrations. The 4 compact directions of the torus generate a sum over a 4-dimensional lattice. With the same approximations (4.1) that we had assumed in section 4.1, we find that the harmonic functions become

$$\begin{aligned} H(\vec{x}, v) &\equiv 1 + \tilde{H} = 1 + \frac{Q}{|\vec{x} - \vec{F}(v)|^2} \\ K(\vec{x}, v) &= \frac{Q|\dot{\vec{F}}|^2}{|\vec{x} - \vec{F}(v)|^2} \\ A_i(\vec{x}, v) &= \frac{Q\dot{F}_i}{|\vec{x} - \vec{F}(v)|^2} \end{aligned} \tag{2.22}$$

where

$$Q = \tag{2.23}$$

Let us now consider the compactification of $x^5 = y$:

$$y \sim y + L \tag{2.24}$$

The string winds n_w times around the y circle, so the total coordinate length along the string is

$$L_T = n_w L \tag{2.25}$$

The profile function thus has the periodicity

$$\vec{F}(v) = \vec{F}(t-y) = \vec{F}(t-(y+L_T)) = \vec{F}(v-L_T) \tag{2.26}$$

Thus $\vec{F}$ is not in general periodic around the y circle; instead, it is periodic after n_w turns around this circle. We will use the coordinate $\tilde{y}$ to cover the full coordinate range $0 \le \tilde{y} < L_T$, and call this range the 'covering space' of the y circle. Thus $\vec{F}(v)$ is a single valued function on this covering space. In the actual space which has the periodicity (2.24), we get n_w different strands, where a strand at the coordinate value $y = L$ joins with the next strand at the coordinate value $y = 0$. We depict this structure in fig.4.1.

The harmonic function H appearing in (2.9) now has a contribution from n_w different strands. We label the strands by an index m, with $0 \le m \le n_w - 1$. The mth strand has a profile function $\vec{F}_m$, with

$$\vec{F}_m(t,y) = \vec{F}(t-\tilde{y}) = \vec{F}(t-(mL+y)), \quad mL \le \tilde{y} < (m+1)L \tag{2.27}$$

We must again perform a sum over strands in the functions (2.21). In the limit of large charges n_w, n_p, there are a large number of closely spaced strands, and we can replace the sum by an integral; this is the same kind of approximation which we used in the lattice sum which gave (2.22) from (2.21). For the quantity $\tilde{H}$ we get

$$\begin{aligned} \tilde{H}(t,y) &= \sum_{m=0}^{n_w-1} \tilde{H}_m(t-(mL+y)) \\ &\approx \int_0^{n_w} dm \tilde{H}_m(t-(mL+y)) \\ &= \frac{1}{L}\int_{\tilde{y}=0}^{L_T} d\tilde{y} \tilde{H}(t-\tilde{y}) \\ &= \frac{1}{L}\int_{v=0}^{L_T} dv \tilde{H}(v) \end{aligned} \tag{2.28}$$

A similar integral is obtained for K and the A_i. Thus the final solution for the NS1-P system is

$$\begin{aligned} ds^2 &= H^{-1}(\vec{x})\left(-dudv + K(\vec{x},v)dv^2 + 2A_i(\vec{x},v)dx^i dv\right) + d\vec{x}\cdot d\vec{x} \\ B_{uv} &= g_{uv} = -\frac{1}{2}H^{-1}(\vec{x}), \quad B_{vi} = g_{vi} = H^{-1}(\vec{x})A_i(\vec{x},v) \\ e^{2\phi} &= H^{-1}(\vec{x}) \end{aligned} \tag{2.29}$$

with

$$
\begin{aligned}
H(\vec{x}) &\equiv 1+\tilde{H} = 1+\frac{Q}{L}\int_{v=0}^{L_T} dv \frac{1}{|\vec{x}-\vec{F}(v)|^2} \\
K(\vec{x}) &= \frac{Q}{L}\int_{v=0}^{L_T} dv \frac{|\dot{\vec{F}}|^2}{|\vec{x}-\vec{F}(v)|^2} \\
A_i(\vec{x}) &= \frac{Q}{L}\int_{v=0}^{L_T} dv \frac{\dot{F}_i}{|\vec{x}-\vec{F}(v)|^2}
\end{aligned}
\tag{2.30}
$$

2.1 The nature of the NS1-P solution

Let us now analyze the geometric stricture of the solution (2.30), which describes a microstate of the 2-charge extremal black hole.

2.1.1 Absence of a horizon

In the Schwarzschild metric, the horizon was at the location $r = 2M$ where g_{tt} vanished. Equivalently, we can say that g^{tt} diverged at the location of the horizon. But consider a more general metric which is independent of t, but where we can have nonvanishing cross terms g_{ti}. In this situation the vanishing of g_{tt} is not the same condition as the divergence of g^{tt}. It turns out that the horizon should be defined by

$$g^{tt} = 0 \tag{2.31}$$

The solution (2.30) is written in terms of $u = t + y, v = t - y$, but we can write it in terms of t, y, and find g_{tt}. We then find

$$g^{tt} = -\frac{H\left(H(1+K) - A_iA_i\right)}{H + 4KA_iA_i} \tag{2.32}$$

We note that

$$\tilde{H} > 0, \quad K > 0 \tag{2.33}$$

Thus the denominator is positive

$$H + 4KA_iA_i = 1 + \tilde{H} + 4KA_iA_i > 0 \tag{2.34}$$

We will now show that the expression $H(1+K) - A_iA_i$ in the numerator is also positive. We have

$$H(1+K) - A_iA_i = (1+\tilde{H})(1+K) - A_1A_1 > \tilde{H}K - A_iA_i \tag{2.35}$$

where we have used (2.58). We have

$$\begin{aligned}\tilde{H}K - A_iA_i &= Q^2\int_{v=0}^{L_T}\int_{v'=0}^{L_T}\frac{1}{|\vec{x}-\vec{F}(v)|^2}\frac{|\dot{\vec{F}}(v')|^2}{|\vec{x}-\vec{F}(v')|^2}\\ &\quad -Q^2\int_{v=0}^{L_T}\int_{v'=0}^{L_T}\frac{\dot{F}_i(v)}{|\vec{x}-\vec{F}(v)|^2}\frac{\dot{F}_i(v')}{|\vec{x}-\vec{F}(v')|^2}\\ &= Q^2\int_{v=0}^{L_T}\int_{v'=0}^{L_T}\frac{\frac{1}{2}\left(|\dot{\vec{F}}(v)^2+|\dot{\vec{F}}(v')|^2\right)-\dot{F}_i(v')\dot{F}_i(v')}{|\vec{x}-\vec{F}(v)|^2|\vec{x}-\vec{F}(v')|^2}\end{aligned} \tag{2.36}$$

where in the second step we have used the symmetrized the first term over the dummy variables v, v'. We now note that

$$\frac{1}{2}\left(|\dot{\vec{F}}(v)|^2+|\dot{\vec{F}}(v')|^2\right)-\dot{F}_i(v')\dot{F}_i(v') \ \geq \ 0 \tag{2.37}$$

by the triangle inequality. Thus we see that $g^{tt}\leq 0$, everywhere, and there is no horizon.

2.1.2 Scales in the geometry

In the limit

$$|\vec{x}|\to\infty \tag{2.38}$$

we find that

$$H\to 1, \quad K\to 0, \quad A_i\to 0 \tag{2.39}$$

and the metric (2.29) reduces to flat spacetime. At smaller $|\vec{x}|$ we have a non-trivial structure at length scales that depend on the choice of profile function $\vec{F}(v)$. We will now analyze these length scales. In particular, we would like to compare the solutions (2.29) with the 'naive' metric (**??**) that we had written for the NS1-P solution assuming an ansatz of spherical symmetry:

$$ds^2_{string} = H^{-1}[-dt^2+dx_1^2+K(dt+dx_1)^2]+[dr^2+r^2d\Omega_3^2]+\sum_{i=1}^{4}dx_idx_i \tag{2.40}$$

where

$$H = 1+\frac{Q}{|\vec{x}|}, \qquad K=\frac{Q_p}{|\vec{x}|^2} \tag{2.41}$$

We proceed in the following steps:

(i) We note that the quantity Q appearing in the geometry (2.29) is the same as the quantity Q appearing in the 'naive' metric (**??**). It has units of $(length)^2$, and is given by

$$Q=\frac{(2\pi)^4g^2\alpha'^4}{V} \tag{2.42}$$

(ii) We have the length scale set by the vibration profile $\vec{F}$. Let us assume that

$$|\vec{F}(v)| \sim F_0 \tag{2.43}$$

Then F_0 sets a length scale that characterizes the region over which the string spreads in its vibrations. The value of F_0 depends on the vibration profile selected, and in particular on the total momentum P that thsi vibration carries.

(iii) The vibration is described by a state of the string

$$|\psi\rangle = a^{\dagger,\mu_1}_{k_1} \ldots a^{\dagger,\mu_n}_{k_n}|0\rangle \tag{2.44}$$

Suppose the harmonics k_i have a typical value

$$k_i \sim k_0 \tag{2.45}$$

The total length of the effective string is $L_T = n_w L$. Thus the wavelength of the typical vibration is

$$\lambda \sim \frac{L_T}{k_0} \tag{2.46}$$

Then we have

$$|\dot{F}_i| = |\frac{d\dot{F}_i(v)}{dv}| \sim \frac{F_0}{\lambda} \sim \frac{F_0 k_0}{L_T} \tag{2.47}$$

(iv) At $|\vec{x}| \to \infty$ we can read off the NS1 and P charges of our solutions (2.29)by comparing with the 'naive' metric (**??**). The Q appearing in the geometry (2.29) is the same as the quantity Q appearing in the 'naive' metric (**??**). It has units of $(length)^2$, and is given by

$$Q = \frac{(2\pi)^4 g^2 \alpha'^4 n_1}{V} \tag{2.48}$$

Comparing the coefficient of dv^2 gives

$$Q|\dot{\vec{F}}|^2 = Q_p = \frac{(2\pi)^6 g^2 n_p \alpha'^4}{V L^2} \tag{2.49}$$

where we have recalled the value of Q_p from (4.1). Using (2.47), we find

$$F_0 \sim \frac{\sqrt{n_w n_p}}{k_0} \tag{2.50}$$

(v) We now recall that for a generic state of the NS1-P system, we have

$$k_0 \sim \sqrt{n_w n_p} \tag{2.51}$$

This gives

$$F_0 \sim \sqrt{\alpha'} = l_s \tag{2.52}$$

i.e., F_0 is order the string length.

(vi) To get a black hole, we hold fixed the moduli L, V, g, while we take the charges n_w, n_p to large values. We have the scaling with the charges

$$Q \sim \sqrt{n_w}, \quad Q_p \sim \sqrt{n_p} \quad F_0 \sim 1 \tag{2.53}$$

Thus if we set

$$n_w \sim n_p \gg 1 \tag{2.54}$$

Then we have relation between length scales

$$\sqrt{Q} \;\sim\; \sqrt{Q_p} \gg F_0 \tag{2.55}$$

(vii) We can now get a picture of the generic microstates in the set (2.29):

(a) For

$$|\vec{x}| \gg \sqrt{Q} \tag{2.56}$$

we approach flat spacetime

(b) For

$$\sqrt{Q} \;\lesssim |\vec{x}| \;\ll F_0 \tag{2.57}$$

we can make the replacement $|\vec{x} - \vec{F}| \to |\vec{x}|$ in the functions (2.30). Then the actual solution (2.29) becomes the 'naive' solution (2.40), whose details do not depend on the profile function $\vec{F}(v)$.

