Cosmological Issues

1 Radiation dominated Universe

Consider the stress tensor of a fluid in the local orthonormal frame where the metric is η_{ab}

$$T_{ab} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{1}$$

We do not often go to such a frame in actual computations, but quite often we have a metric that is diagonal. For example, in Cosmology we will work with the metric

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] \tag{2}$$

When we change coordinates to arbitrary new coordinates, the above stress tensor will no longer be diagonal in general. But when the metric is diagonal, we do have nice values for the components $T_a{}^b$. We will get

$$T_a{}^b = \frac{\partial x^c}{\partial \xi^a} \frac{\partial \xi^b}{\partial x^d} T_c{}^d \tag{3}$$

If the coordinate transformation between the local orthonormal frame and the actual coordinates is diagonal, then we get

$$T_a{}^b = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{4}$$

Let us now consider the conservation equation $T_a{}^b{}_{;b} = 0$. We get

$$T_0{}^b{}_{;b} = -\Gamma^c_{0b} T_c{}^b + \frac{1}{\sqrt{-g}} (T_0{}^b \sqrt{-g})_{,b} = 0 \tag{5}$$

We have

$$\Gamma^i_{0j} = \delta^i_j \frac{\dot{a}}{a}, \quad \sqrt{-g} = a^{D-1}, \quad T_0{}^0 = -\rho, \quad T_i{}^j = \delta^i_j p \tag{6}$$

and we find

$$-(D-1)p\frac{\dot{a}}{a} - \frac{1}{a^{D-1}}(\rho a^{D-1})_{,t} = 0 \tag{7}$$

Let us now take the equation of state

$$p = w\rho \tag{8}$$

where w is a constant. Then we get

$$\dot{\rho} + \frac{\dot{a}}{a}\rho(D-1)(1+w) \tag{9}$$

which has the solution

$$\rho = \frac{C}{a^{(D-1)(1+w)}} \tag{10}$$

For radiation, we have $w = \frac{1}{D-1}$, and we get

$$\rho = \frac{C}{a^D} \tag{11}$$

In particular, for our $3+1$ dimensional Universe, we get

$$\rho = \frac{C}{a^4} \tag{12}$$

2 Expansion of the radiation filled Universe

Let us solve for the expansion of the Universe with radiation. We have

$$G_{tt} = 3(\frac{\dot{a}}{a})^2 = 8\pi G T_{tt} = 8\pi G \frac{A}{a^4} \tag{13}$$

This gives

$$\dot{a}^2 a^2 = \frac{8\pi G A}{} \tag{14}$$

which has the solution

$$a = (\frac{32\pi G A}{3})^{\frac{1}{4}} t^{\frac{1}{2}} \tag{15}$$

We thus find

$$\rho = \frac{A}{a^4} = \frac{3}{32\pi G}\frac{1}{t^2} \tag{16}$$

Restoring powers of c, we have

$$\rho = \frac{3c^4}{32\pi G}\frac{1}{t^2} \tag{17}$$

We can relate this energy density to temperature. The energy density of black body radiation is given by

$$\rho = \frac{\pi^2 k^4}{30(\hbar c)^3} f T^4 \tag{18}$$

where f is the number of massless 'species'. For photons, we have 2 polarizations, so $f = 2$. Thus given f for the early Universe, we can find the temperature T as a function of time t after the big bang.

3 Light propagation in the Universe

We will soon see that besides the radiation dominated and matter dominated phases of expansion, we will also look for an *exponentially* expanding 'inflation' phase. To see why we might be interested in such an exponentially inflating phase in our Universe, let us look at how signals propagate in Cosmology. Let us start again with the ansatz

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] \tag{19}$$

A particle can be 'at rest' at a location (x_0, y_0, z_0); its worldline would just move in the t direction as the Universe expands. Thus we call the coordinates x, y, z *comoving* coordinates. The *distance* between two values of comoving coordinates will increase as the Universe expands. Thus two particles of dust can sit at two different fixed values of the comoving coordinates, and their separation will keep growing with the scale factor of the Universe.

