Lecture notes 1

The no-hair theorem

1.1 The no-hair theorem

We have seen that the no-hair theorem of black holes is central to the information paradox. This theorem says that the Schwarzschild metric

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.1}$$

gives the unique structure for the black hole. In other words, one cannot have any deformations of the horizon.

From this description it may appear that the no-hair theorem is a *classical* theorem; i.e., it says that the classical Einstein equations admit no deformation of the classical metric (1.1). But this fact alone would not be very significant for the information paradox. Hawking pairs are produced by a *quantum* effect around the horizon of the metric (1.1). In deriving this pair creation, Hawking assumed that the quantum state around the horizon was the same as the vacuum state of empty spacetime. If we could have a *different* quantum state around the horizon, then we would not be able to claim that the the state of the produced pairs is necessarily the entangled state found by Hawking, and we would not be able to argue that there is a paradox.

In this section we will see that quantum state around the metric (1.1) admits no deformations, as long as assume the usual perturbative structure of quantum field theory. This is the form of the 'no-hair' theorem that will be useful in understanding the information paradox. Later, we will see how non-perturbative effects in string theory violate the no-hair theorems, and allow a resolution of the information paradox.

1.1.1 Quantizing fields around a star

Consider a scalar quantum field $\hat{\phi}$. In flat spacetime, we quantize this field by taking a large box of volume V and setting periodic boundary conditions. This gives the following expansion of $\hat{\phi}$ in modes

$$\hat{\phi} = \sum_{\vec{k}} \left(f_{\vec{k}}\, \hat{a}_{\vec{k}} + f^*_{\vec{k}}\, \hat{a}^\dagger_{\vec{k}} \right) \tag{1.2}$$

Here the functions

$$f_{\vec{k}} = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}} e^{-i\omega t} \tag{1.3}$$

satisfy the wave-equation for the scalar field ϕ

$$\Box f_{\vec{k}} = 0 \tag{1.4}$$

The vacuum $|0\rangle$ is the state annihilated by all the $\hat{a}_{\vec{k}}$

$$\hat{a}_{\vec{k}}|0\rangle = 0 \tag{1.5}$$

We can add one particle of momentum $\vec{k}_1$ to this vacuum state by acting with a creation operator

$$|\psi\rangle = \hat{a}^\dagger_{\vec{k}_1}|0\rangle \tag{1.6}$$

We can add several particles by using multiple creation operators

$$|\psi\rangle = \hat{a}^\dagger_{\vec{k}_n} \dots \hat{a}^\dagger_{\vec{k}_2} \hat{a}^\dagger_{\vec{k}_1}|0\rangle \tag{1.7}$$

The entire space of states near the vacuum state $|0\rangle$ can be obtained in this way. What we cannot obtain this way are nonperturbative effects that generate objects like topological solitons, but we normally assume that such objects have a large energy, and so should not included in defining the class of states that are 'close' to the vacuum.

Now consider a neutron star, which has a metric of the form

$$ds^2_{star} = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.8}$$

When the metric of this star has settled down to the time independent form (1.8), the quantum state around the star also settles down to a state that we can call the vacuum state $|0\rangle_{star}$. The quantum field $\hat{\phi}$ can again be expanded in a manner similar to (1.2). But the coefficient functions f should now satisfy the scalar waveequation $\Box_g \phi = 0$, where $\Box_g$ is the d'Alembertian operator on a spacetime with metric ds^2_{star}. Since the star is spherically symmetric, it is useful to use spherical polar coordinates, in which the f have the form

$$f_{nlm} = f_{nlm}(r) Y_{lm}(\theta, \phi) e^{-i\omega t} \tag{1.9}$$

and the field $\hat{\phi}$ can be written as

$$\hat{\phi} = \sum_{n,l,m} \left(f_{nlm}\, \hat{a}_{nlm} + f^*_{nlm}\, \hat{a}^\dagger_{nlm} \right) \tag{1.10}$$

We again have

$$\hat{a}_{\vec{k}}|0\rangle_{star} = 0 \tag{1.11}$$

and we can add particles to the vacuum state by the action of creation operators

$$|\psi\rangle = \hat{a}^\dagger_{n_k l_k m_k} \dots \hat{a}^\dagger_{n_2 l_2 m_2} \hat{a}^\dagger_{n_1 l_1 m_1}|0\rangle_{star} \tag{1.12}$$

In this way we can explore the entire space quantum states close to the ground state of the star.

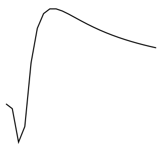

Figure 1.1: caption ...