(c) In the region

$$|\vec{x}| \sim F_0 \tag{2.58}$$

we have a complicated geometry whose details depend on the choice of the profile function $\vec{F}(v)$. We call this region the 'cap' region, since it 'caps off' the geometry instead of letting it progress to a horizon or singularity at $\vec{x} = 0$. We call the geometry in the region (2.58) the 'fuzzball', since it is a complicated object, with large quantum fluctuations around the metric (2.29).

2.1.3 The size of the fuzzball

We have seen for a generic microstate, the fuzzball region has a coordinate extent

$$|\vec{x}| \sim F_0 \sim \sqrt{\alpha'} \tag{2.59}$$

We would now like to get a more geometric picture of the size of this fuzzball. In particular, we will compute the surface area A of the region

$$|\vec{x}| = \sqrt{\alpha'} \tag{2.60}$$

Interestingly, we will find that A/G is of order the entropy of the 2-charge system; thus a Bekenstein tyoe relation emerges from the size of the fuzzball. We proceed as follows:

(i) We have to find the area of the surface bounding the fuzzball region. The fuzzball region joins smoothly to the 'naive geometry' (**??**), and we can thus imagine our bounding surface to be placed at the location $|\vec{x}| = \sqrt{\alpha'}$ in this naive geometry. The area of the surface has three components: one from the noncompact directions, one from the compact T^4, and one from the compact S^1. We have written the metric (**??**) in the string metric g^S_{ab}, while for the Bekenstein relation, we must use the Einstein metric. We will perform this converstion, and then divide the area in the Einstein metric by G to obtain our quantity of interest.

(ii) We introduce polar coordinates in the noncompact spatial directions $x^1, \ldots x^4$

$$d\vec{x} \cdot d\vec{x} = dr^2 + r^2 d\Omega_3^2 \tag{2.61}$$

At the location $r = \sqrt{\alpha'}$ we get from the angular S^3 an area

$$A_{S^3} \sim \alpha'^{\frac{3}{2}} \tag{2.62}$$

From the T^4 we get an area

$$A_{T^4} \sim V \tag{2.63}$$

The S^1 has a coordinate length L, and so it has a proper length $L_y = (g_{yy})^{\frac{1}{2}} L$. In the region $|\vec{x}| \ll \sqrt{Q}$, we find that the dominant contribution to g_{yy} comes from the coefficient of dv^2 in (**??**). In this limit, we find

$$L_y \sim \sqrt{HK} L \sim \sqrt{\frac{Q_p}{Q_1}} L \tag{2.64}$$

Thus the area of the 8-D surface bounding the region occupied by the string is given, in the string metric, by

$$A^S \sim \sqrt{\frac{Q_p}{Q_1}} L V \alpha'^{\frac{3}{2}} \tag{2.65}$$

(iii) The Einstein metric is related to the string metric by

$$g^E_{ab} = e^{-\frac{\phi}{4}} g^S_{ab} \tag{2.66}$$

Thus the area in Einstein metric will be related to the area in the string metric by

$$A^E = A^S e^{-2\phi} \tag{2.67}$$

Note that the dilaton ϕ becomes very negative at the surface of interest

$$e^{-2\phi} = H \approx \frac{Q}{r^2} \sim \frac{Q}{\alpha'} \tag{2.68}$$

We thus find

$$A^E \sim \sqrt{QQ_p} LV \alpha^{\frac{\prime 1}{2}} \sim g^2 \alpha'^4 \sqrt{n_w n_p} \tag{2.69}$$

where we have used the values of Q, Q_p from (2.42),(2.49). Now we observe that

$$\frac{A^E}{G_{10}} \sim \sqrt{n_1 n_p} \sim S_{micro} \tag{2.70}$$

where we have used that $G_{10} = 8\pi^6 g^2 \alpha'^4$. This is very interesting, since it shows that the surface area of our 'fuzzball' region satisfies a Bekenstein type relation $A/4G = S_{bek}$.

2.2 Dipole charges

We have seen that the NS1-P source in the naive metric (**??**) spreads into a fuzzball to give the microstates (2.29). Let us analyze the physics behind this spreading.

2.2.1 Longitudinal vs transverse vibrations

We have noted that the fuzzball acquires its nonzero size because the NS1 has only transverse vibration; there is no longitudinal mode. If we did have a longitudinal mode, then we could orient the string along the direction $x^5 = y$, and let the momentum be carried by the longitudinal mode. In that case the NS1-P system would sit at one point $\vec{x} = 0$ in the noncompact space, and would generate the naive metric (**??**). So let us understand in more detail why the NS1 has no longitudinal vibrations.

A guitar string has both longitudinal and transverse vibration modes. In the longitudinal mode, the atoms on the string come closer together near one location, and separate away at another location, generating the 'compressions and rarefactions' that propagate along the string.

But the NS1 is a fundamental object of string theory, and so is not made out of a chain of 'atoms'. The action of the string illustrates this fact: it is given in terms of the area of the worldsheet: $S = -TA$. Thus we cannot change the action by moving the points on the string along the string itself: that would generate the same surface and therefore the same area. If we cannot change the action by longitudinal motions, then we cannot have a longitudinal mode of vibration.

A transverse vibration must either break the symmetry of the solution in the angular directions Ω of noncompact space, or the symmetry in the compact T^4 directions. Thus we see that the fundamental nature of the string has a direct role to play in invalidating the naive solution (**??**) which starts by assuming both these symmetries.

2.2.2 Dipole charges

In fig.4.1(a) we depict the string carrying a transverse vibration, opened up to the covering space. Consider the small segment of the string marked A. Since the segment is short, we think of it as a straight line. Let the transverse vibration at this point of the string be in the noncompact direction x^1. Then this segment of the string has a component in the direction y and a component in the direction x^1.

The component in the direction y is part of the NS1 winding charge of the string. Indeed, of we look at the string in the actual space rather than the covering space, then this component leads to the string winding multiple times around the y circle. The component in the direction x^1 can be thought of an a NS1 charge in the direction x^1. We do not however have any *net* NS1 charge in the direction x^1. Consider the segment marked B in the figure; this segment has an NS1 charge in the direction y, but also an NS1 charge in the *negative* x^1 direction. Thus the NS1 charges in the x^1 direction cancel out between different parts of the solution. Such charges are called 'dipole charges' of the solution, since their net value is zero. Dipole charges have an important role to play in the structure of the fuzzball. We can see this from the fact that the wandering of the string in the positive and negative x^1 directions is what leads to the nonzero transverse spread of the NS1-P state in the direction x^1.

In fig.4.1(b) we depict the vibrating string in the actual space rather than the covering space. At the location of the segment A, we see that the singular curve – the location of the string in the space transverse to y – is moving in the x^1 direction. Thus the NS1 dipole charge is in the direction of the singular curve at each point of this curve.

The NS1-P state also has a second kind of dipole charge. Consider again the segment A of the string in fig.4.1(a). The vibration profile is a function $\vec{F}(v)$ of $v = t - y$. Thus the segment A is moving at the speed of light in the y direction. This velocity can be decomposed into two parts: one along the segment, and one perpendicular to the segment. The motion along the segment does nothing; as we noted in section 4.1 above, the string is not a 'line of atoms', and so there is no dynamical motion of a string along its own length. But the motion perpendicular to the string is meaningful, so the momentum carried by the segment A points normal to this segment as shown in the figure.

We can now decompose this momentum into two components. One component is along the direction y, and contributes to the P charge carried by the NS1-P solution. But the other component is in the negative x^1 direction. A segment like B, on the other hand, has a component of P in the y direction, and a *positive* component of P in the x^1 direction. Thus the momentum charge P in the x^1 direction is again a dipole charge; there is no net momentum in this direction.

Since the dipole P charge of the segment A is in the negative x^1 direction, we see that the dipole P charge is also in a direction along the singular curve, though in the direction opposite to the directions of the dipole NS1 charge.

To summarize, the NS1-P state has the following set of 'true' and 'dipole' charges:

(i) A true charge n_w for NS1 winding in the direction y, and a true charge n_p for momentum P in the direction y.

(ii) A dipole NS1 charge along the singular curve, and a dipole P charge along this curve in the opposite direction.

We will see that when we make microstates with more kinds of true charges, then we also get more kinds of dipole charges.

Lecture notes 3

The D1D5 system

Let us again examine the 2-charge extremal bound state, but change the NS1-P charges to other charges by performing S and T dualities on the gravitational solution (2.29). Since these dualities are exact symmetries of the theory, it may seem that no new information can be obtained by examing the states in a new duality frame. But as we will see, some computations may be easier to perform or visualize in one duality frame as compared to another. The actual construction of 2-charge microstates was done in the NS1-P frame becuase we can visualize all states as arising from different vibration profiles $\vec{F}(v)$ on the string. We will now perform dualities to map the NS1 and P charges to a 5-brane and 1-brane charges respectively. The new frame will allow a useful perspective on the dynamics of the 2-charge system, and also allow us to progress towards the 3-charge system later on.

3.1 The D1D5 system

In section 4.1 we had performed a set of dualities to get the map

$$NS1-P \quad \to \quad D5-D1 \tag{3.1}$$

The D5 branes wrapped the $T^4 \times S^1$, while the D1 branes wrapped the S^1. We can perform these duality steps on the gravitational solution (2.29). We find the solution

$$ds^2_{string} = \frac{1}{1+K}[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2] + H dx_i dx_i + dz_a dz_a \tag{3.2}$$

The functions H, K, A_i are the essentially the same as in (2.30). But the integral over v in these functions ran over the range $(0, L_T)$, where $L_T = n_w L$ was the length of the NS1 string, and the waveform $\vec{F}(v)$ was periodic around this string. In the D5-D1 duality frame, we do not have the picture of a waveform over an interval, so the parameter L_T has no direct meaning. Thus we scale the parameter v to a parameter $\hat{v}$ with range $(0,1)$. Then we have

$$\begin{aligned} H(\vec{x}) &\equiv 1+\tilde{H} = 1 + Q_5\int_{\hat{v}=0}^{1} d\hat{v}\frac{1}{|\vec{x}-\vec{F}(\hat{v})|^2} \\ K(\vec{x}) &= \frac{Q_5}{\nu^2}\int_{\hat{v}=0}^{L_T} d\hat{v}\frac{|\dot{\vec{F}}|^2}{|\vec{x}-\vec{F}(\hat{v})|^2} \\ A_i(\vec{x}) &= \frac{Q_5}{\nu}\int_{\hat{v}=0}^{L_T} d\hat{v}\frac{\dot{F}_i(v)}{|\vec{x}-\vec{F}(\hat{v})|^2} \end{aligned} \tag{3.3}$$

The functions B_i are defined as the 'electric-magnetic dual' potential of the A_i

$$\partial_i B_j - \partial_j B_i = \epsilon_{ij}{}^{kl}(\partial_k A_l - \partial_l A_k) \tag{3.4}$$

where all indices are raised and lowered by the flat metric δ_{ij} in the 4-d noncompact space $\vec{x}$. At each step of the duality chain, the coupling g and the lengths of compact directions change. Following these changes carefully, we find

$$\nu = \frac{(2\pi)^2 Q_5}{L} \tag{3.5}$$

where L is the length of the S^1 along the directoon $x^5 = y$, in the D5-D1 duality frame.