Now consider a particle at comoving coordinates (x_0, y_0, z_0) at time t_0, and suppose that it emits a photon at this time. The photon can head in any direction in the x, y, x space; by choice of coordinates we can assume that it is heading in the x direction. By symmetry, it cannot bend its path towards either positive or negative y, or towards positive or negative z, so its direction of motion will remain in the x direction as the Universe expands. We now ask: where will the photon be at time t?

Since the photon travels along a null geodesic, we get

$$ds^2 = -dt^2 + a^2(t)dx^2 = 0 \tag{20}$$

which gives

$$dx = \frac{dt}{a(t)}, \quad x(t) = \int_{t_0}^{t} \frac{dt}{a(t)} \tag{21}$$

Thus given the expansion $a(t)$, we can find out where the photon would be at any time t.

One quantity we can immediately compute with the above expression is the comoving distance travelled by a photon from the time the Universe began. Thus let the photon start at $(x, y, z) = (0, 0, 0)$, at time $t = 0$ which we take to be the 'big bang' when $a(t) = 0$. Then we will have

$$x = \int_0^t \frac{dt}{a(t)} \tag{22}$$

Suppose that we have a dust filled Universe with $a = a_0 t^{\frac{2}{3}}$. Then we will get

$$x = a_0 \int_0^t \frac{dt}{t^{\frac{2}{3}}} = 3a_0^{-1} t^{\frac{1}{3}} \tag{23}$$

No influence can spread out from the point $(0, 0, 0)$ faster than the speed of light, so the entire zone that can be influenced by this point at time t is covered by the points

$$r = \sqrt{x^2 + y^2 + z^2} \leq 3a_0^{-1} t^{\frac{1}{3}} \tag{24}$$

We see that this zone of influence has zero comoving size at $t = 0$, and then grows with time.

The early Universe is actually *radiation* filled, which gives $a = a_0 t^{\frac{1}{2}}$. We again get a similar behavior for the zone of influence

$$x = a_0 \int_0^t \frac{dt}{t^{\frac{1}{2}}} = 2a_0^{-1} t^{\frac{1}{2}} \tag{25}$$

The problem arises when we look at the sky today. We get microwave background photons streaming to us from all points on the sky. Suppose one photon comes from a point (x_1, y_1, z_1), and another from a point (x_2, y_2, z_2). These photons have been streaming to us unimpeded from a definite time t_d: the 'decoupling' time when the matter density fell to a value so low that the typical photon did not interact with anything after that and is being picked up in our detector today. We should now ask if these two photons should come from regions with the *same* temperature. This temperature can be same if these two points (x_1, y_1, z_1) and (x_2, y_2, z_2) could communicate with each other in the past, and in some way came to a temperature equilibrium with each other. This would need that

$$|\vec{x}_2 - \vec{x}_1| \le 3a_0^{-1} t_d^{\frac{1}{3}} \equiv d_{max} \tag{26}$$

where we have used the dust filled Universe expansion for illustration. If on the other hand we have points separated by *more* than this distance, then there is no reason for the temperatures of the two photons to agree.

Let us ask how far apart two points would look in the sky if their comoving coordinates were separated by d_{max}. The photons coming from $\vec{x}_1$ has, in reaching to us today (time t), travelled a comoving distance

$$d_{travel} = \int_{t_d}^t \frac{dt}{a(t)} = 3a_0^{-1}[t^{\frac{1}{3}} - t_d^{\frac{1}{3}}] \tag{27}$$

Thus for the angular separation of the two points $\vec{x}_1, \vec{x}_2$ we get

$$\Delta\theta \approx \frac{d_{max}}{d_{travel}} = \frac{t_d^{\frac{1}{3}}}{t^{\frac{1}{3}} - t_d^{\frac{1}{3}}} \approx \Big(\frac{t_d}{t}\Big)^{\frac{1}{3}} \tag{28}$$

where in the last step we have used the fact that $t \gg t_d$. We have

$$t_d \approx 3 \times 10^5 \ yrs, \quad t \approx 3 \times 10^{10} \ yrs \tag{29}$$

so that we get

$$\Delta\theta \sim .02 \ radians \approx 1.2^0 \tag{30}$$

Thus the sky should show uniformity of temperature only across a few degrees at most. But observations tell us that the entire sky has the same temperature to a first approximation. How can we understand that?

