TOPIC II

SOME BASIC TOOLS

In this part we will take a tour through the two main theories of physics that are needed to set up the information paradox.

We will first make a brief pass through general relativity, which gives a relativistic theory of gravity. After understanding the idea of a metric, we will explore the properties of the black hole metric which will be relevant to the information paradox.

We will then discuss quantum field theory, which is a relativistic theory of quantum mechanics. In particular we will discuss the nature of the vacuum, since this plays a crucial role in the phenomenon of Hawking radiation.

Putting these two tools together will allow us to understand the origin of Hawking radiation from the black hole.

Lecture notes 1

General relativity

Let us take a brief tour through Einstein's theory of gravity – general relativity.

1.1 The essential idea

Newton taught us that the motion of a particle is affected by the force F acting on it, through the relation

$$F = ma \tag{1.1}$$

To use this relation one needs to know F. For the case of gravity, Newton gave the needed expression. The gravitational force on a mass m due to a mass M at a distance r is an attractive force of magnitude

$$F = \frac{GMm}{r^2} \tag{1.2}$$

The interesting thing is that m cancels out, giving

$$a = \frac{GM}{r^2} \tag{1.3}$$

so that we get the same trajectory for all values of the mass m. This is of course the lesson of the fabled experiment of Galileo, where he dropped bodies with different values of m from the Tower of Pisa, and found that they reached the ground at the same time.

In Newtonian mechanics this cancellation of m remains little more than a curiosity, but Einstein sought to make it the starting point for his formulation of gravity. If all particles will follow the same trajectory, then why don't we try to find a simple characterization of these universal trajectories? Following this line of thought, he arrived at a formulation whose statement at first looks startling: *gravity is not a force at all.*

How can this be? We think we know quite well when the force F is nonzero, and gravity certainly feels like a force in everyday life!

In fig.1.1(a) we draw particle trajectories on an $x - t$ graph, for the case $F = 0$. There is no acceleration, so the trajectories are straight lines. Since straight lines can meet at most once, we recover the physical fact that in the absence of any forces, two bodies which meet once will then drift apart and never come back to meet again.

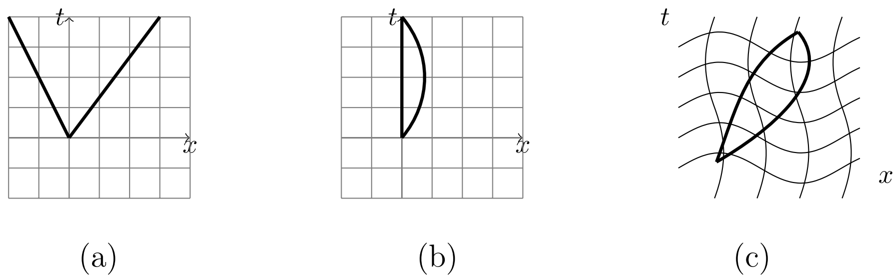


Figure 1.1: (a) No force: particle trajectories are straight lines in the $x - t$ plane, and particle trajectories can meet no more than once. (b) A situation with force: the vertical line is the surface of the earth, and the curved line represents a ball thrown up from the ground and allowed to fall back. (c) Einstein's view of gravity: spacetime $x - t$ is curved, while the trajectories of bodies are 'straight lines' (geodesics) on this curved spacetime.

In fig.1.1(b) we draw trajectories in a situation with F nonzero. The parabola is the trajectory of a ball which is thrown up and allowed to fall back to earth. The vertical line on the graph marks the surface of the earth. The force of gravity acts between the earth and the ball, and curves the trajectories so that they do meet more than once, at points A and B. So it seems that the effect of a force is clear: it changes straight trajectories to curved ones.

Einstein invited us to think of the situation in fig.1.1(b) in a different way: not as curved lines on a flat $x - t$ plane, but as a set of *straight* lines drawn on *curved* graph paper. What do we mean by a 'straight' line on a curved surface? As an example, consider the surface of the earth. Between any two points, we can ask for the *shortest* path, and define such paths to be our 'straight lines'. Such shortest distance paths are called 'geodesics' in general, and in the case of the earth are given by the great circles on a sphere. If a person starts walking on a curved surface, and at every infinitesimal step 'goes straight forward', then his trajectory would be a geodesic. But note that these 'straight lines' on a curved surface can meet more than once; for example two circles of longitude meet at the North pole and again at the South pole.