3.1.1 Dipole charges

The dualities change the 'true charges' in the manner (3.1). Let us now ask how they change the dipole charges. Pick a point on the singular curve $\vec{x} = \vec{F}(v)$, and look at the geometry (2.29) in a small neighborhood of this point. Let the singular curve be in the direction x^1 at this point. Thus in the NS1-P frame we have the dipole charges $NS1_{x1}$ and P_{x1}. Let us now follow the dualities of section **??**:

(a) The S-duality converts the NS1 to a D1, and leaves P unchanged. Thus we get $D1_1$ and P_1.

(b) The T-dualities along x^6, x^7, x^8, x^9 convert the D1 to a D5, while the P is unaffected. Thus we get $D5_{16789}$ and P_1.

(c) The S-duality converts the D5 to an NS5, leaving the P unchanged. Thus we get $NS5_{16789}$ and P_1.

(d) We perform a T-duality in the direction x^5. This direction is perpendicular to the NS5, and so converts the NS5 to a KK-monopole, where the nontrivially fibered circle of the monopole is the direction $x^5 = y$. The monopole structure extends uniformly in the directions 16789. The P charge is left unchanged. The T-duality beings us to IIA theory, and the dipole charges are $KK_{16789;5}$ and P_1.

The appearance of the KK monopole is very interesting, since the monopole is a geometric object: its charge is carried by the nontrivial topology of the fibre y, rather than by a string or brane source. At this stage of the dualities the true charges are NS5-NS1, and we already have the physical picture of true and dipole charges that we need. But we can perform the two additional steps to bring the system to the D5-D1 duality frame, as this is the frame that was historically the one that was first used to study the 5-brane and 1-brane system.

(e) We perform a T-duality in the direction x^6. This leaves the dipole charges unchanged at $KK_{16789;5}$ and P_1, but brings the theory to IIB.

(f) We perform an S-duality. The dipole charges remain $KK_{16789;5}$ and P_1. The true charges are now D5 and D1.

3.2 The KK dipole charge

It is very interesting to see how the metric (3.2) turns out to be a metric with no singularities. We will now check this fact explicitly. As we will see, this smoothness will require all the numbers of string theory to be used correctly.

We proceed in the following steps:

(i) Focus on the neighborhood of a point on the curve $\vec{x} = \vec{F}(\hat{v})$, we set $\hat{v} = 0$ at this point. Let the curve be along the direction x^1 at this point, and let

$$\left|\dot{\vec{F}}(\hat{v} = 0)\right| = \dot{F}_0 \tag{3.6}$$

Then we have in our neighborhood

$$x^1 \approx \dot{F}_0 \hat{v} \tag{3.7}$$

In the 3-d space perpendicular to this curve (i.e., the space along x^2, x^3, x^4) we introdce polar coordinates centered at the curve

$$(dx^2)^2 + (dx^3)^2 + (dx^4)^2 = d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.8}$$

(ii) We can now find the harminic functions in our neighborhood:

$$\begin{aligned}
H &\approx Q_5 \int_{-\infty}^{\infty} \frac{d\hat{v}}{\rho^2 + (x^1)^2} = \frac{Q_5}{\dot{F}_0} \int_{-\infty}^{\infty} \frac{dx^1}{\rho^2 + (x^1)^2} = \frac{Q_5 \pi}{\dot{F}_0} \frac{1}{\rho} \\
K &\approx \frac{Q_5}{\nu^2} \int_{-\infty}^{\infty} \frac{d\hat{v}(\dot{F}_0)^2}{\rho^2 + (x^1)^2} = \frac{Q_5 \pi \dot{F}_0}{\nu^2} \frac{1}{\rho} \\
A_1 &\approx \frac{Q_5}{\nu} \int_{-\infty}^{\infty} \frac{d\hat{v} \dot{F}_0^2}{\rho^2 + (x^1)^2} = \frac{Q_5 \pi}{\nu} \frac{1}{\rho}
\end{aligned} \tag{3.9}$$

Let us write

$$\tilde{Q} = \frac{Q_5 \pi}{\nu} \tag{3.10}$$

From the field A_1 we find the 'field strength'

$$F_{x^1\rho} = \tilde{Q}\,\frac{1}{\rho^2} \tag{3.11}$$

This is the radial electric field of a point charge. The dual potential will then describe a magnetic charge, and we find from its definition (4.1)

$$B_\phi = \tilde{Q}(1-\cos\theta) \tag{3.12}$$

Consider the metric (3.2) in our neighborhood, restricting for the moment to the 4 space directions $y.\rho, \theta, \phi$

$$\begin{aligned} ds^2 &\to \sqrt{\frac{H}{1+K}}(dy + B_i dx^i)^2 + \sqrt{\frac{1+K}{H}}(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)) \\ &\approx \frac{\rho}{\tilde{Q}}(dy - \tilde{Q}(1-\cos\theta)d\phi)^2 + \frac{\tilde{Q}}{\rho}(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)) \end{aligned} \tag{3.13}$$

This is the metric near the core of a Kaluza-Klein monopole. thoiugh there is an apparent singularity at $\rho = 0$, we have seen that the meric is smooth if the length of the y circle is $4\pi\tilde{Q}$; i.e.,

$$L = 4\pi\frac{Q_5\pi}{\nu} \tag{3.14}$$

Using (4.1), we see that this relation is exactly satisfied. The change of coordinates

$$\begin{aligned} &\tilde{r}^2 = \rho, \quad \tilde{\theta} = \frac{\theta}{2}, \quad \tilde{y} = \frac{y}{2\tilde{Q}}, \quad \tilde{\phi} = \phi - \frac{y}{2\tilde{Q}} \\ &0 \le \tilde{\theta} < \frac{\pi}{2}, \quad 0 \le \tilde{y} < \frac{\pi R'}{\tilde{Q}} = 2\pi, \quad 0 \le \tilde{\phi} < 2\pi \end{aligned} \tag{3.15}$$

makes manifest the locally R^4 form of the metric

$$ds^2 = 4\tilde{Q}[d\tilde{r}^2 + \tilde{r}^2(d\tilde{\theta}^2 + \cos^2\tilde{\theta}d\tilde{y}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2)] \tag{3.16}$$

For the coordinate change (3.15) to be consistent with identifications on the periodic coordinates we need the condition (**??**), which we have seen is satisfied.

The (t, x^1) part of the geometry gives

$$\begin{aligned} ds^2 &\to -\sqrt{\frac{H}{1+K}}(dt - A_1 dx^1)^2 + \sqrt{\frac{1+K}{H}}(dx^1)^2 \\ &\approx -\frac{\rho}{\tilde{Q}}dt^2 - 2dtddx^1 \approx -2dtdx^1 \end{aligned} \tag{3.17}$$

which is regular. The T^4 part gives

$$ds^2 \to \dot{F}_0|dz_a dz_a \tag{3.18}$$

and is thus regular as well.

3.3 A special example

The above general solution looks rather complicated. To get a feeling for the nature of these D1-D5 solutions let us start by examining in detail a simple case. Start with the NS1-P solution which has the following vibration profile for the NS1 string

$$F_1 = \hat{a}\cos(\frac{2\pi k v}{L_T}), \quad F_2 = \hat{a}\sin(\frac{2\pi k v}{L_T}), \quad F_3 = F_4 = 0 \tag{3.19}$$

where $\hat{a}$ is a constant, and $k = 1, 2, \ldots$. This makes the NS1 swing in a uniform helix in the $x_1 - x_2$ plane, with k complete turns of the helix. We then find

$$\begin{aligned} H &= 1 + \frac{Q}{L_T}\int_0^{L_T} \frac{dv}{(x_1 - \hat{a}\cos(\frac{2\pi k v}{L_T}))^2 + (x_2 - \hat{a}\sin(\frac{2\pi k v}{L_T}))^2 + x_3^2 + x_4^2} \\ &= 1 + \frac{Q}{2\pi}\int_0^{2\pi} \frac{d\xi}{(x_1 - \hat{a}\cos(k\xi))^2 + (x_2 - \hat{a}\sin(k\xi))^2 + x_3^2 + x_4^2} \\ &= 1 + \frac{kQ}{2\pi}\int_0^{\frac{2\pi}{k}} \frac{d\xi}{(x_1 - \hat{a}\cos(k\xi))^2 + (x_2 - \hat{a}\sin(k\xi))^2 + x_3^2 + x_4^2} \\ &= 1 + \frac{Q}{2\pi}\int_0^{2\pi} \frac{d\xi'}{(x_1 - \hat{a}\cos(\xi'))^2 + (x_2 - \hat{a}\sin(\xi'))^2 + x_3^2 + x_4^2} \end{aligned} \tag{3.20}$$

where in the second step we have defined $\xi = 2\pi v/L_T$, and in the last step we have written $\xi' = k\xi$. To compute the integral we introduce polar coordinates in the $\vec{x}$ space

$$\begin{aligned} x_1 &= \tilde{r}\sin\tilde{\theta}\cos\tilde{\phi}, \qquad x_2 = \tilde{r}\sin\tilde{\theta}\sin\tilde{\phi}, \\ x_3 &= \tilde{r}\cos\tilde{\theta}\cos\tilde{\psi}, \qquad x_4 = \tilde{r}\cos\tilde{\theta}\sin\tilde{\psi} \end{aligned} \tag{3.21}$$

Then we find

$$H = 1 + \frac{Q}{\sqrt{(\tilde{r}^2 + \hat{a}^2)^2 - 4\hat{a}^2\tilde{r}^2\sin^2\tilde{\theta}}} \tag{3.22}$$

The above expression simplifies if we change from $\tilde{r}, \tilde{\theta}$ to coordinates r, θ:

$$\tilde{r} = \sqrt{r^2 + \hat{a}^2\sin^2\theta}, \qquad \cos\tilde{\theta} = \frac{r\cos\theta}{\sqrt{r^2 + \hat{a}^2\sin^2\theta}} \tag{3.23}$$

($\tilde{\phi}$ and $\tilde{\psi}$ remain unchanged). Then we get

$$H = 1 + \frac{Q}{r^2 + \hat{a}^2\cos^2\theta} \tag{3.24}$$

Similarly we get

$$K = \frac{\hat{a}^2}{n_w^2 R^2}\,\frac{Q}{(r^2 + \hat{a}^2\cos^2\theta)} \tag{3.25}$$

With a little algebra we also find

$$\begin{aligned} A_{x_1} &= \frac{Q\hat{a}}{2\pi R n_1}\int_0^{2\pi}\frac{d\xi\sin\xi}{(x_1-\hat{a}\cos\xi)^2+(x_2-\hat{a}\sin\xi)^2+x_3^2+x_4^2} \\ &= \frac{Q\hat{a}}{2\pi R n_1}\int_0^{2\pi}\frac{d\xi\sin\xi}{(\tilde{r}^2+\hat{a}^2-2\tilde{r}\hat{a}\sin\tilde{\theta}\cos(\xi-\tilde{\phi}))} \\ &= \frac{Q\hat{a}^2}{Rn_1}\sin\tilde{\phi}\frac{\sin\theta}{(r^2+a^2\cos^2\theta)}\frac{1}{\sqrt{r^2+a^2}} \end{aligned} \tag{3.26}$$

$$A_{x_2} = -\frac{Q\hat{a}^2}{Rn_1}\cos\tilde{\phi}\frac{\sin\theta}{(r^2+a^2\cos^2\theta)}\frac{1}{\sqrt{r^2+a^2}} \tag{3.27}$$

$$A_{x_3}=0, \qquad A_{x_4}=0 \tag{3.28}$$

We can write this in polar coordinates

$$\begin{aligned} A_{\tilde{\phi}} &= A_{x_1}\frac{\partial x_1}{\partial\tilde{\phi}}+A_{x_2}\frac{\partial x_2}{\partial\tilde{\phi}} \\ &= -\frac{Q\hat{a}^2}{Rn_1}\frac{\sin^2\theta}{(r^2+a^2\cos^2\theta)} \end{aligned} \tag{3.29}$$

We can now substitute these functions in (**??**) to get the solution for the NS1-P system for the choice of profile (3.19).

