Lecture notes 1

The small corrections theorem

1.1 Overview

The fuzzball paradigm provides a clear and concrete resolution of the information paradox. We have taken a consistent theory of quantum gravity – string theory – and found by explicit computation that black hole microstates have the structure of 'fuzzballs' where no traditional horizon exists. The fuzzball radiates from its surface like any normal body, and there is no problem with entanglement or information loss.

In spite of this success, many people were reluctant at first to accept the fuzzball paradigm. Years of work with the traditional picture of the black hole had led to a strong prejudice that the horizon should be a region where the local quantum state is the vacuum. Further, many people working with standard general relativity were reluctant to accept the additional structure – strings, branes and extra dimensions - brought in by string theory. Without this additional structure one does not get fuzzballs, and the natural conclusion is that black hole evaporation leads to the information of the hole getting locked in a planck mass 'remnant'.

In string theory we believe that remnants do not exist. Many string theorists, however, were also reluctant to accept that the nature of the horizon was severely altered away from the vacuum. What then was their resolution of the puzzle?

We will see in this chapter that there was a pervasive belief that the puzzle could be resolved through the *cumulative effect of small corrections*. In this view the physics at the horizon can continue to be the physics of the local vacuum at leading order. Quantum gravity effects will, however, lead to small subleading effects to this vacuum dynamics. We know that the evaporation of the hole involves a very large number of radiated quanta. It was hoped that these small subleading effects would some how introduce 'delicate correlations' among all the quanta involved in the process, and that these small correlations would resolve the information paradox.

In 2004 Hawking conceded that information was probably not lost in the process of black hole evaporation, using a version of this small corrections argument. But most other relativists did not concur. Their belief was that if we make a tiny change in the evolution at the horizon, then we will get only a tiny change in the entanglement between the emitted quanta and the remaining hole. How then could small corrections resolve the problem?

In 2009 a theorem was proved showing that the paradox *cannot* be resolved by the idea of small corrections. Thus to avoid the ever-increasing entanglement between the radiated quanta and the hole, we will need corrections of *order unity* to the low energy dynamics at the horizon. In other words, the horizon cannot be local vacuum to leading order; we need a significant departure from the traditional picture of the hole.

The proof of this theorem used a nontrivial tool from quantum information theory: the strong subadditivity of quantum entanglement entropy. The power of the theorem lies in its generality: given general assumptions like the usual locality of interactions in situations with weak curvature, the theorem says that small corrections cannot stop the entanglement from growing monotonically, regardless of the nature or origin of these small corrections.

This theorem did much to put the idea of fuzzballs on a firm footing, and increase its acceptance as the resolution of the paradox in string theory. After all, if severe changes are needed at the horizon, and one has found that specific states in the theory are fuzzballs which have no horizon, then it is reasonable to accept that all states of the hole will have a similar nature.

We will describe the small corrections belief in some detail. Then we will explain the strong subadditivity relation, and sketch the proof of the small corrections theorem. We will then discuss some misconceptions about the information puzzle that relate in one way or another to the small corrections belief.

1.2 The origins of the small corrections belief

A black hole radiates energy, and in this regard seems like behave like any other thermodynamic system. What can we say about information and entanglement in usual thermodynamic systems?

Thermodynamic systems are *large*, in the sense that they have many degrees of freedom. This leads to a principal characteristic of such systems: they obey the law of large numbers. Consider a box of volume V, containing N atoms of a gas at temperature T. Suppose we ask: what is the pressure P of this gas? There are two ways of addressing this question:

(a) We consider the $3N$ degrees of freedom arising from the motion of the N atoms, and compute the pressure they create.

(b) We use the ideal gas law $PV = NkT$ and find the pressure P.

Clearly method (b) is the easier one to use. It gives an approximate answer, which misses the detailed information contained in the full description (a). One may say that the answer provided by (b) is a 'corase-grained' answer, while 'fine-grained' details can be obtained by doing the full computation (a).