Einstein asked us to think of the situation in fig.1.1(b) in the description fig.1.1(c). Now the effect of gravity is encoded in the curvature of the paper on which we are drawing our trajectories, The particle trajectories themselves are given by the simplest of all rules: they are geodesics – the 'straight line' – on this curved surface. Since each infinitesimal segment of the geodesic is a 'straight line segment', we can say that gravity is not a force at all; it just manifests itself in the curvature of spacetime.

Of course we need to know how much the spacetime should curve. This is given by Einstein's equation of gravity, which says that curvature is proportional to mass; or rather, energy, since even massless particles like photons create a gravitational field.

To summarize, general relativity is made of two rules:

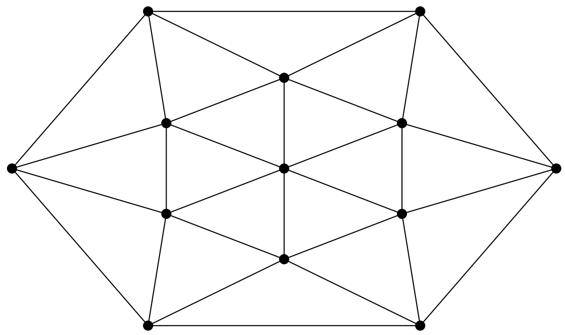


Figure 1.2: A 2-dimensional surface. The shape of the surface is known once we specify the length of each link depicted by a line.

(a) Energy curves spacetime.

(b) Particles move in geodesics on this curved spacetime.

Our next task is to understand how to describe curved spacetime.

1.1.1 The metric

In fig.1.2 we depict a surface made with a child's builder set: sticks of different lengths are joined together to make a lattice. It is clear that if we fix the length of each stick, we would fix the shape of the surface. Thus to define a surface all we need to do is specify the distance between any two infinitesimally separated points on the surface.

On a flat plane, the distance between two points is given by the Pythagorean theorem

$$\Delta s^2 = \Delta x^2 + \Delta y^2 \tag{1.4}$$

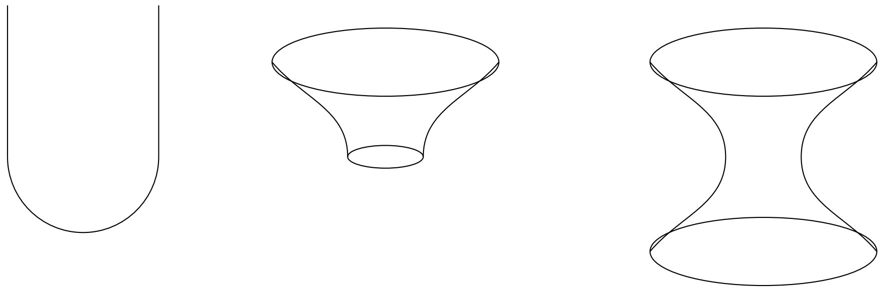
On a curved surface, the Pythagorean theorem does not work. But an infinitesimal piece of a curved manifold is essentially flat, and there the Pythagorean theorem does work. On such an infinitesimal piece we can introduce Cartesian coordinates x, y and look at infinitesimal distances

$$ds^2 = dx^2 + dy^2 \tag{1.5}$$

The advantage of looking at infinitesimal distances becomes clear if we look at the flat plane again but in polar coordinates $\{r, \theta\}$. At any point in this plane, moving towards larger r and moving towards larger θ give two orthogonal directions. The former gives a length dr, while the latter gives a length $rd\theta$. Then

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{1.6}$$

Note that there is no simple expression for the distance between two points with a non-infinitesimal separation, since the factor r^2 does not have a fixed value



(a) (b) (c)

Figure 1.3: (a) The surface corresponding to the metric (**??**. (b) The surface for the metric (1.13). (c) The metric is (b) can be continued smoothly past its boundary.

along the path joining the two points. If we do want the distance between two finitely separated points, then we have to choose a path between them, break up this path into infinitesimal segments, find the length ds for each segment using (1.6) and add up these contributions. As an example, consider the length of a circle at radius r. Each infinitesimal segment has a length $ds = rd\theta$, so the total length of the circumference is

$$C(r) = \int_{\theta=0}^{2\pi} r\, d\theta = r \int_{\theta=0}^{2\pi} d\theta = 2\pi r \tag{1.7}$$

An expression like (1.5) or (1.6) is called the *metric*, since it allows us to measure distances on this space. The general feature we see in these examples is that (a) the metric is quadratic in the infinitesimal displacements and (b) the coefficients of these quadratic terms can depend on position; for example the coefficient of $d\theta^2$ is r^2.