3.3.1 The solution in the D1D5 frame

Let us now get the corresponding D1-D5 solution. Recall that the harmonic functions remain essentially unchanged with the charges Q_5, Q_1 replacing Q, P. The harmonic function is now written as

$$F_1 = a\cos(\hat{v}), \quad F_2 = a\sin(\hat{v}), \quad F_3 = F_4 = 0 \tag{3.30}$$

Thus

$$\left|\dot{\vec{F}}\right|^2 = \left(\frac{dF_1}{d\hat{v}}\right)^2 + \left(\frac{dF_1}{d\hat{v}}\right)^2 = a^2 \tag{3.31}$$

Performing the same manipulations as in the NS1-P case, we find

$$\begin{aligned} H &= 1+\frac{Q_5}{f} \\ K = &= \frac{Q_5 a^2}{\nu^2 f} \end{aligned} \tag{3.32}$$

where

$$f = r^2 + a^2\cos^2\theta \tag{3.33}$$

A large $|\vec{x}|$ we should have $\frac{K \approx Q_p}{|\vec{x}|^2}$. This gives

$$Q_p = \frac{Q_5 a^2}{\nu^2} \tag{3.34}$$

Using (4.1) we find

$$a = \frac{\sqrt{Q_1 Q_5}}{R} \tag{3.35}$$

where R is the radius of the y circle.

To finish writing the D1-D5 solution we also need the functions B_i defined through (**??**). In the coordinates $r, \theta, \tilde{\phi} \equiv \phi, \tilde{\psi} \equiv \psi$ we have

$$A_\phi = -a\sqrt{Q_1 Q_5}\frac{\sin^2\theta}{f} \tag{3.36}$$

We can check that the dual form is

$$B_\psi = -a\sqrt{Q_1 Q_5}\frac{\cos^2\theta}{f} \tag{3.37}$$

To check this, note that the flat 4-D metric in our coordinates is

$$dx_i dx_i = \frac{f}{r^2 + a^2} dr^2 + f d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2 \tag{3.38}$$

We also have

$$\epsilon_{r\theta\phi\psi} = \sqrt{g} = fr\sin\theta\cos\theta \tag{3.39}$$

We then find

$$F_{r\psi} = \partial_r B_\psi = a\sqrt{Q_1' Q_5'}\frac{2r\cos^2\theta}{f^2} = -\epsilon_{r\psi\theta\phi} g^{\theta\theta} g^{\phi\phi}[\partial_\theta A_\phi] = -(*dA)_{r\psi} \tag{3.40}$$

$$F_{\theta\psi} = \partial_\theta B_\psi = a\sqrt{Q_1' Q_5'}\frac{r^2\sin(2\theta)}{f^2} = -\epsilon_{\theta\psi r\phi} g^{rr} g^{\phi\phi}[\partial_r A_\phi] = -(*dA)_{\theta\psi} \tag{3.41}$$

verifying (**??**).

Putting all this in (**??**) we find the D1-D5 (string) metric for the profile (3.19)

$$\begin{aligned} ds^2 &= -\frac{1}{h}(dt^2 - dy^2) + hf\left(d\theta^2 + \frac{dr^2}{r^2 + a^2}\right) - \frac{2a\sqrt{Q_1' Q_5'}}{hf}\left(\cos^2\theta dy d\psi + \sin^2\theta dt d\phi\right) \\ &+ h\left[\left(r^2 + \frac{a^2 Q_1' Q_5' \cos^2\theta}{h^2 f^2}\right)\cos^2\theta d\psi^2 + \left(r^2 + a^2 - \frac{a^2 Q_1' Q_5' \sin^2\theta}{h^2 f^2}\right)\sin^2\theta d\phi^2\right] \\ &+ \sqrt{\frac{Q_1' + f}{Q_5' + f}}\, dz_a dz_a \end{aligned} \tag{3.42}$$

where

$$f = r^2 + a^2\cos^2\theta, \qquad h = \left[\left(1 + \frac{Q_1'}{f}\right)\left(1 + \frac{Q_5'}{f}\right)\right]^{1/2} \tag{3.43}$$

3.3.2 Analyzing the D1D5 metric

At large r this metric goes over to flat space. Let us consider the opposite limit $r \ll (Q_1'Q_5')^{1/4}$ (we write $r' = r/a$):

$$\begin{aligned} ds^2 &= -(r'^2+1)\frac{a^2dt^2}{\sqrt{Q_1'Q_5'}} + r'^2\frac{a^2dy^2}{\sqrt{Q_1'Q_5'}} + \sqrt{Q_1'Q_5'}\frac{dr'^2}{r'^2+1} \\ &+ \sqrt{Q_1Q_5}\left[d\theta^2 + \cos^2\theta\left(d\psi - \frac{ady}{\sqrt{Q_1'Q_5'}}\right)^2 + \sin^2\theta\left(d\phi - \frac{adt}{\sqrt{Q_1'Q_5'}}\right)^2\right] \\ &+ \sqrt{\frac{Q_1'}{Q_5'}}dz_adz_a \end{aligned} \tag{3.44}$$

Let us transform to new angular coordinates

$$\psi' = \psi - \frac{a}{\sqrt{Q_1'Q_5'}}y, \qquad \phi' = \phi - \frac{a}{\sqrt{Q_1'Q_5'}}t \tag{3.45}$$

Since ψ, y are both periodic coordinates, it is not immediately obvious that the first of these changes makes sense. The identifications on these coordinates are

$$(\psi \to \psi + 2\pi, \quad y \to y), \qquad (\psi \to \psi, \quad y \to y + 2\pi R') \tag{3.46}$$

But note that we have the relation (**??**), which implies that the identifications on the new variables are

$$(\psi' \to \psi' + 2\pi, \quad y \to y), \qquad (\psi' \to \psi' - \frac{a2\pi R'}{\sqrt{Q_1'Q_5'}} = \psi' - 2\pi, \quad y \to y + 2\pi R') \tag{3.47}$$

so that we do have a consistent lattice of identifications on ψ', y. The metric (3.44) now becomes

$$\begin{aligned} ds^2 &= \sqrt{Q_1'Q_5'}\left[-(r'^2+1)\frac{dt^2}{R^2} + r'^2\frac{dy^2}{R^2} + \frac{dr'^2}{r'^2+1}\right] \\ &+ \sqrt{Q_1'Q_5'}\left[d\theta^2 + \cos^2\theta d\psi'^2 + \sin^2\theta d\phi'^2\right] + \sqrt{\frac{Q_1'}{Q_5'}}dz_adz_a \end{aligned} \tag{3.48}$$

This is just $AdS_3 \times S^3 \times T^4$. Thus the full geometry is flat at infinity, has a 'throat' type region at smaller r where it approximates the naive geometry (**??**), and then instead of a singularity at $r = 0$ it ends in a smooth 'cap'. This particular geometry, corresponding to the profile (3.19), was derived earlier in [**?**, **?**] by taking limits of general rotating black hole solutions found in [**?**]. We have now obtained it by starting with the particular NS1-P profile (3.19), and thus we note that it is only one member of the complete family parametrized by $\vec{F}$. It can be shown that all the metrics of this family have the same qualitative structure as the particular metric that we studied; in particular they have no horizons, and they end in smooth 'caps' near $r = 0$. We will review the argument for this smoothness below.

3.4 Energy gaps

Here we solve the wave equation for a massless scalar. We look for the solution of the Klein–Gordon equation

$$\Box\Phi = 0 \tag{3.49}$$

in the metric (**??**).

We write

$$\Phi(t, r, \chi, \theta, \psi, \phi) = \exp(-i\omega t)H(r) \tag{3.50}$$

Then we get the following equations for H and Θ (see [**?**] for details):

$$\frac{1}{r}\frac{d}{dr}\left(r(\frac{r^2}{L^2}+\gamma^2)\frac{dH}{dr}\right)+\frac{\omega^2}{\frac{r^2}{L^2}+\gamma^2}H = 0 \tag{3.51}$$

$$\tag{3.52}$$

The solution regular at $r = 0$ is [**?**]:

$$H(x) = (r^2+\gamma^2)^q F(q, q+1; 1; -\frac{r^2}{\gamma^2}), \tag{3.53}$$

$$\text{where} \quad q = \frac{\omega L}{2\gamma}, \tag{3.54}$$

For large u we have

$$F[q, q+1, 1; -u] \approx u^{-q}\frac{q\pi}{\sin(q\pi)} \tag{3.55}$$

Thus normalizability at infinity requres

$$q = \pm 1, \pm 2, \ldots \tag{3.56}$$

Taking the positive sign for teh frequencies we get which implies

$$\omega_k = \frac{2\gamma}{L}k \quad k = 1, 2, \ldots \tag{3.57}$$

The functions $H(r)$ becomes a rational function for these values of ω.

3.5

Since we are looking at BPS states, we do not change the count of states by taking R to be vary large. In this limit we have small transverse vibrations of the NS1. We can take the DBI action for the NS1, choose the static gauge, and obtain an action for the vibrations that is just quadratic in the amplitude of vibrations. The vibrations travel at the speed of light along the direction

y. Different Fourier modes separate and each Fourier mode is described by a harmonic oscillator. The total length of the NS1 is

$$L_T = 2\pi R n_1 \tag{3.58}$$

Each excitation of the Fourier mode k carries energy and momentum

$$e_k = p_k = \frac{2\pi k}{L_T} \tag{3.59}$$

The total momentum on the string can be written as

$$P = \frac{n_p}{R} = \frac{2\pi n_1 n_p}{L_T} \tag{3.60}$$

First focus on only one of the transverse directions of vibration. If there are m_i units of the Fourier harmonic k_i then we need to have

$$\sum_i m_i k_i = n_1 n_p \tag{3.61}$$

Thus the degeneracy is given by counting *partitions* of the integer $n_1 n_p$. The number of such partitions is known to be $\sim Exp(2\pi\sqrt{\frac{n_1 n_p}{6}})$. We must however take into account the fact that the momentum will be partitioned among 8 bosonic vibrations and 8 fermionic ones; the latter turn out to be equivalent to 4 bosons. Thus there are $\frac{n_1 n_p}{12}$ units of momentum for each bosonic mode, and we must finally multiply the degeneracy in each mode. This gives

$$\mathcal{N} = [Exp(2\pi\sqrt{\frac{n_1 n_p}{72}})]^{12} = Exp(2\pi\sqrt{2}\sqrt{n_1 n_p}) \tag{3.62}$$

which again gives the entropy (**??**).