All this is of course correct and well known. But it suggested the following line of thought about black holes:

(a) Hawking took the classical metric of the black holes, and computed the evolution of quantum modes on this metric to leading order. This computation leads to an ever growing entanglement between the emitted quanta and the remaining hole, and therefore leads to a problem at the endpoint of evaporation.

(b) There can, however, be small corrections to Hawking's leading order evolution; such corrections may arise, for example, from quantum gravitational effects. The correction to each created pair must be small, since we are assuming the traditional picture of the hole to leading order. But the number of created pairs N is very large; we have $N \sim (M/m_p)^2$.

(c) The small corrections can induce delicate correlations among this large number of quanta. Near the endpoint of evaporation, we are required to do the following computation. We have to look at the wavefunction of the entire system: the quanta that have been radiated and their partners that fell into the hole. We have to then find the entanglement between the quanta outside the hole and everything that is inside the hole.

(d) It is possible that when we compute this entanglement, the small correlations between all the quanta give rise to a significant correction to the entanglement computed at leading order by Hawking. In fact, by the time we reach near the endpoint of evaporation, the entanglement of the radiation with the remaining hole might go to *zero*. In this case the hole can evaporate away completely, with no violation of quantum mechanics: all the information in the matter which initially made the hole would be encoded in the delicate correlations among the emitted quanta, and there would be no troublesome entanglement of these radiated quanta with a tiny remnant.

If the above scenario was correct, then there would be no information paradox in the first place. We would say that Hawking did a computation using the evolution to leading order, but this was analogous to a 'coarse-grained' approximation in statistical physics. The 'fine-grained' details were hidden small quantum correlations caused by subleading effects. When we take these small corrections into account, the troublesome growth of entanglement vanishes.

This looks like an easy way out of the paradox. It is very appealing, in part because there are so many possible sources of small corrections to the leading order computation. How can we be sure that one or more of these will not invalidate the original Hawking argument? To investigate the issue, let us begin by formulating it in more mathematical terms.

1.2.1 A bit model for small corrections

We have modeled the emitted radiation quantum as having two states, $|0\rangle$ and $|1\rangle$. At leading order we assumed that the vacuum at the horizon produces an entangled pair of quanta. The emitted quantum is called b, and its partner that falls into the hole is called c. The state of this entangled pair was modeled by

$$|\psi\rangle_{pair} = |0\rangle_b|0\rangle_c + |1\rangle_b|1\rangle_c \tag{1.1}$$

In the leading order Hawking process, each pair is produced independently, in the same entangled state $|\psi\rangle_{pair}$. Thus the overall state of the entangled pairs becomes

$$\begin{aligned}|\Psi\rangle &= \frac{1}{\sqrt{2}}\Big(|0\rangle_{b_1}|0\rangle_{c_1}+|1\rangle_{b_1}|1\rangle_{c_1}\Big)\\ &\otimes \frac{1}{\sqrt{2}}\Big(|0\rangle_{b_2}|0\rangle_{c_2}+|1\rangle_{b_2}|1\rangle_{c_2}\Big)\\ &\quad\dots\\ &\otimes \frac{1}{\sqrt{2}}\Big(|0\rangle_{b_N}|0\rangle_{c_N}+|1\rangle_{b_N}|1\rangle_{c_N}\Big)\end{aligned} \tag{1.2}$$

The tensor product sign $\otimes$ tells us that the state of each pair is completely uncorrelated with the state of all other pairs, in this leading order Hawking computation.

Let us now allow for small corrections that do introduce correlations among the pairs. Let the state of the first pair be

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big((1+\epsilon_1)|0\rangle_{b_1}|0\rangle_{c_1}+(1-\epsilon_1)|1\rangle_{b_1}|1\rangle_{c_1}\Big) \tag{1.3}$$

where the correction parameter is small

$$|\epsilon_1| \ll 1 \tag{1.4}$$

The value of ϵ_1 can depend on the initial matter that made the hole, so the correction above helps the emitted quantum carry a small amount of information about the hole.