The metric (1.6) described a flat plane, but we can now have fun making new surfaces. Let us try

$$ds^2 = dr^2 + \frac{r^2}{r^2+1} d\theta^2 \tag{1.8}$$

For small r, we have

$$\frac{r^2}{r^2+1} \approx r^2 \tag{1.9}$$

so the surface will look the usual flat plane near $r = 0$. But at large r, we have

$$\frac{r^2}{r^2+1} \approx 1 \tag{1.10}$$

so the metric becomes

$$ds^2 = dr^2 + d\theta^2 \tag{1.11}$$

Now the circumference of a circle at a fixed value of r is

$$C(r) = \int_{\theta=0}^{2\pi} d\theta = 2\pi \tag{1.12}$$

Thus the surface looks like a cylinder, where the circumference of the circle does not change as we go up. The overall surface is depicted in fig.1.3(a). Even though this surface is infinite in extent, we will see that surfaces like this can be found inside the black hole, and are a key to understanding the information paradox.

Now let us try the metric

$$ds^2 = dr^2 + (r^2 + 1)d\theta^2 \tag{1.13}$$

For large r, we have $r^2 + 1 \approx r^2$, so the metric looks like the metric (1.6) of a plane. For $r \to 0$, we have $r^2 + 1 \approx 1$, so the metric looks like the metric of a cylinder (1.11). Over the range $0 < r < \infty$, the surface will look as depicted in fig.1.3(b).

But this surface seems to end suddenly on a circle at $r = 0$. Can we continue further past this circle? Looking at the metric (**??**), we see no problem in continuing r to negative values. If we explore the complete range

$$-\infty < r < \infty, \qquad 0 \le \theta < 2\pi \tag{1.14}$$

then we get the surface in fig.1.3(c). There are *two* planes, joined by a 'wormhole'.

This notion of extending a surface will be central to our study of the black hole. We will ask if the metric outside the hole can be extended to cover the inside of the hole, and if it can extended even further. In the maximally extended metric that we can make, we will find a wormhole joining two infinite flat spacetimes.

1.1.2 Lorentzian spaces

The coefficients in the metric need not be all positive. Einstein taught us to think of time as an additional dimension, but with a negative sign in the metric

$$ds^2 = -c^2 dt^2 + dx^2 \tag{1.15}$$

Now the distance between two points can be positive, negative or zero. In these cases we say the separation is spacelike, timelike and null respectively. The *light cone* is the set of directions that are null

$$dx = \pm c dt \tag{1.16}$$

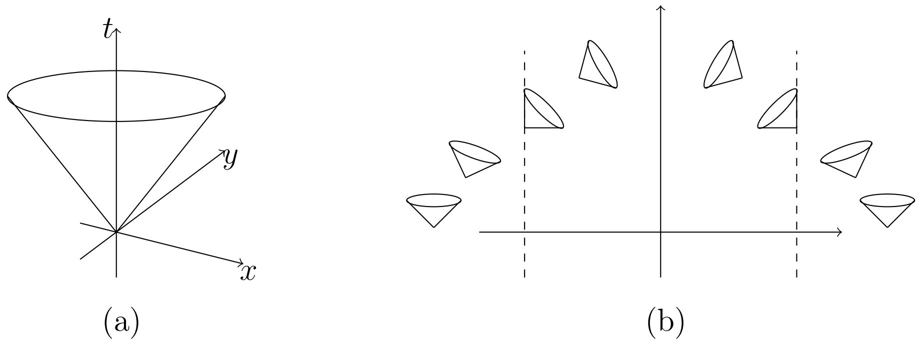


Figure 1.4: (a) The light cone; the trajectories of particles must always have a slope that lies within this light cone. (b) A toy model of a black hole. As we come inwards, the light cones tilt inwards as well. The vertical dashed lines are horizons; no particle trajectory can emerge from inside the horizon to the outside.

Massless particles like photons travel at the speed of light, so their trajectories lie on the light cone. Massive particles travel at speeds less than c, so their trajectories must lie inside the light cone. We depict the lightcone in fig.1.4(a).