(c) We can look at the vibrations of (b) above as a 1-dimensional *gas* of massless quanta traveling on the NS1 string. The gas lives in a 'box' of length $L_T = 2\pi R n_1$. All quanta in the gas travel in the same direction, so the gas has a total energy and momentum

$$E = P = \frac{n_p}{R} = \frac{2\pi n_1 n_p}{L_T} \tag{3.63}$$

Further there are 8 bosonic degrees of freedom and 8 fermionic degrees of freedom. We can write a partition function Z for the bosonic and fermionic modes

$$Z = \sum_{states} e^{-\beta E_{state}} \tag{3.64}$$

For a bosonic mode of harmonic k each quantum has energy $e_k = 2\pi k/L_T$, so its contribution to the partition function is

$$Z_k^B \to \sum_{m_k=0}^{\infty} e^{-\beta m_k e_k} = \frac{1}{1-e^{-\beta e_k}} \tag{3.65}$$

Similarly a fermionic mode in the harmonic k contributes

$$Z_k^F \to \sum_{m_k=0}^{1} e^{-\beta m_k e_k} = 1 + e^{-\beta e_k} \tag{3.66}$$

We consider the log of Z, so that we add the logs of the individual contributions above. We then approximate the sum over k by an integral ($\sum_k \to \int dk = \frac{L_T}{2\pi}\int de_k$) getting for bosonic modes

$$\log Z^B \to -\frac{L_T}{2\pi}\int_0^\infty de_k \ln[1 - e^{-\beta e_k}] = \frac{L_T}{2\pi\beta}\frac{\pi^2}{6} \tag{3.67}$$

and for fermionic modes

$$\log Z^F \to \frac{L_T}{2\pi}\int_0^\infty de_k \ln[1 + e^{-\beta e_k}] = \frac{L_T}{2\pi\beta}\frac{\pi^2}{12} \tag{3.68}$$

If we have f_B bosonic degrees of freedom and f_F fermionic degrees of freedom we get

$$\log Z = (f_B + \frac{1}{2}f_F)\frac{\pi L_T}{12\beta} \equiv c(\frac{\pi L_T}{12\beta}) \tag{3.69}$$

We can see explicitly in this computation that a fermionic degree of freedom counts as half a bosonic degree of freedom. From the 8 transverse bosonic vibrations and 8 fermionic vibrations we get $c = 12$.

We determine β by

$$E = -\partial_\beta(\ln Z) = \frac{c\pi L_T}{12\beta^2} \tag{3.70}$$

which gives for the temperature

$$T = \beta^{-1} = [\frac{12E}{\pi L_T c}]^{\frac{1}{2}} \tag{3.71}$$

The entropy is

$$S = \ln Z + \beta E = \frac{c\pi L_T}{6\beta} = [\frac{c\pi L_T E}{3}]^{\frac{1}{2}} \tag{3.72}$$

Substituting the values of c, E we again find

$$S = 2\sqrt{2}\pi\sqrt{n_1 n_p} \tag{3.73}$$

From the above computation we can however extract a few other details. The average energy of a quantum will be

$$e \sim T \sim \frac{\sqrt{n_1 n_p}}{L_T} \tag{3.74}$$

so that the generic quantum is in a harmonic

$$k \sim \sqrt{n_1 n_p} \tag{3.75}$$

on the multiwound NS1 string. Given that the total energy is (3.70) we find that the number of such quanta is

$$m \sim \sqrt{n_1 n_p} \tag{3.76}$$

The occupation number of an energy level e_k is

$$< m_k >= \frac{1}{1 - e^{-\beta e_k}} \tag{3.77}$$

so for the generic quantum with $e_k \sim \beta^{-1}$ we have

$$< m_k > \sim 1 \tag{3.78}$$

To summarize, there are a large number of ways to partition the energy into different harmonics. One extreme possibility is to put all the energy into the lowest harmonic $k = 1$; then the occupation number of this harmonic will be

$$m = n_1 n_p \tag{3.79}$$

At the other extreme we can put all the energy into a single quantum in the harmonic $n_1 n_p$; i.e.

$$k = n_1 n_p, \quad m_k = 1 \tag{3.80}$$

But the *generic* state which contributes to the entropy has its typical excitations in harmonics with $k \sim \sqrt{n_1 n_p}$. There are $\sim \sqrt{n_1 n_p}$ such modes; and the occupation number of each such mode is $< m_k > \sim 1$. These details about the generic state will be important to us later.

3.6 'Size' of the bound state

Lecture notes 4

The development of the microstsate program

We have seen that the 2-charge extremal hole gives a simple setting where we can completely understand the quantum structure of black holes. It can be seen that none of the microstates ahve a traditional horizon which would have a vacuum statge in its vicinity. If a similar situation existed for all black holes, we would have no information paradox, since it is the vacuum horizon which leads to the troublesome creation of entangled pairs.

So the question becomes: do more general black holes behave like the 2-charge hole? Given the similarity in the behavior of the entropy expression between 2-charge and other holes, it would seem plausible that the microsstates of all black holes would have the feature that there is no vacuum horizon. At the same time, it should be recognized that such a situation would be a radical change to our conventional piucture of black holes. Thus one must investigate the nature of microstates of more general holes in string theory.

We will now note the steps that have been taken in the study of microstates for more general black holes. It is not necessary to construct eplicitly all microstates of all holes, to arrive at an understanding of how the information paradox is resolved in string theory. What we must ask is the following: are there two categories of microstates, with one category having no horizon and the other category having a traditional vacuum horizon?

4.1 The simplest 3-charge extremal states: perturbative addition of P charge

The 2-charge extremal hole has a large entropy, but R^2 terms in the gravity action are as important at the horizon scale as the traditional Einstein action term R. Thus we should ask: could it be that these R^2 quantum corrections are what remove the traditional horizon for the 2-charge extremal case? The 3-charge extremal hole has a larger horizon where the R^2 term is small. Thus could it be that all 3-charge states do have a regular horizon?

To tackle this question, the first step would be to consider the a simple 3-charge extremal state, and see if it has a horizonless structire similar to the 3-charge microstates. This was done in [**?**]. We proceed in the following steps:

(i) We start with the 2-charge state (4.1). In the gravity description, we have seen that in the AdS region, we can do a spectral flow map to regard this geometry as just global AdS space. In the field theory description, all components strings are singly wound, and carry no spin.

(ii) We add an excitation to this global AdS, which described a supergravity particle localized near the center of this global AdS. The supergravity fields involved are the 2-form field $B_{\mu\nu}$, which couples to a scalar ϕ. Such localized solutions for supergravity fields were found in []. The field theory state corresponding to this state is given in fig.4.1(a); we call it $|\psi\rangle_{NS}$

(c) We rotate this state around in the angular S^3 directions, by application of the rotation generator $J_0^+ = J_0^x + iJ_0^y$. Thus we have the state $J_0^+|\psi\rangle_{NS}$.

(d) We do a reverse spectral flow. The operator J_0^+ changes under spectral flow to J_{-1}^+. Thus it acts to increase the energy of the left movers on the effective string. The right movers stay in the ground state. Thus the state acquires a net momentum charge

$$n_p - \bar{n}_p = 1 - 0 = 1 \tag{4.1}$$

and we have a state with D1D5P charges.

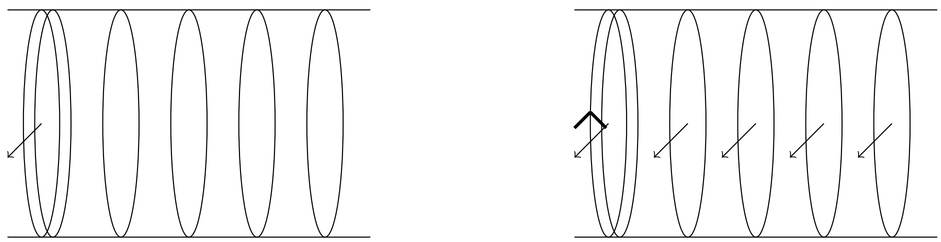
(e) The crucial question now is the following. Suppose the 3-charge state of fig.4.1(a) did *not* correspond to regular solution of gravity; i.e., it had a horizon and/or a singularity. The gravity solution above describes $J_{-1}^+|\psi\rangle_R$ in the cap region. If an overall regular solution did not exist, then we should find that this solution should diverge at infinity; i.e. there should be no way to join this solution to a solution of the field equation that dies off at infinity in a normalizable way. If, on the other hand, the state $J_{-1}^+|\psi\rangle_R$ corresponded to a regular solution of gravity, then it *should* be possible to join the solution in the cap region to a normalizable solution at infinity.

In [1] it was found that it was indeed possible to match the cap solution to a solution that was normalizable at infinity. This matching was carried out in a perturbation expansion, to several orders in the parameter $a = \frac{\sqrt{Q}}{R}$. (More recently, in [2] the exact solution to the supergravity equations describing such a perturbation was found, at the *nonlinear* level; this corresponmds to having an arbitrary number of components strings carrying the state $J_{-1}^+|\psi\rangle_R$.)

The 3-charge extremal state constructed here is very simple, but it shows that the idea of miocrostates without horizons is not limited to 2-charge extremal states.

4.2 3-charge extremal states with nonperturbative amount of P charge

In the above example we added only a small amount of P charge: $n_p = 1$. The natural next question is: if we have a large amount of P charge, then can we still get an extremal D1D5P state with no horizon?



(a) (b)

Figure 4.1

This question was addressed by the following construction:

(i) Start with the 2-charge extremal state (4.1). This state has the quantum numbers

$$h = \frac{N}{4}, \quad j = \frac{N}{2}, \quad \bar{h} = \frac{N}{4}, \quad \bar{j} = \frac{N}{2} \tag{4.2}$$

Consider the spectral flow operation applied to the left movers of this state, by the spectral flow parameter $\alpha = 2$. From (4.1), this gives another state in the R sector, the sector where the fermions are periodic around the y circle. But the quantum numbers change to

$$h' = h + 2j + N = \frac{7N}{4}, \quad j' = j + N = \frac{3N}{2}, \quad \bar{h}' = \frac{N}{4}, \quad \bar{j}' = \frac{N}{2} \tag{4.3}$$

The P charge now is

$$n_p = h' - \bar{h}' = \frac{3N}{2} \tag{4.4}$$

Since $N = n_1 n_5 \gg 1$, this is a state with a large amount of P charge.