The corrections to the second pair can depend on the initial matter that made the hole, and also on the state of the first pair. Suppose the first pair was created in the state $|0\rangle_{b_1}|0\rangle_{c_1}$. Then we let the state of the second pair be

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big((1+\epsilon_2)|0\rangle_{b_2}|0\rangle_{c_2}+(1-\epsilon_2)|1\rangle_{b_2}|1\rangle_{c_2}\Big) \tag{1.5}$$

If, on the other hand, the state of the first pair was $|1\rangle_{b_1}|1\rangle_{c_1}$, then we let the state of the second pair be

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big((1+\epsilon_3)|0\rangle_{b_2}|0\rangle_{c_2}+(1-\epsilon_3)|1\rangle_{b_2}|1\rangle_{c_2}\Big) \tag{1.6}$$

Thus we have allowed the earlier created pair to influence the later pair through a small correction in the state of the later pair. The overall state for both pairs looks quite complicated

$$\begin{aligned}|\Psi\rangle &= \frac{1}{\sqrt{2}}(1+\epsilon_1)|0\rangle_{b_1}|0\rangle_{c_1}\Big(\frac{1}{\sqrt{2}}\Big((1+\epsilon_2)|0\rangle_{b_2}|0\rangle_{c_2}+(1-\epsilon_2)|1\rangle_{b_2}|1\rangle_{c_2}\Big)\Big)\\ &+ \frac{1}{\sqrt{2}}(1-\epsilon_1)|1\rangle_{b_1}|1\rangle_{c_1}\Big(\frac{1}{\sqrt{2}}\Big((1+\epsilon_3)|0\rangle_{b_2}|0\rangle_{c_2}+(1-\epsilon_3)|1\rangle_{b_2}|1\rangle_{c_2}\Big)\Big)\end{aligned} \tag{1.7}$$

But all we need to note is that the number of correction parameters ϵ_i grows very rapidly with the number of emission. After N steps of emission we have 2^{N-1} correction parameters in the state. Note that

$$2^N \;=\; e^{N\log 2} \;\sim\; e^{\alpha\left(\frac{M}{m_p}\right)^2} \tag{1.8}$$

where α is order unity. Here we have recalled that the number of quanta emitted by the hole is $N \sim \left(\frac{M}{m_p}\right)^2$.

This exponentially large number of correction parameters create an exponentially large number of correction terms to the leading order state (**??**). Thus to get an effect of order unity, it would appear that we can take each correction term to be exponentially small; i.e., we can take

$$|\epsilon_i| < \epsilon, \quad \epsilon \sim e^{-\alpha\left(\frac{M}{m_p}\right)^2} \tag{1.9}$$

Corrections this small can of course have many sources. A simple one is is the idea of nonperturbative quantum gravitational fluctuations of the black hole metric. A full understanding of such effects would need a detailed computation in a full theory of quantum gravity, but we can estimate the order of the corrections by a statistical argument using phase space arguments, as follows.

A black hole has a large number of states, given by $Exp[S_{bek}(M)]$. The matter which made this black hole had, in comparison, a very low entropy, $S_{matter}(M)$, and therefore a small number of possible states $Exp[S_{matter}(M)]$. Suppose we ask for the probability P that the black hole change suffers a quantum fluctuation to a state of ordinary matter. Then we may expect that

$$P \;\sim\; \frac{e^{S_{matter}}}{e^{S_{bek}}} \;\sim\; e^{-(S_{bek}-S_{matter})} \;\sim\; e^{-S_{bek}} \;\sim\; e^{-4\left(\frac{M}{m_p}\right)^2} \tag{1.10}$$

This argument would suggest corrections of order $\epsilon \sim P$, which is is compatible with the requirement (1.9) on the magnitude of these corrections.