When we curve a spacetime, the light cones also distort. In fig.1.4(b) we depict a toy model of a black hole. We have let the light cones to 'tilt inwards' towards $r = 0$. At a point like A, the light cone allows particles to move towards either larger or smaller values of r. But at the point B the light cone has tilted so much that no particle can escape towards $r > r_h$; a massless particle heading 'outwards' would stay at $r = r_h$, and all other particles would fall in towards $r < r_h$. The locations $r = \pm r_h$ are the 'horizons'. At a point like C, the light cone has tilted even further, and all particle trajectories must fall towards $r_h = 0$.

This toy model answers a common question about black holes. We have said that a star will collapse to a black hole when it no longer be held up by neutron degeneracy pressure. Can't there be other forces, yet undiscovered, that will hold up the star when the neutron degeneracy pressure cannot, so that a black hole will never form?

In fig.1.5 we depict a star which has shrunk to a radius smaller than the horizon radius r_h. Because of the tilt of the light cones, all points on this star *must* continue to move towards smaller and smaller values of $|r|$. No force can stop this collapse because it would require making particles move outside the light cone; i.e., with a speed more than c. As long as our theory only allows velocities $|v| \leq c$, there cannot be any force that stops this collapse.

If we cannot halt the collapse after the star has shrunk inside the radius r_h, then we might try to argue that some force halts the collapse *before* the radius becomes less than r_h. But here we run up against a fundamental problem: we can make a star fit inside the radius r_h even when its density is very low, and at

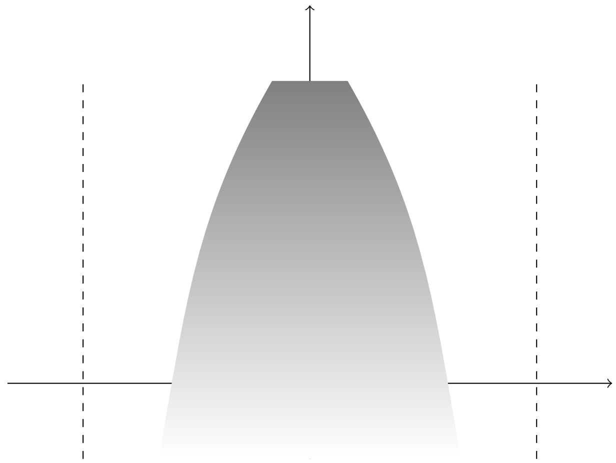


Figure 1.5: A star which is inside the horizons depicted in fig.1.4(b). All points of this star must keep moving inwards, since otherwise their trajectories would lie outside the local light cone.

low densities we believe we already know all the forces that could be relevant. Recall that

$$r_h = \frac{2GM}{c^2} \tag{1.17}$$

Assuming the matter forms a uniform ball of radius R, we have

$$M \sim \rho R^3 \tag{1.18}$$

where ρ is the density. Eqs. (1.17) and (1.18) give

$$\rho \sim \frac{c^2}{Gr_h^2} \sim \frac{c^4}{G^3M^2} \tag{1.19}$$

So we can take ρ to be as small as we like, provided we make the ball be big enough to contain a sufficiently large overall mass M. To emphasize this point, let the matter consist of dust grains, of mass 1 gm each, placed in a cubical lattice with separation 100 cm between neighboring grains in each direction. This gives $\rho \sim 10^{-6}$ gm/cc. One might imagine that at this low a density, there are no unknown forces between the dust grains, and in fact none of the forces we know would prevent us from making such a lattice of dust grains. Now assume that this low density dust forms a ball of radius $R = 10^{18}$ cm. Then $M \sim 10^{48}$ gm, and

$$r_h \sim 10^{20}\,\text{cm} > R \tag{1.20}$$

So the ball of dust is *already* inside its critical radius r_h, and we cannot stop its collapse to a point.

This example illustrates the crux of the black hole puzzle. It seems easy to make very large, very low density balls of dust, which will cause the light cones to tilt in the manner shown by the toy example of fig.**??**(c). If we cannot find a way to prevent the existence of such low density balls, then we must accept that black holes form. And then we have to face the paradoxes that come with them.