(ii) We observe that this spectral flowed state is the state with the largest possible value of j for the value $h = \frac{7N}{4}$, and in fact it is the only possible state with these quantum numbers. This can be seen by recalling that spectral flow adds fermions which all have the same spin, and which fill the fermi sea without any gaps. Thus if we look for a gravity solution with the quantum numbers (4.3), then this gravity solution will be unique; i.e., it will have no entropy.

We now follow the procedure of (4.1). We take the set of all axisymmetric Cvetic-Youm solutions, which are parametrized by different values of $h, j, \bar{h}, \bar{j}$ and set these quantum numbers equal to the values (4.3). We expect a unique state, so we do expect to find a solution, and we expect this solution to have no horizon. Indeed, after some careful work with taking limits near the horizon, we indeed find a unique solution that is completely regular: it has no horizon and no singularity. The CFT state for this solution is depicted in fig.4.1(a), and the gravity solution in (4.1)(b).

(iii) We can extend this construction to get states for values of the spectral flow parameter

$$\alpha = 2n, \quad n = 1, 2, \ldots \tag{4.5}$$

We again find regular D1D5P solutions with no horizon or singularity.

(iv) We can start with the 2-charge extremal microstates (4.1) where each component string has a winding $k > 1$ and the same spin state. We have noted that such solutions have a resolvable conical defect singularity, but no horizon. Applying spectral gives the states depicted in fig.4.1. Proceeding in the same way as in (ii) above, we find gravity solutions with the required quantum numbers with the same conical defect singularity, but no horizon.

Thus we find that we can consider 3-charge extremal states with a large value for all charges, and the gravity solutions for these examples have no horizon.

4.3 Nonextremal solutions

Extremal states do not radiate, so the more intersesting black hole microstates are the nonextremal ones. Such states have more energy than charge, so they can radiate the excess energy away, while still preserving the bound $M \geq |Q|$. Thus the natural next question is: can we make nonextremal states with no horizon?

In [] a 2-charge extremal geometry was taken, and excitations were added in terms of supergarvity quanta. The excitations carried energy but no charge, so the resulting state was nonextremal,. The field theory state is depicted in fig.4.1. It was found that these quanta stayed tarpped for long times in the throat region of the geometry, bonucing between the cap and the neck. Each time the quanta reached the neck, there was a small probability of escaping to infinity; this represented the radiation from the state.

The amount of nonextremal excitation was low however; the supergarvitry quanta in the throat were treated as test particles, so their backreaction on the geometry was ignored. In this sense these solutions were like the perturbative construction of section 4.1, where again the backreaction of the added excitation was ignored. One may ask: is it possible that when we take into account the backreaction from the quanta giving nonextremality, a horizon develops? To argue that this does not happen, we need to find gravity solutions where the amount of nonextremality is large, and its backreaction is fully taken into account.

Such solutions were found in [], by following the method used for extremal states in section**??**:

(i) Start again with the 2-charge state (4.2).

(ii) Perform a spectral flow on the left movers by the spectral flow parameter $\alpha_L = 2n_L$, and on the right movers by $\alpha_R = 2n_R$, where n_L, n_R are integers.

This gives a state with quantum numbers

$$h = (n_L^2 + n_L + \frac{1}{4})N, \quad j = (\frac{1}{2} + n_L)N, \quad \bar{h} = (n_R^2 + n_R + \frac{1}{4})N, \quad j = (\frac{1}{2} + n_R)N \tag{4.6}$$

The momentum charge P is $\mathrm{P} = \frac{n_p}{R}$, where

$$n_p = h - \bar{h} = (n_L(n_L + 1) - n_R(n_R + 1))\, N \tag{4.7}$$

while the added energy is

$$\Delta E = \frac{1}{R}\,(n_p + \bar{n}_p) = \frac{1}{R}\,(n_L(n_L + 1) + n_R(n_R + 1))\, N \tag{4.8}$$

We see that we add mode energy ΔE than momentum charge P, so the state is nonextremal. In particular, if we take

$$n_L = n_R \tag{4.9}$$

then we get no P charge, but we do get an added energy ΔE. The field theory states (4.6) are depicted in fig.4.1.

(iii) It can be seen that the value of j in (4.6) is the largest possible for the given value of h, and the value of $\bar{j}$ is the largest possible for the given value of $\bar{j}$; further the state with these quantum numbers is unique, since it is given by filling the fermi sea with no gaps on both the left and right sides. We therefore expect a unique state with these quantum numbers. We take the family of Cvetic-Youm family of soutions, and look for the solution with these quantum numbers. We indeed find a solution with no horizon or singularity.

(iv) We can extend this class of solutions as in section **??**, by starting with the 2-charge states (4.1), where all component strings have a winding $k > 1$. We again get solutions with no horizon, and the solutions are regular apart from the resolvable conical defect present in the sarting 2-charge soluition.

Thus we see that we can get solutions with D1D5P charges and a large amount fo nonextremality, but no horizon. An interesting roperty of these solutions is that while they possess an *ergoregion*. It is possible for a timelike path to come out of an ergoregion, so the boundary of an ergoregion is not a horizon that traps information. But an ergoregion nevertheless leads to particle craetion, in a manner very similar to the partucle creatio at a horizon. We will see later that in the fuzzball paradigm, emission from such an ergoregion takes the place of Hawking emission, for microstates of the kind described above. Thus the black holes do radiate, but this radiation does not lead to an information paradox.

4.4 States with low rotation

We have seen that the entropy of extremal black holes carrying angular momentum j is given by the expression

$$S = 2\pi\sqrt{n_1 n_5 n_p - j^2} \tag{4.10}$$

For the 3-charge extremal states with quantum numbers (4.3), we have

$$n_p = N = n_1 n_5, \quad j = \frac{3}{2}N = \frac{3}{2}n_1 n_5 \tag{4.11}$$

Thus

$$n_1 n_5 n_p - j^2 = -\frac{9}{4}(n_1 n_5)^4 < 0 \tag{4.12}$$

Thus there is no well defined entropy (4.10). This reflects the fact that this state is overrotating, in the same sense that the 2-charge extremal state (4.1) was overrotating. In other words, the value of j is too large for this state to correspond to a classical black hoe, metric with horizon.

One may therefore be concerned that perhaps it is the overrotation that is responsible for the absence of a horizon, and if we look at states that are not overrotating; i.e., states with

$$n_1 n_5 n_p > j^2 \tag{4.13}$$

then we may again get a horizon. Note that in the 2-charge extremal case, the overrotation was not responsible for the absence of a horizon; even states with zero rotation had no horizon. We now check that the same is true for 3-charge extremal states, by the following example:

(i) Start with a 2-charge extremal state that has $j = 0$. An example of such a state is depicted in fig.4.1(a): we have taken half the component strings to have spin $j = \frac{1}{2}$ and the other half to have $j = -\frac{1}{2}$. We get a 2-charge solution with no horizon; and example of such a solution was given in [] in Appendix .

(ii) Add one unit of P charge by the method of section 4.1. That is, spectral flow to the NS sector, add a wavefunction describing a supergravity quantum $|\psi\rangle$ in the cap region, then spectral flow back to the R sector. The supergravity excitation add the quantum number

$$\Delta n_p = 1, \quad \delta j = 1 \tag{4.14}$$

The state remains a regular one, with no singularity or horizon.

(iii) We now find that

$$n_1 n_5 n_p - j^2 = n_1 n_5 - 1 > 0 \tag{4.15}$$

Thus we see that states which are not overrotaring can also have a structure with no horizon.

4.5 Stringy physics

The microstates we have constructed above have either been decsribed by classical supergravity solutions, or have supergravity quanta added to such solutions. A classical supergravity solution, of course, represents a quantum state: the values of the classical fields just give the peaks of a gaussian distribution that gives the actual wavefunctional of the full state. But we may exepect that when the quantum theory of gravity is string theory, then we will see a more direct manifesation of quantum objects like strings in the construction of general microstates. It is in fact easy to find states where strings are explicitly present, as the following construction shows.

(i) Start with the 2-charge extremal state (4.1).

(ii) Perform a spectral flow to the NS sector: this gives global AdS in the throat+cap region. The field theory state is given in fig.4.1.

(iii) Join together k component strings, and fill up the fermi sea to make an excitation with quantm numbers

$$h = j = \bar{h} = \bar{j} = \frac{k-1}{2} \tag{4.16}$$

(iv) Add further excitations to both the left and right sectors. In [] it was shown that such states correspond to a string livibg in global AdS.

(v) Spectral flow back to the R sector. The state we have constructed has the field theory structure given in fig.4.1.

4.6 The base + fibre split

We have seen above some explicit examples of gravity solutions that have the quantum numbers of an extremal D1D5P state, but which have no horizon. We would now lke to extract some lessons from these examples, which might tell us how more general states may be found.

Several years ago, the following question was asked; given a supersymmetric theory, is there a way to find the most general solution of the theory preserving some supersymmetry? early work in this direction focused on 4-d supergravity [?, ?], and in [?] the method was extended to 5-d theories. Our interest will be in the 6-d case, for the fllowing reason. In our microstate constructions above, we can set $Q_1 = Q_5$, and then the torus T^4 decouples from the solution:

$$ds^2 = ds_6^2 + ds_{T^4}^2 \tag{4.17}$$

and we can focus on solutions in a supersymmetric 6-d theory.

4.6.1 Solutions in 6-d

In [**?**], a class supersymmetric solutions to teh 6-d theory was constructed as folows:

(i) We spilt the 6-d spacetime into a 4-d 'base' and a 2-d 'fiber'.

(ii) The 4-d base if a hyperkahler manifold. This base describes, roughly, the four noncompact spatial directions. Let the metric of this hyperkahler base be

$$ds^2_{HK} = h_{mn}dx^m dx^n \tag{4.18}$$

(iii) The fiber contains, again roughly speaking, the coordinates t, y; here t is the time and y is the coordinate along the S^1. Write

$$u = t + y, \quad v = t - y \tag{4.19}$$

(iv) Then the supersymmtric solutions have the form

$$ds^2 = -H^{-1}(dv+\sqrt{2}\beta_a dx^a)\left(du + \sqrt{2}\omega_a dx^a + \frac{F}{2}(dv + \sqrt{2}\beta_a dx^a)\right)+Hh_{mn}dx^m dx^n \tag{4.20}$$

Here the functions H, F and the vectors β, ω are functions of the base coordinates only. They satisfy a set of four relations among them; these relations are however nonlinear.

Thus if we choosde a hyperkahler base, and then solve the equations giving H, F, β, ω, we would get a supersymetric solution of the 6-d theory. Adding in the T^3 metric trivially, we would get a supersymmetric solution of the full 10-d theory.

4.6.2 Decompsing the 3-charge solutions into base + fiber form

While the above constructions gives supersymmetric solutions in principle, it is not obvious how to solve the nonlinear equations to obtain an actual solution. But we already have the set of supersymmetric solutions described in section **??**; these were obtained by taking special limits in the family of Cvetic-Youm black hole solutions. Thus we can decompose these solutions into the base + fiber form (4.20), and obtain particular solutions to the coupled equations for the functions appearing in this ansatz.

This procedure was carried out in []. But here a curiousity was found, that would have deep consequences for the understanding of black hole microstates.