At first it may seem that we will get a similar story for the black hole metric (1.1). But here we have to contend with the singularity in the metric coefficients at $r = 2M$. Wavefunctions of the form (1.9) which have a time dependence $e^{-i\omega t}$ are called 'stationary state wavefunctions'; since the time evolution for such wavefunctions is just a change of overall phase, the probability density is time-invariant. For the black hole, we have seen that in the interior region $r < 2M$ the light cones point inwards. If we assume that no effects can propagate outside the light cone, then we find that everything moves inexorably towards smaller values of r. Thus we do not expect to describe the interior region $r < 2M$ by stationary wavefunctions of the form (1.9).

Let us therefore restrict attention to the region outside the horizon $r > 2M$. What we will see now is that a stationary wavefunction in this region will necessarily have a singularity in the limit $r \to 2M$. This will mean that we cannot find stationary wavefunctions to describe the quantum field around the Schwarzschild hole. In consequence, we cannot act with creation operators to make new states that are close to the vacuum. This is what makes the state around the horizon unique.

To see the problem, consider a spherically symmetric wavefunction

$$f = f(r)e^{-i\omega t} \tag{1.13}$$

and examine it at a point $r_0 = 2M + \epsilon$ just outside the horizon. We can set up a local orthonormal coordinate system around this point. The relevant directions are r, t, where the local coordinates will be

$$d\tilde{t} = g_{tt}^{\frac{1}{2}}(r_0)dt = (1 - \frac{2M}{r_0})^{\frac{1}{2}} dt \approx \frac{(r_0 - 2M)^{\frac{1}{2}}}{(2M)^{\frac{1}{2}}} dt \tag{1.14}$$

$$d\tilde{r} = g_{rr}^{\frac{1}{2}}(r_0)dr \approx \frac{(2M)^{\frac{1}{2}}}{(r_0 - 2M)^{\frac{1}{2}}} dr \tag{1.15}$$

The waveequation gives

$$\frac{\partial^2 f}{\partial \tilde{t}^2} = \frac{\partial^2 f}{\partial \tilde{r}^2} \tag{1.16}$$

with solutions of the form

$$f \sim e^{i\tilde{\omega}(\tilde{r} - \tilde{t})} \tag{1.17}$$

With the above scaling of t, we have

$$\tilde{\omega}^2 f = \frac{\partial^2 f}{\partial \tilde{t}^2} = \frac{2M}{(r_0 - 2M)}\frac{\partial^2 f}{\partial t^2} = -\frac{2M}{(r_0 - 2M)}\omega^2 f \tag{1.18}$$

The local energy density ρ and pressure p are given by $\tilde{\omega}$

$$\rho = p \sim \tilde{\omega} \approx \frac{(2M)^{\frac{1}{2}}}{(r_0 - 2M)^{\frac{1}{2}}}\omega \tag{1.19}$$

A stationary wavefunction is characterized by a definite value of ω. We then find that the energy density and pressure of this wavefunction diverge as $r_0 \to 2M$. Since the problem arises in a small neighborhood of the horizon, we are looking at points very close to the horizon, it is easy to see that a similar analysis holds for any other spherical harmonic Y_{lm}. Thus there are no regular solutions f to the waveequation $\Box\phi = 0$.

We are therefore unable to write the field operator $\hat{\phi}$ in the form (1.10), and so cannot find a set of stationary excitations of the form (1.12).

We considered a scalar field ϕ above, but a similar analysis can be done for a vector field like the photon field A_μ, and again one finds no stationary excitations of the vacuum. Most importantly, we can write the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{1.20}$$

where $\bar{g}_{\mu\nu}$ is the Schwarzschild metric (1.1) and $h_{\mu\nu}$ is a small perturbation. We can then regard $h_{\mu\nu}$ as a quantum field. If we explore the quantum states of $\hat{h}_{\mu\nu}$, then we will explore the space of all perturbative *quantum* deformations of the black hole. The waveequation for $h_{\mu\nu}$ is a little more complicated that the one for a scalar ϕ, but in a suitable gauge it again reduces to the d' Alembertian operator acting on $h_{\mu\nu}$, and we again find no stationary wavefunctions f that are regular at the horizon.

To summarize, we can regard the stationary solutions f to an equation like $\Box\phi = 0$ in two different ways:

(i) As solutions to the classical wave equation. In the case of the graviton field $h_{\mu\nu}$, this will tell us that there are no small, time-independent deformations of the metric (1.1) that give a smooth deformation of the horizon. In this classical domain, one can make additional assumptions and prove that the metric (1.1) is unique even if we allow deformations that are not small.