1.1.3 More metrics

Let us consider more examples of metrics. Consider the surface of a sphere of radius R, using spherical polar coordinates θ, ϕ. An infinitesimal change in θ moves us down along a longitude, and gives a length $Rd\theta$. The angle ϕ runs along a latitude circle with radius $R\sin\theta$; thus an infinitesimal change in ϕ gives a length $R\sin\theta d\phi$. Further, the θ and ϕ displacements are orthogonal. Thus

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.21}$$

If we also include the radial direction, then we have length dr in a direction perpendicular to the $d\theta, d\phi$ directions. Thus we get the metric in 3-dimensional space in spherical polar coordinates

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.22}$$

Including time t, we get the metric of 3+1 dimensional spacetime

$$ds^2 = -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.23}$$

We can now write the metric of a black hole. A black hole of mass M with zero angular momentum, in 3+1 spacetime dimensions, has the metric

$$ds^2 = -(1 - \frac{2GM}{rc^2})dt^2 + (1 - \frac{2GM}{rc^2})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.24}$$

This metric will occupy us for most of this book. It is very fundamental to general relativity, since it is the metric created by a point mass. Its analogue in electromagnetism would be the electric field of a point charge

$$\vec{E} = k\frac{Q}{r^2}\hat{r} \tag{1.25}$$

But while this electric field $\vec{E}$ is straightforward to understand, analyzing (1.24) will lead to many surprises.

1.1.4 The structure of the black hole

All the mysteries associated with the black hole lie in the metric (1.24). We will now start uncovering some of the features of this metric. To simplify things, in what follows we choose units where

$$c = 1, \quad G = 1, \quad \hbar = 1 \tag{1.26}$$

The Schwarzschild metric (1.24) then becomes

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1.27)$$

We can see that the metric has potential problems at the horizon radius r_h

$$r_h = 2M \quad (1.28)$$

To get started, we first look at the region far from the horizon: $r \gg r_h$. Then we find

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1.29)$$

so we recover flat spacetime; this makes sense since all gravitational effects should fall at large distances from the hole.

To explore the metric closer to the horizon, we first note the different kinds of surfaces in general relativity.

1.1.5 Spacelike and timelike surfaces

Consider flat 3+1 dimensional spacetime with metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad (1.30)$$

(i) Consider the surface spanned by the directions x, y, z. If we move in any direction on this surface, we find that $ds^2 > 0$; i.e., the direction behaves as a spacelike direction. Thus we call this a spacelike surface. The normal to this surface is in the direction t, which is a timelike direction. So another way to define a spacelike surface is to say that its normal is timelike.

(ii) Consider the surface spanned by the directions t, x, y. If we move along this surface in the t direction, then $ds^2 < 0$; i.e. this direction is timelike. If we move in the x or y directions, then $ds^2 > 0$, as before. Since one of the directions in the surface is timelike, we call this a timelike surface. (It is not possible to have more than one direction be timelike, since there is only one time direction overall.) The normal to this surface is in the z direction, which is spacelike; so another way to define a timelike surface is to say that its normal is spacelike.

Spacelike surfaces are of fundamental important in physics. We give initial data on such a surface, and then the evolution equations determine all variables at all times. Thus in the flat metric (1.30), the equation

$$t = 0 \quad (1.31)$$

defines a spacelike surface, spanned by the directions x, y, z. In classical field theory, we can specify the value of the field and its normal derivative on this surface. In quantum theory, we would specify a wavefunction on the spacelike surface. In each case, the physics is then completely determined by the evolution equations, both to the future and past of this surface.

(iii) Finally, we can have a 'null' surface, where two directions are spacelike, but the third is null: i.e., with $ds^2 = 0$. An example of this surface would be the 3-dimensional surface formed by taking all values of x, y, but restricting to the line $t - z = 0$ in the $t - z$ plane. The direction

$$\delta x = 0, \quad \delta y = 0, \quad \delta t = \delta z \tag{1.32}$$

along this surface is null. This same direction is also normal to the surface, since it has vanishing dot product with all vectors tangent the surface. Thus a null surface has a normal which is also null, and this normal happens to lie within the surface itself.

1.1.6 Spacelike surfaces in the Schwarzschild metric

To understand the evolution of the black hole, we must look for a spacelike surface in the Schwarzschild metric. Outside the horizon, i..e, at $r > r_h$, we see that we get a spacelike surface by fixing the value of t:

$$t = t_0 \tag{1.33}$$

This is because in the metric (1.27) all the three infinitesimal directions of displacement within this surface – those in the r, θ, ϕ directions have a positive coefficient, and are thus spacelike.