It would therefore seem that there is no paradox at all; the black hole geometry is expected to have exponentially small corrections from its natural quantum fluctuations, and these small corrections will induce subtle corrections to the quantum state of the large number of created particles. Taking these corrections into account would remove the large entanglement between the $\{b_i\}$ and $\{c_i\}$ found in the leading order Hawking state (1.2), and would also encode in the radiation the information about the initial matter which made the hole.

A version of this idea was suggested in [**?**]. In 2004 Hawking himself endorsed a similar reasoning, using the language of Euclidean field theory. If we rotate time to Euclidean signature, then the black hole is described by a smooth instanton. But there are also instantons with larger action, which give exponentially small subleading corrections to the full Euclidean path integral. Once these subleading corrections were taken into account, the paradoxes created by the leading instanton – the traditional black hole solution – would be removed.

1.2.2 The counterargument

In 1996 Hawking and Kip Thorne had jointly taken a bet with John Preskill; Hawking and Thorne maintained that information was lost in black holes, while Preskill bet that when a final understanding of black holes was obtained, one would find that quantum mechanics was preserved. Based on his argument of 2004, Hawking conceded the bet to Preskill, sending him a set of sports encyclopedias. But Thorne refused to concede the bet; after all no one had shown that the small corrections *do* act in the required way to resolve the paradox. In fact most relativists were unconvinced by Hawking's 2004 argument. Their belief can be summarized as follows. The leading order Hawking state (1.2) exhibits an entanglement between the radiation and the hole of order $N\ln 2$. Suppose each step of emission had a small correction of order ϵ. Then we should expect no more than a small reduction in the entanglement, say

$$N\ln 2 \;\to\; (1-\epsilon)N\ln 2 \tag{1.11}$$

This small reduction in entanglement would not resolve the paradox, since we need this entanglement to go to *zero* at the endpoint of evaporation.

These opposing views on the role of small corrections created a very strange split in the field of black hole physics, for several years. Many relativists believed that (1.11) was true. With traditional approaches to quantum gravity (i.e., without using the additional structure of string theory), they could find no way to alter the vacuum nature of the horizon, and were therefore led to conclude that black hole evaporation would lead to remnants. Many string theorists, on the other hand, thought that the relations (1.9) and (1.10) implied that there was no information puzzle; the hole could have the traditional vacuum horizon to leading order, and yet completely evaporate away with no problem since subleading corrections would remove the troublesome entanglement.

Interestingly, no one wrote down an explicit choice of the small parameters $\{\epsilon_i\}$ which would remove the entanglement between the $\{b_i\}$ and $\{c_i\}$. In 2009, it was proved that *no* such choice exists; thus the intuition (1.11) is correct, and small corrections do not resolve the information puzzle. This proof used a non-trivial relation in quantum information theory, called the strong subadditivity of quantum entanglement entropy.

We start with some definitions related to quantum entanglement, and then note the strong subadditivity relation between entanglement entropies. We will then use this relation to prove the 'small correction theorem', which shows that small corrections can generate only a small reduction in entanglement.

1.3 Entanglement entropy

Consider two physical systems, A and B. We require that these systems be *disjoint*; i.e, they are characterized by different degrees of freedom, though they can interact with each other. An example would be two boxes, each filled with a gas. The boxes can exchange heat energy, but the degrees of freedom for box

A describe the atoms in box A and the degrees of freedom for box B describe the atoms in box B.

We are interested in the overall state of the union of the two boxes; we will call this system $A + B$. Let $|\psi\rangle$ be a possible state for system A and $|\chi\rangle$ a possible state for system B. The simplest state for $A + B$ is of the type

$$|\Psi\rangle = |\psi\rangle \otimes |\chi\rangle \tag{1.12}$$

Such states are called *factored* states.