(ii) As a set of modes f that enter into the field expansion of a quantum operator like $\hat{\phi}$. The nonexistence of regular, stationary solutions f then tell us that the quantum state around the black hole cannot be altered to a nearby stationary quantum state. Thus a black hole horizon will settle down to a situation where the quantum state around the horizon is the vacuum state that that Hawking used in his derivation of pair creation.

What we will see later is that in string theory, the assumptions made in the classical no-hair theorems turn out to be violated in string theory. As a consequence a horizon never really forms, and then we are not forced to the uniqueness of the quantum state near the surface of the hole.

1.1.2 Counting states

We have seen that there are no quantum states 'near' the standard vacuum state for the region near the horizon of a black hole. But it is interesting that *if* we had indeed been able to find regular solutions f to the waveequation, and thus been able to construct new states of the form (1.12), then the number of such states would have been of the correct order to account for the Bekenstein entropy of the black hole. Let us review this estimate, since when we do find 'hair' for the black hole, a similar argument will tell us that the entropy of this hair will naturally be of the correct order to account for the entropy of the hole.

Consider a scalar field ϕ, and recall the form (1.9) of the field modes f. In the spherical harmonic Y_{lm}, we have $-l \le m \le l$, so there are $(2l+1)$ states for each value of $l = 0, 1, 2, \ldots$. The function Y_{lm} oscillates as $e^{im\phi}$, so the angular wavelength of oscillation is

$$\Delta\phi \sim \frac{1}{m} \sim \frac{1}{l} \tag{1.21}$$

At the horizon radius $r_h = 2GM$, this corresponds to a physical wavelength

$$\lambda \sim r_h \Delta\phi \sim \frac{r_h}{l} \tag{1.22}$$

Suppose we allow all wavelengths upto the planck scale; i.e., $\lambda \gtrsim l_p$. This corresponds to

$$l \lesssim \frac{r_h}{l_p} \equiv l_{max} \tag{1.23}$$

This number of Y_{lm} in the range (1.23) is $\sim l_{max}^2$. Suppose we take a take the lowest allowed value of n in f_{nlm}; this corresponds to deformations close to the horizon. Let us assume that we have ~ 1 excitations 0 or 1 quantum in each mode $\hat{a}^\dagger_{n_k l_k m_k}$; for simplicity let us just take the occupation numbers of the modes to be 0 or 1. Then the number of allowed states is

$$\mathcal{N} \sim 2^{l_{max}^2} \tag{1.24}$$

and the entropy is of order

$$S = \ln \mathcal{N} \sim l_{max}^2 \sim (\frac{r_h}{l_p})^2 \sim \frac{A}{G} \tag{1.25}$$

where $A \sim r_h^2$ is the area of the horizon and $G \sim l_p^2$. Thus we indeed recover the Bekenstein entropy.

Even though we have found no regular modes f in the present treatment, we will find structure at the horizon when we consider the fuzzball construction in string theory. What we will do is find analogues of the deformations

f_{nlm} for modes $l = 0, 1, 2.\ldots$. We cannot reasonably hope to get an accurate wavefunction for all the states of the hole; in fact we cannot write down all the wavefunctions even for a planet like the earth, since the best we can do today is write wavefunctions for molecules with a few atoms. But once we find that the states analogous to the low angular harmonic modes are 'fuzzballs' with no horizon, we can extrapolate the picture to one where the angular wavelength becomes as small as the planck scale. This will again yield the estimate (1.25), giving us a qualitative picture for the general states of the black hole.

1.1.3 Summary

Our discussion above addresses a common misconception about the no-hair theorem. The proofs of many of the no-hair theorems were classical, so one might think that the metric of the black hole is unique only at the classical level. This would then seem to leave open the possibility that the hole could have many different quantum states when we move away from the classical approximation, and that the entropy of the hole would arise from these possibilities for the quantum state. If such were the case, then the information of the hole would reside near the horizon, and this information could be encoded in the radiation from the surface of the hole. In that case here would be no information paradox.

But we see that the same functions f that give classical perturbative deformations of fields around the hole, also serve to give the possible wavefunctions that can be added to the hole to change its quantum state. The absence of regular functions f then tells us that the quantum state of the hole is unique around the horizon. Thus we need a nontrivial change in our picture of the hole to resolve the information paradox. In string theory we will find that the horizon does not really form; then it is indeed possible to have different quantum states near the horizon, and these will account for the entropy of the hole.

Bibliography