But now consider the region inside the horizon. For $r < r_h$, we find that the coefficient of the t coordinate becomes *positive*, and the coefficient of the r component becomes *negative*:

$$ds^2 = (\frac{2M}{r} - 1)dt^2 - (\frac{2M}{r} - 1)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.34}$$

Thus the time coordinate t and the space coordinate r have interchanged roles. A spacelike surface is now given by fixing a value for r

$$r = r_0 \tag{1.35}$$

This follows because the directions along this surface are t, θ, ϕ. The t direction is now spacelike, and θ, ϕ continue to be spacelike as before. We draw the surface (1.35) in fig.**??**. The angular directions are depicted as a circle, while the vertical direction is t.

How large is this surface? The angular directions θ, ϕ form a 2-sphere of radius r_0. In the t direction, we can consider any range $t_1 < t < t_2$. The length in this direction can be read off from the metric (1.34)

$$s = \int_{t_1}^{t_2} (\frac{2M}{r_0} - 1)dt = (\frac{2M}{r_0} - 1)(t_2 - t_1) \tag{1.36}$$

But we can take $t_2 - t_1$ as large as we want, so we can make the cylinder arbitrarily long. This is very interesting, because this cylinder sits inside the horizon radius r_h. So we can fit an infinitely long cylinder inside the horizon of the black hole! This sounds puzzling, but is a simple consequence of the fact that the metric coefficients of t and r changed signs at the horizon.

This unbounded length of the cylinder answers a question that could have been asked when we made our first pass at discussing remnants. We had started with a mass M, and then cancelled this down to zero by adding negative energy particles inside the horizon radius r_h. But each time we add a negative energy particle the overall mass goes down, and so r_h decreases. After we put more and more negative energy particles inside the horizon, the horizon will become smaller and smaller; yet it has to contain the initial mass M as well as all the negative energy particles we have added. But how will all these particles fit inside the horizon? Could it not be that when the particles are squeezed into too small a region, they somehow 'choke up' the space inside the horizon, so that we cannot add another negative energy particle? If we could not add more particles inside the horizon, then we could not reduce the mass further, and we would not have the problem of remnants.

But now we see that there is actually an *infinite* amount of space inside a horizon with any radius r_h. The cylinder depicted in fig. **??**(a) has finite cross sectional area, but it has an infinite length. Thus we can place an arbitrarily large number of particles along this cylinder, without having to place them close to each other. This is depicted in fig.**??**(b). The Hawking radiation process will, in fact, automatically deposit the negative energy particles along this cylinder in this manner; the initial matter M making the hole and the negative energy particles will all be well separated from each other along the cylinder.

To summarize, the horizon is a place where the time and space coordinates interchange roles. This interchange results in the existence of an infinitely long cylindrical spacelike surface inside the horizon. It is this infinite space which allows the black hole to keep absorbing so many negative energy particles, without choking up and ending its radiation process.

1.1.7 The smoothness of the horizon

We still have to understand what is happening at the horizon $r = 2M$. The metric coefficients are singular here; the coefficient of t vanishes while the coefficient of r diverges. If this is a real singularity, then we may not be able to continue from the outer region $r > 2M$ to the inner region $r < 2M$. In that case the picture of infinitely long slices inside the horizon (and the consequent existence of remnants) would be quite meaningless.

To see the problems that we face at the horizon, let us write the metric (1.27) for the approximation