More complicated states can be obtained by taking linear superpositions of factored states. Let the Hilbert space of A have dimension N_A and the Hilbert space of B have dimension N_B. Let ψ_i, $i = 1. \ldots N_A$ be an orthonormal basis for A and χ_j, $j = 1, \ldots N_B$ be an orthonormal basis for B. Consider a state of the form

$$|\Psi\rangle = \sum_{i=1}^{k} \frac{1}{\sqrt{k}} |\psi_i\rangle \otimes |\chi_i\rangle \tag{1.13}$$

with any $k \geq 1$. This is called an *entangled* state: the state of A is entangled with the state of B. A is in the state $|\psi_i\rangle$ *if* is in the state $|\chi_1\rangle$. A is in the state $|\psi_2\rangle$ *if* is in the state $|\chi_2\rangle$, and so on. We define a quantity $S_{ent}(A)$: the entanglement of the system A with everything else; in the present case the 'everything else' is just the system B. The number of terms in the sum in (1.13) is a measure of how entangled A is with everything outside itself, and for the state (1.13) we will get

$$S_{ent}(A) = \ln k \tag{1.14}$$

If $k = 1$ then we get the unentangled state (1.12), and then the above expression gives $S_{ent}(A) = 0$.

Note that in our situation (1.13) where the only degrees of freedom involved are those in A and B, we also have

$$S_{ent}(B) = \ln k \tag{1.15}$$

In general we have

$$S_{ent}(A) = S_{ent}(A^c) \tag{1.16}$$

where A_c is the complement of A; i.e., the set containing all the degrees of freedom in our total physical system, excluding the degrees of freedom in A.

We can of course write states more general than (1.13); the most general state of $A + B$ can be written as

$$|\Psi\rangle = \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} C_{ij} |\psi_i\rangle |\chi_j\rangle \tag{1.17}$$

where we have dropped the symbol $\otimes$ for compactness of expression. The fact that $|\Psi\rangle$ is normalized – $\langle\Psi|\Psi\rangle = 1$ – gives

$$\sum_{i,j=1}^{N} |C_{ij}|^2 = 1 \tag{1.18}$$

1.3.1 Density matrices

Suppose the system $A + B$ is in the entangled state (1.17). Now suppose we want to measure the expectation value of an operator $\hat{O}_A$ which involves only the degrees of freedom in A. Is there any simplification from the fact that $\hat{O}_A$ does not involve the degrees of freedom in B?

We have for the expectation value of $\hat{O}_A$

$$\langle\Psi|\hat{O}_A|\Psi\rangle = \sum_{i'=1}^{N_A}\sum_{j'=1}^{N_B}\sum_{i=1}^{N_A}\sum_{j=1}^{N_B} C^*_{i'j'}C_{ij}\,\langle\chi_{j'}|\langle\psi_{i'}|\,\hat{O}_A\,|\psi_i\rangle|\chi_j\rangle \tag{1.19}$$

Since there is no operator acting on the system B, the χ part of the dot product gives a simple contribution

$$\langle\chi_{j'}|\chi_j\rangle = \delta_{jj'} \tag{1.20}$$

Thus we get

$$\langle\Psi|\hat{O}_A|\Psi\rangle = \sum_{i'=1}^{N_A}\sum_{i=1}^{N_A}\sum_{j=1}^{N_B} C^*_{i'j}C_{ij}\,\langle\psi_{i'}|\,\hat{O}_A\,|\psi_i\rangle \tag{1.21}$$

We wish to extract from the full entangled state $|\Psi\rangle$ the part that is relevant to operations acting on the degrees of freedom in A alone. To do this we define a matrix $\hat{\rho}(A)$ as follows

$$(\hat{\rho}(A))_{ii'} = \sum_{j=1}^{N_B} C^*_{i'j}C_{ij} \tag{1.22}$$

The indices i, i' range over $1, \dots N_A$, since they both refer to the system A. The system B has been 'traced out' to leave us with a quantity that describes A alone. In terms of $\hat{\rho}_A$ we find

$$\langle\Psi|\hat{O}_A|\Psi\rangle = \sum_{i'=1}^{N_A}\sum_{i=1}^{N_A} (\hat{\rho}(A))_{ii'}\,\langle\psi_{i'}|\,\hat{O}_A\,|\psi_i\rangle \tag{1.23}$$