A hyperkahler manifold is a Euclidean space, with signature $(++++)$. But when the solutions of (4.1) were decomposed into base + fibre form, the base did *not* have signature $(++++)$ everywhere. Instead, the situation was as follows:

(i) One region of the base was hyperkahler with signature $(++++)$; this part included the region near infinity.

(ii) Another region of the base, closer to the center of the geometry, had signature $(----)$. The metric again satisfies the condition to be hyperkahler; thus this region has a normal hyprekahler metric with an overall change of sign: $ds^2 \to -ds^2$.

(iii) The two regions are separated by a codimension-1 boundary in the base. The metric on the fibre degenerates on this boundary. The overall 6-d metric however remains regular everywhere, with signature $(-++++++)$. Thus the base + fibre decomposition becomes singular, while the metric itself has no pathologies.

A base of the above kind was termed 'pseudo-hyperkahler'. As we will see, this extension from hyperkahler bases to pseudo-hyperkahler bases is crucial; it allows the existence of nontrivial solutions without horizon carrying the required D1D5P charges.

A second feature was observed in the base + fibre decomposition of the known microstates, and this had to do with the structure of the base itself. The base in this instance turned out to be a special kind of hyperkahler manifold, balled a Gibbons-Hawking space. A Gibbons-Hawking manifold is a 4-d Euclidean space with the following structure:

(i) The 4-d space can be written as a 3-d Gibbons-hawking base times a 1-dimensional fibre. The base is a 3-d space spanned by coordinates $x = \{x^1, x^2, x^3\}$. The fibre is a circle ψ, with

$$0 \le \psi < 4\pi \tag{4.21}$$

This circle is an isometry direction; i.e., all metric coefficients depend on the Gibbons-Hawking base coordinates x^i.

(ii) The metric of the 4-d space has the form

$$ds^2 = V(\vec{x})(dx_1^2 + dx_2^2 + dx_3^2) + V^{-1}(d\psi + A_i dx^i)^2 \tag{4.22}$$

where V is a harmonic function on the flat x space

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i} V = 0 \tag{4.23}$$

apart from pointlike sources. The vector potential $\vec{A}$ satisfies

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} V \tag{4.24}$$

For our extremal solution (4.1) where the left movers were given a spectral flow by n units we find

$$V = \frac{n+1}{|\vec{x}-\vec{x}_1|} - \frac{n}{|\vec{x}-\vec{x}_2|} \tag{4.25}$$

We call this a 2-center solution since there are two point sources for the potential V.

Recall that we are rewriting a solution that was obtained by taking limits in the family of Cvetic-Youm black holes. These black are in 4+1 dimensions, where the angular sphere has the metric

$$ds^2 = d\theta^2 + \cos^2\theta d\chi^2 + \sin^2\theta d\phi^2 \tag{4.26}$$

The rotation of the hole destroys spherical symmetry, but retains the axisymmetry in the directions ψ, ϕ. Thus in the base+fibre form of this solution, we should still find two axisymmetries. These are the following:

(a) The symmetry along the Gibbons Hawking fiber ψ. This corresponds to motion along $\chi + \phi$ in the coordinates (4.26).

(b) The metric of the Gibbons-Hawking base is invariant under rotation around the line joining the two centers. This corresponds to motion along $\chi - \phi$.

If we look for solutions that preserve both these axisymmetries, then there are no other microstates to find other than the ones discussed above. We now wish to relax these (and other) and symmetries of the solution, and thereby move towards more general microstates.

4.7 Using the base + fiber structure to get new solutions

In 2005, two sets of authors: Berglund, Gimon and Levi [] and Bena and Warner [] achieved something quite remarkable. They managed to use the base + fibre split to generate a large class of supersymmetric solutions with the charges required of the extremal black hole. Let us outline their method:

(i) First, we consider the general case which has a hyperkahler base (i.e., a base not necessarily of the Gibbons-Hawking subclass). We introduce additional harmonic functions, so that we are not limited to 6-d solutions with a constant T^4, instead we make solutions directly in the full 11-d M theory. The D1D5P charges now appear as D2 brane charges. The compactification is

$$M_{10,1} \to M_{4,1} \times T^6 \tag{4.27}$$

The T^6 has directions $x^1, \ldots x^6$. There are three sets of M2 branes, wrapping the directions (12), (34) and (56). The metric ansatz has the form

$$\begin{aligned} ds^2 &= -(Z_1 Z_2 Z_3)^{-\frac{2}{3}}(dt + k_m dx^m)2 + (Z_1 Z_2 Z_3)^{\frac{1}{3}} h_{mn} dx^m dx^n \\ &+ \left(\frac{Z_2 Z_3}{Z_1^2}\right)^{\frac{1}{3}} (dx_1^2 + dx_2^2) + \left(\frac{Z_1 Z_3}{Z_2^2}\right)^{\frac{1}{3}} (dx_3^2 + dx_4^2) + \left(\frac{Z_1 Z_2}{Z_3^2}\right)^{\frac{1}{3}} (dx_5^2 + dx_6^2) \end{aligned} \tag{4.28}$$

and the gauge field is

$$C_{m12} = A_m^{(1)}, \quad C_{m34} = A_m^{(2)}, \quad C_{m56} = A_m^{(3)} \tag{4.29}$$

The metric h_{mn} again describes a 4-d hyperkahler base, and the indices $m, n, \ldots$ run over the coordinates of this base.

With this ansatz, the authors made several important observations:

(i) The unknowns objects are $k_m, h_{mn}, Z_1, Z_2, Z_3, A_m^{(1)}, A_m^{(2)}, A_m^{(3)}$. These satisfy a set of coupled nonlinear equations. But suppose h_{mn} and k_m were known. Then there is a certain order on which we can solve the remaining equations, so that the equation to be solved is *linear* at each stage.

This is an important observation. The first linear equation to be solved in linear homogeneous. The solution of this equation then provides a source term to the next linear equation., and so on. But linear equations with a source can be solved in terms of Green's functions, so we have a way of obtaining the full solution given h_{mn}, k_m.

(ii) Let us now specialize to the case where the hyperkahler base is a Gibbons-Hawking manifold. The potential (4.25) can be generalized to one with more than two centers; for example we can take

$$V = \frac{q_1}{|\vec{x} - \vec{x}_1|} + \frac{q_2}{|\vec{x} - \vec{x}_2|} + \ldots \frac{q_n}{|\vec{x} - \vec{x}_n|} \tag{4.30}$$

where the q_i are integers, and

$$\sum_i q_i = 1 \tag{4.31}$$

We can see that at least one of the q_i must be negative. Thus the base will have a pseudo-hyperkahler form; the region where a negative q_i dominates in V will have a signature $(- - --)$.

In [**?**, **?**] the field equations were solved completely for general choices of the potential V, and explicit microstate solutions obtained. Such solutions break one of the two axial symmetries present in the solutions of []. Since the base is Gibbons-Hawking, we still have the isometry along the circular fiber ψ; this is the symmetry (a) listed in section 4.6.2. But we lost the isometry (b) that was present in the Gibbons-Hawking base in the case where we had just two centers. Thus we have been able to make a class of solutions with less symmetry than the black hole metrics of the Cvetic-Youm class, a remarkable achievement, given the complexity of the equations involved.

4.8 Microstates in 3+1 noncompact dimensions

The Strominger-Vafa black hole was constructed using three brane changes, in 4+1 noncompact spacetime dimensions. The microstates above have been constructed for the case of 4+1 noncompact dimensions. One may now ask: can we also make microstates for the case of 3+1 noncopact dimensions? It turns out that there is a relatively simple way to extend the construction of section 4.7 to the 3+1 dimensional case.

Consoder the Gibbons-Hawking base (4.22), with a general choice of potential (4.30). With the condition (4.31), at large r we have

$$V \approx \frac{1}{|x|} \tag{4.32}$$

We write

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = dr'^2 + r'^2(d\theta'^2 + \sin^2\theta' d\phi'^2) \tag{4.33}$$

The solution to (4.24) is

$$A_{\phi'} = (1 - \cos\theta') \tag{4.34}$$

We then see that the metric (4.22) of the Gibbons Hawking base near spatial infinity has the same form as we found for the KK-monopole near the origin. Thus by the coordinate transformation (4.1), we see that near infinity, this base has the metric of R^4. This is of course as expected, since this base described microsates in 4+1 noncomact spacetime dimensions.

But we can now see how to modify the Gibbons-Hawking base so that the region $r \to \infty$ becomes $R^3 \times S^1$ rather than R^4; this will then give us a microstate in 3+1 noncompact dimensions. We change the potential (4.30) by adding a constant equal to unity

$$V = 1 + \frac{q_1}{|\vec{x} - \vec{x}_1|} + \frac{q_2}{|\vec{x} - \vec{x}_2|} + \dots \frac{q_n}{|\vec{x} - \vec{x}_n|} \tag{4.35}$$

Now we find at $|x| \to \infty$

$$V \approx 1 + \frac{1}{r'} \tag{4.36}$$

and the metric (4.22) of the Gibbons-Hawking base goes over to the metric (4.1) of the KK-monoopke near infinity. This achieves two things:

(i) The fibre of the Giboons-Hakwing base now becomes a compact circle at infinity, so we have acompactified one additional direction, and we get microstates in 3+1 noncompact spacetime .

(ii) We have added an extra charge – a KK-monopole charge – to the microstate. Recall from the discussion in section 4.1 that black holes in 3+1 noncompact dimensions haad 4 charges, where the extra charge compared to the 3-charge D1D5P case was the KK-monopole charge.

Thus by this process we obtain microstates for the 3+1 dimensional extremal case.

4.9 Time dependent solutions

Our goal is to show that in string theory there exist solutions with no horizons that carry the same quantum numbers as the black hole. We have done this by finding a variety of states. Some of these states are exhibited as classical solutions, which can be understood as describing the peak value of a coherent state wavefunctional in the full quantum theory. At other times we have added quanta to such a coherent state: for example we added a particle wavefunction in section 4.1 and string states in section 4.1.

When we write a quantum wavefunction, the time evolution is explicit in the ansatz through a factor $\sim e^{-i\omega t}$. For a string moving in a background, as in section 4.1, the time dependence is manifest in the position of the string.

For coherent states, the classical solution describing its peak may or may not be time dependent. For example consider a neutron star that is rotating about its axis, and which is axisymmetric about this axis. The metric produced by this star will be time independent. But consider the metric of a bar galaxy; this object also rotates, but has no symmetry about its rotation axis. The metric is then time dependent.

We now ask: can we find classical microstate solutions that are time *dependent*?

First recall the solutions discussed in section []. We can think of these solutions as a spectral flow of a special set of 2-charge extremal solutions. This set of 2-charge solutions was axisymmetric, and the spectral flow operation respected symmetry around this axis. As a result, the 3-charge solutions obtained were also time-independent.

But we can start with a 2-charge extremal soluton that does not possess symmetry about any axis. The operation of spectral flow then results in a solution that is time dependent. Such solutions were found in []. The metric was a function of

$$v = t - y \tag{4.37}$$

where y is the coordinate along the S^1 direction. The fact that there is no dependence on $u = t + y$ reflects the fact that the solutions are extremal; a general nonextremal solution would depend on both u and v.