$$r = 2M + r', \qquad |r'| \ll 2M \tag{1.37}$$

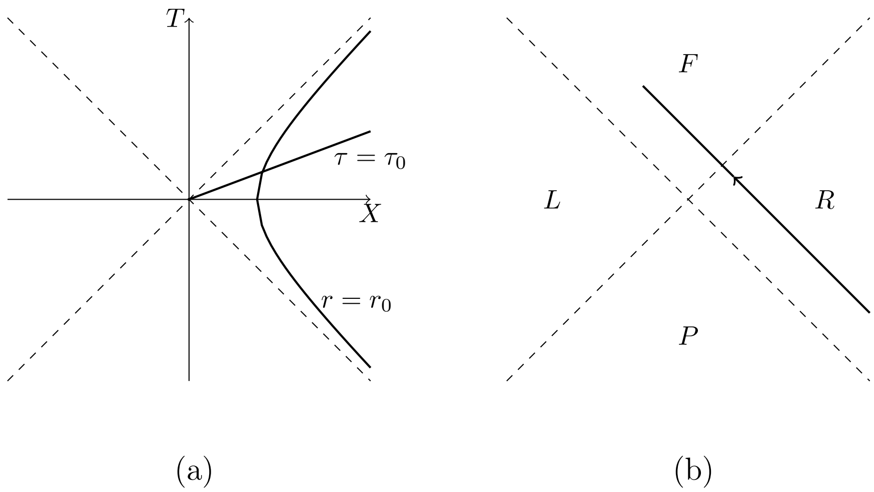


Figure 1.6: caption ...

We find

$$ds^2 \approx -\frac{r'}{2M}dt^2 + \left(\frac{2M}{r'}\right)dr'^2 + (2M)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.38}$$

where we have noted that $dr = dr'$.

The coefficient of dt^2 vanishes at $r' = 0$; this is part the issue we are addressing. But coefficients in the metric can sometimes vanish without there being anything wrong with the space itself. Consider the 2-dimensional plane with metric

$$ds^2 = dx^2 + dy^2 \tag{1.39}$$

Put polar coordinates on this plane

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1.40}$$

Then we get

$$dx = \cos\theta dr - r\sin\theta d\theta, \qquad dy = \sin\theta dr + r\cos\theta d\theta \tag{1.41}$$

This gives

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{1.42}$$

as we had found by direct geometrical intuition in (1.6). At $r = 0$, the coefficient of $d\theta^2$ vanishes. But there is nothing wrong with the space itself; its just that the coordinates r, θ are not a good set around the point $r = 0$, since all values of θ correspond to the same point at $r = 0$.

The Schwarzschild metric (1.27) has Lorentzian signature. So let us repeat the above computation starting not with the metric (1.39) but with the metric

$$ds^2 = -dT^2 + dX^2 \tag{1.43}$$

We write

$$T = r\sinh\tau, \qquad X = r\cosh\tau \tag{1.44}$$

This gives

$$ds^2 = -r^2 d\tau^2 + dr^2 \tag{1.45}$$

This metric has a vanishing coefficient for the time direction $d\tau^2$.

What will be very important for us is the following feature of (1.45). In the Euclidean case (1.42), the polar coordinates r, θ covered the entire plane $x - y$. But in the Lorentzian case, the 'polar' coordinates r, τ naturally cover only one quadrant of the full T, X plane. As in the Euclidean case, these coordinates fail at $r = 0$, so let us restrict to $r > 0$. For τ, we can take the full range $-\infty < \tau < \infty$. We then see that we have

$$\begin{aligned} X &> 0 \\ |T| &< X \end{aligned} \tag{1.46}$$

We depict this domain by the shaded region in fig.**??**.

Let us return to our metric of interest, which in the region near the horizon had the form (1.38). This metric not only has a vanishing of the coefficient of the time direction t, it also has a divergence of the metric in the space direction dr'. Thus we need a further change of variables. We write

$$\left(\frac{2M}{r'}\right)^{\frac{1}{2}} dr' = d\tilde{r} \tag{1.47}$$

which gives

$$r' = \frac{\tilde{r}^2}{8M} \tag{1.48}$$

The metric (1.38) becomes

$$ds^2 \approx -\frac{\tilde{r}^2}{16M^2} dt^2 + d\tilde{r}^2 + (2M)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.49}$$

We can clean this up a little by writing

$$\frac{t}{4M} = \tilde{t} \tag{1.50}$$

which gives

$$ds^2 \approx -\tilde{r}^2 d\tilde{t}^2 + d\tilde{r}^2 + (2M)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.51}$$

To recap: we have taken the Schwarzshild metric (1.34), and examined the metric around the horizon; i.e., in the approximation $r \approx 2M$. The metric in this region has split onto two factors:

(a) The angular part gives

$$ds^2 \to (2M)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.52}$$

which is just the metric on a sphere S^2 of radius $r = 2M$.

(b) A part

$$ds^2 \to -\tilde{r}^2 d\tilde{t}^2 + d\tilde{r}^2 \tag{1.53}$$

This is just the metric (1.45), which describes one quadrant of a full plane of flat 1+1 dimensional spacetime. .

Bibliography