The above relations can be written a little more compactly in terms of matrices. Define the matrix $\hat{C}$ as

$$\left(\hat{C}\right)_{ij} = C_{ij} \tag{1.24}$$

Then

$$\hat{\rho}(A) = \hat{C}^\dagger\hat{C} \tag{1.25}$$

Note that

$$\mathrm{Tr}\hat{\rho} = 1 \tag{1.26}$$

where the trace is only over the Hilbert space of A since $\hat{\rho}$ is an operator taking states of A to states of A. The expectation value (1.23) becomes

$$\langle\hat{O}_A\rangle = \mathrm{Tr}\left[\hat{\rho}_A\,\hat{O}_A\right] \tag{1.27}$$

where again that trace is only over the states of A.

1.3.2 Entanglement entropy

We can now get a give a full characterization of entanglement. The entanglement of A with the remainder of the system is given by the entanglement entropy of the set A, defined as

$$S_{ent}(A) = -\mathrm{Tr}\,[\hat{\rho}\,\ln\hat{\rho}] \tag{1.28}$$

This expression may look complicated in terms of the definition of $\hat{\rho}$ in (1.22). It turns out however that the form of the state $|\Psi\rangle$ in (1.17) can be simplified by a change of basis. While it is not immediately obvious, it can be shown that one can make an appropriate choice of orthonormal basis for A and an appropriate choice of orthonormal basis for B so that $\Psi\rangle$ can be written as

$$|\Psi\rangle = \sum_{i=1}^{N} C_i |\psi_i\rangle \otimes |\chi_i\rangle \tag{1.29}$$

where

$$N = \mathrm{Min}\,(N_A,\, N_B) \tag{1.30}$$

With this choice of basis, we get

$$S_{ent}(A) = -\sum_{i=1}^{N} |C_i|^2 \ln |C_i|^2 \tag{1.31}$$

When two large systems A amd B are allowed to interact with each other, then the overall state of $A+B$ tends towards one of maximal entanglement. This happens on purely entropic grounds: there are more states that are highly entangled than states that are less entangled. A maximally entangled state of $A+B$ has the form Take example of maximally entangled state

$$|\Psi\rangle = \sum_{i=1}^{N} \frac{1}{\sqrt{N}} |\psi_i\rangle \otimes |\chi_i\rangle \tag{1.32}$$

Thus the number of terms in the sum is the maximal possible, and each term has the same coefficient. This state has the form (1.13), so let us check that we reproduce (1.14) from our general expression (1.28). We have

$$\hat{C} = \mathrm{diag}\{\frac{1}{\sqrt{N}}, \dots \frac{1}{\sqrt{N}}\} \tag{1.33}$$

$$\hat{\rho}(A) = \hat{C}^{\dagger}\hat{C} = \mathrm{diag}\{\frac{1}{N}, \dots \frac{1}{N}\} \tag{1.34}$$

and we get

$$S_{ent}(A) = -\mathrm{Tr}\,\hat{\rho}\ln\hat{\rho} = -\sum_{i=1}^{N} \frac{1}{N} \ln \frac{1}{N} = \ln N \tag{1.35}$$

1.3.3 Properties of entanglement entropy

As we have already noted, by its very definition the entanglement entropy of a set A is the same as the entropy of the complement of A

$$S_{ent}(A) \quad = \quad S_{ent}(A^c) \tag{1.36}$$

Consider two disjoint systems A and B, but also allow these to be entangled with a third system C. (We can call C a 'heat bath' that is interacting with A and B.) Then we have the subadditivity relation

$$S_{ent}(A+B) \quad \leq \quad S_{ent}(A) + S_{ent}(B) \tag{1.37}$$

Note that if there was no system C, then by (1.36) we would get $S_{ent}(A+B) = 0$, and the inequality would be trivial.