How should we find more general v dependent solutions? In [], the following approach was taken to look for gravity solutions produced by D-branes:

(i) Start with a collection of D1 and D5 branes in flat space.

(ii) Add excitations of open strings to these branes to represent the state of a 2-charge D1D5 system or a 3-charge D1D5P system.

(iii) Using string worldsheet computations, find the metric and gauge fields produced by these branes, near $r \to \infty$. The fields in this region are linear perturbations on flat space, but even in these perturbations one can see the different behavior of different brane states. Thus we explore the asymptotic form of microstates near infinity.

The generic D1D5P solutions obtained this way had a nontrivial dependence on v, so they were time dependent solutions.

Since these solutions are determined only in the large r regime, we do not see the interesting structure in the small r region where the gravity equatiions are nonlinear. A way to find large sets of general v dependent microstates was found in [], as follows:

(i) Consider the case of D1D5P charges. Assume a T^4 is compactified, and has a constant metric. Thus we focus on the supergravity solution in the remaining 6 dimensions.

(ii) The general ansatz for supersymmetric v dependent solutions is

$$ds^2 = 2H^{-1}(dv+\beta_m dx^m)\left(du + \omega_m dx^m + \frac{1}{2}\mathcal{F})dv + \beta_m dx^m)\right) - Hh_{mn}dx^m dx^n \tag{4.38}$$

The quantities $H, \mathcal{F}, \beta_m, \omega_m$ depend on x^m, v but not on u.

(iii) The h_{mn} define a 4-d 'base', and the directions u, v define a 2-d fibre, so the structure of this metric looks similar to (4.20). But now that the functions in the metric have a nontrivial depndence on v, the base is no longer hyperkahler.

(iv) What is remarkable is that there is still a way to reduce much of the problem to one involving linear equations. The situation is similar to what we had found in section **??**. The base h_{mn} and β_m have to be given first. But then the other unknown functions in the solution are obtained by solving a linear equation at each stage; the output of one stage provides a source at the next stage, giving a linear equations with a known source.

4.10 Neck modes vs. near horizon states

We depict again the structure of the traditional extremal hole in fig.4.1(a), and the fuzzball picture for the extremal hole in fig.4.1(b). In the fuzzball paradigm, there is no horizon, and the details of the microstate are encoded in the gravity solution of fig.4.1(b). But where are these details localized?

Sen had considered the traditional picture of the extremal hole, fig.4.1(a). In this picture we have a regular horizon, and by the usual no-hair theorems, there is no information about the microstate near this horizon. But he argued that a limited number of states could be localized at a different region of the geometry – at the 'neck' where the AdS throat joins the asymptotically flat region. He termed these states 'hair', since they altered the full black hole solution in a way that made this solution non-unique.

Note that this 'hair' is quite different from the 'hair' that the relativists were speaking about in the no-hair theorems. In proving the no-hair theorems we are looking for states localized near the *horizon*, not the neck. Note that in a

Schwarzschild hole there is no throat region, and so no sharp separation betwen the neck and the hear horizon regions. But in extremal and near extremal holes we do have such a separation. Thus for the extremal microstates we have considered we should ask: are the details of our microstate localized near the neck, or near the place where the horizon would have been?

This question is important for the following reason. In the above examples of microsate constructions, we construct special families of microstates, and use these to guess the physics of the black hole. But if the families we construct are limited to states localized near the neck, then we have not really explored the microstates that describe the physics of the region closer to the horizon. It is the latter states that are really of interest for black hole physics, so we need to ensure that among the microstates that we construct, there are families that are *not* neck modes.

In [] a family of 3-charge etremal solutions were found in 4+1 noncompact dimensions. In this famly the throat region became very long; in fact the length of the throat is infinite in the classical theory, but is rendered finite by quantum effects. These microstates had no horizon; instead they had a cap at the end of the throat, and a structure in this cap region that distinguished different such microstates. Thus in this case we can say that the microstate structure lives near the horizon region, and not at the neck. A further aspect of this analysis was the length of the throat after it was regulated by quantum effects: the depth of the throat matched the length of the maximally wound effective string, just as in the 2-charge case discussed in section 4.1.

The question we now ask is: is there a definite way to classify excitations into those localized near the neck and those localized near the horizon? It turns out that we can indeed make such a separation in the field theory picture of states. We can these use this picture to construct explicit examples of modes that live at the neck. Finally, we can use this picture to isolate microstates that do *not* have their structure at the neck; we can consider these microstates as describing true degrees of freedom at the horizon.

4.10.1 Neck modes in the field theory

We begin with an idea of Brown and Henneaux []. Consider a 2+1 dimensional gravity solution which at large r is asymptotic to AdS_3 spacetime. Now apply a diffeomorphism to this solution. Normally we would not consider the diffeomorphed solution as a new solution; it is, after all the same spacetime with new coordinates placed on it. But the AdS_3 spacetime has a boundary at large r. Suppose the diffeomorphism did not die off as we approached this boundary. Then we would deform the boundary of the spacetime, and obtain a new state of the theory.

How many new states can we obtain this way? Let the starting state be called $|\psi\rangle$. The diffeomorphisms which deform the boundary turn out to form an algebra, given by two copies of the Virasoro algebra. This is not surprising: the boundary is a 1+1 dimensional manifold, and diffeomorphisms of AdS_3 turn out to induce conformal transformations on the 1+1 dimensional boundary. But conformal transformations in 1+1 dimensions factorise into a left moving set and a right moving set, and each set forms a Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} \tag{4.39}$$

Thus the diffeomorphisms applied to the initial state $|\psi\rangle$ generate states

$$L_{-m}|\psi\rangle, \quad L_{-n}L_{-m}|\psi\rangle, \quad L_{-p}L_{-n}L_{-m}|\psi\rangle, \ldots \tag{4.40}$$

If we consider the supersymmetric theory, then we get additional fermionic supersymmetry generators $G^{\alpha}_{\dot{A},-r}$ with $\dot{A} = \pm, \alpha = \pm$ and a triplet of currents $J^{\pm}, J^3$.

But what is the significance of such boundary states? This becomes clear if we add back the asymptotically flat part of the spacetime. We proceed as follows:

(i) We start with the microstate corresponding to the state $|\psi\rangle$, where we include the neck and asymptotically flat spacetime in the microstate solution; this is depicted in fig.4.1(a).

(ii) We cut off the neck and flat spacetime part, so we are left with just the asymptotically AdS region (fig.4.1(b)).

(iii) We apply a diffeomorphism to this asymptotically AdS spacetime, creating in the process a deformation of the boundary of the AdS region (fig.4.1(c)).

(d) We glue this asymptotically AdS region back to the neck and flat spacetime region. But due to the deformation of the boundary of the AdS region, there is an alteration in the gravity solution in the neck region, compared to the solution that we had for the state $|\psi\rangle$. This alteration is a real change; it is not a diffeomorphism, and it carries energy. If we had applied the diffeomorphisms corresponding to L_{-n}, the the new solution would describe $L_{-n}|\psi\rangle$.

In [] this procedure was carried out to make states $L_{-n}|\psi\rangle$ and $J^3_{-n}|\psi\rangle$. The theory also has 4 U(1) currents $\tilde{J}^i_{-n}$ corresponding to translation along the 4 directions of T^4; these currents are just the boson oscillators

$$\tilde{J}^i = \alpha^i_{-n}, \quad i = 1, \ldots 4 \tag{4.41}$$

where the index i runs over the 4 directions of the T^4.

In [**?**]his process was extended to the nonlinear level, so that one generates states

$$e^{\sum_n \mu^i_n \tilde{J}^i_{-n}}|\psi\rangle \tag{4.42}$$

for arbitrary values of the parameters μ^i_n. A similar nonlinear construction was achieved for the nonabelian currents J^a_{-n} in [].

The analysis of the field equations at the neck also tell us that there are no *other* excitations at the neck. We have therefore arrived at a full understanding of the excitations that Sen had termed 'hair' – the excitations that reside at the neck of the extremal geometry. In the field theory these excitations are obtained by application of the symmetry currents $L_{-n}, G^{\alpha}_{\dot{A}}, J^{\pm}, J^3, \tilde{J}^i$ to any state $|\psi\rangle$. In the gravity description, the hair excitations are deformations of the gravity solution that are localized in the neck region of the geometry. In this neck region the deformation is not a diffeomorphism, and thus carries energy. In particular we can take the limit of large R, where R is the radius of the S^1 on which the field theory lives. In this limit we find that in the gravity solution the energy added by the deformation corresponding to L_{-n} is

$$\Delta E = \frac{n}{R} \tag{4.43}$$

in accordance with the energy of L_{-n} in the field theory.

Having understood the 'hair' modes, we can now look for states that can be called 3-charge microstates with structure at the horizon. Let $|\psi\rangle$ be a 2-charge extremal D1D5 state. The states obtained by the action of current generators on $|\psi\rangle$ will not have any 'new' structure at the horizon; the new structure will appear in the neck region. Thus we look for field theory states that are *not* of the form

$$A_{-n_1} \dots A_{-n_k}|\psi\rangle \tag{4.44}$$

where the A_{-n} are any of the chiral algebra generators. A set of such states were studied in []. The gravity solutions for these states had been found by Gimon and Levi []. What was now noted was that the field theory duals for these gravity solutions had the form

$$|\psi\rangle \tag{4.45}$$

Thus on each component string, fermions were added with no gaps, upto a certain fermi level. But this level was *not* the level that would be reached if we had applied a spectral flow to the 2-charge state $|\psi\rangle$. Rather, the state on each string can be said to arise from a fractional spectral flow, where the spectral flow parameter is a fraction rather than an integer. It was then shown that the states (4.1) cannot be written in the form (4.44), so we have obtained 3-charge microstates that are not of the form of Sen's neck hair added to a 2-charge state.

4.11 Multi-particle states in AdS vs black hole states

In fig.4.1 we depict global AdS_3. In this AdS space we can add some quanta of supergravity; this is indicated in fig.4.1(b). Finally we can consider adding enough mass M that we get a black hole in AdS; this is depicted in fig.4.1(c).

Now the question is: are the multi-particle states in fig.4.1(b) different in principle from the black hole states in fig.4.1(c)? If these two kinds of states lie in two different categories, then we have to check if out microstate constructions capture examples of both kinds of states. The general idea behind the fuzzball program is, of course, that all states are fundamentally similar, so there should be no sharp difference between different categories of states. More precisely, there should be no category of states where the gravity solution has a vacuum horizon.

We will proceed in three steps. First, we will see that the field theory suggests a way to define multiparticle states. We will then see how the gravity solutions for these states can be constructed. Finally we will observe that there are gravity solutions that correspond to field theory states that are *not* in the multiparticle state category.

4.11.1 Multiparticle states in the field theory

Bibliography

[1] Borun D. Chowdhury, Stefano Giusto, and Samir D. Mathur. A Microscopic model for the black hole - Black string phase transition. *Nucl. Phys.*, B762:301–343, 2007.

[2] Stefano Giusto, Luca Martucci, Michela Petrini, and Rodolfo Russo. 6D microstate geometries from 10D structures. *Nucl. Phys.*, B876:509–555, 2013.