We also have the 'triangle inequality'

$$S_{ent}(A+B) \quad \geq \quad \Big| S_{ent}(A) - S_{ent}(B) \Big| \tag{1.38}$$

The relation we will really be using is called the strong-subadditivity of quantum entanglement entropy. Consider three disjoint systems A, B, C, entangled with a fourth system D. Then

$$S_{ent}(A+B) + S_{ent}(B+C) \quad \leq \quad S_{ent}(A) + S_{ent}(C) \tag{1.39}$$

Again, we note that even though the relation involves only A, B and C, it would be trivially true if there were no system D; this is because in that case (1.36) would give $S_{ent}(A+B) = S_{ent}(C)$ and $S_{ent}(B+C) = S_{ent}(A)$.

1.4 Proof of the small corrections theorem

We can now return to our goal: to prove that small corrections to the leading order Hawking state (1.2) cannot remove the growing entanglement between the emitted radiation and the remaining hole.

Consider first the entanglement in the leading order state (1.2).

(a) Let the quanta emitted in emission steps $1, 2, \dots N$ be denoted $\{b_1, b_2, \dots b_N\} \equiv \{b\}$. The entanglement of the radiation with the hole at step N is then

$$S_N = S(\{b\}) \tag{1.40}$$

where $S(A)$ for any set A denotes the entanglement of A with the remainder of the system.

(b) The bits in the hole evolve to create an 'effective bit' b_{N+1} and an 'effective bit' c_{N+1}. (The bit b_{N+1} has not yet left the region $r < 10M$.) The entanglement of the earlier emitted quanta $\{b\}$ does not change in this evolution.

(If two parts of a system are entangled, and we make a unitary rotation on one part, the entanglement between the parts does not change.)

(c) The effective bits b_{N+1}, c_{N+1} must approximate the properties of the Hawking pair (**??**). In (**??**) we have $S(b_{N+1}, c_{N+1}) = 0$, since the pair is not entangled with anything else. We also have $S(c_{N+1}) = \ln 2$. Thus for our model we must have

$$S(b_{N+1} + c_{n+1}) < \epsilon_1 \tag{1.41}$$

$$S(c_{N+1}) > \ln 2 - \epsilon_2 \tag{1.42}$$

for some $\epsilon_1 \ll 1$, $\epsilon_2 \ll 1$.

(d) The bit b_{N+1} now moves out to the region $r > 10M$. The value of S_{ent} at timestep $N+1$ is

$$S_{N+1} = S(\{b\} + b_{N+1}) \tag{1.43}$$

since now b_{N+1} has joined the earlier quanta $\{b\}$ in the outer region $r > 10M$.

(e) We now recall the strong subadditivity relation

$$S(A + B) + S(B + C) \geq S(A) + S(C) \tag{1.44}$$

We wish to set $A = \{b\}$, $B = b_{n+1}$, $C = c_{N+1}$. We note that these sets are made of independent bits: (i) The quanta $\{b\}$ have already left the hole and are far away (ii) The quantum b_{n+1} is composed of some bits, but as it moves out to the region $r > 10M$, it is independent of the bits remaining in the hole and also the bits $\{b\}$ (iii) The quantum c_{N+1} is made of bits which are left back in the hole. Applying the strong subadditivity relation, we get

$$\begin{aligned} S(\{b_i\} + b_{N+1}) + S(b_{N+1} + c_{N+1}) &\geq \\ S(\{b_i\}) &+ S(c_{N+1}) \end{aligned} \tag{1.45}$$

Using (1.41),(1.42),(1.43) we get

$$S_{N+1} > S_N + \ln 2 - (\epsilon_1 + \epsilon_2) \tag{1.46}$$

Thus for $\epsilon_1, \epsilon_2 \ll 1$, the entanglement keeps growing in the manner of fig.**??**(b) and cannot behave like that of a normal body (fig.**??**(a)).

Thus we conclude that having an 'approximate emergent space-time' instead of the smooth space-time used in Hawking's original calculation [**?**] does not resolve the information paradox.

Bibliography