4 Inflation

Let us first see how we can get an exponentially expansing scale factor in our Universe; next w will see how it solves some of the difficulties with our picture of Cosmology.

Consider a scalar field with action

$$S = \int d^d\xi\sqrt{-g}L = \int d^d\xi\sqrt{-g}[-\frac{1}{2}\phi_{,a}\phi^{,a} - V(\phi)] = \int d^d\xi\sqrt{-g}[-\frac{1}{2}\phi_{a}\phi_{,b}g^{ab} - V(\phi)] \quad (31)$$

The equation of motion is

$$\phi_{,a}{}^{;a} - \frac{\partial V}{\partial \phi} = 0 \quad (32)$$

We have

$$\delta S = \int d^d\xi \Big([-\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}][-\frac{1}{2}\phi_{,c}\phi^{,c} - V(\phi)] - \frac{1}{2}\sqrt{-g}\phi_{,a}\phi_{,b}\delta g^{ab} \Big) \quad (33)$$

Writing

$$\delta S = -\int d^d\xi\sqrt{-g}\frac{1}{2}T_{ab}\delta g^{ab} \quad (34)$$

We find

$$T_{ab} = \phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi_{,c}\phi^{,c} - g_{ab}V(\phi) \quad (35)$$

Suppose that $V(\phi)$ has a minimum at ϕ_0

$$\frac{\partial V}{\partial \phi}(\phi_0) = 0, \quad \frac{\partial^2 V}{\partial \phi^2}(\phi_0) > 0 \quad (36)$$

From the first of these equations, we see that a solution to the field equation (32) is

$$\phi = \phi_0 \quad (37)$$

For this solution we have the stress tensor

$$T_{ab} = -g_{ab}V(\phi_0) \quad (38)$$

Thus the stress tensor is proportional to the metric.

Let us now see what the Universe may look like with such a stress tensor. Let us try the ansatz of a Universe with flat spatial slices

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] \quad (39)$$

We have

$$\begin{aligned} G_{tt} &= 3(\frac{\dot a}{a})^2 = 8\pi G V(\phi_0) \\ G_{xx} &= -2a\ddot a - \dot a^2 = -8\pi G a^2 V(\phi_0) \quad (40) \end{aligned}$$

with $G_{yy} = G_{zz} = G_{xx}$. The first equation has the solution

$$a = a_0 e^{H_0 t} \tag{41}$$

where a_0 is a constant and

$$H_0 = \sqrt{\frac{8\pi G}{3} V(\phi_0)} \tag{42}$$

We then observe that the second equation is satisfied as well, so we have a valid solution of Einstein's equations. This is the inflating Universe.

5 The effect of inflation

We had seen above that based on the traditional big band scenario with its radiation and matter dominated phases we had a problem with understanding how the Universe could be homogeneous across the entire width of the visible sky. But if can add an inflating phase in our Cosmology, then small distances get 'stretched' to larger distances, and homogeneity across small scales can become homogeneity across much larger scales.

We split the evolution of the Universe into different stages, and study each in turn.

5.1 From $t = 0$ to the GUT era

The first phase of the Universe is a radiation dominated phase, which we follow from time $t = 0$ to $t = t_{GUT}$, the time when strong, weak and electromagnetic interactions are all equally strong. At this time, the comoving distance across which physics could have equilibriated is given by (25)

$$|\vec{x}_2 - \vec{x}_1| = 2ca_0^{-1} t_{GUT}^{\frac{1}{2}} \tag{43}$$

where we have now restored the factor of c since we will be computing actual lengths in what follows. The physical size of this comoving interval is

$$d_{GUT} = a(t_{GUT})|\vec{x}_2 - \vec{x}_1| = a_0 t_{GUT}^{\frac{1}{2}} 2ca_0^{-1} t_{GUT}^{\frac{1}{2}} = 2ct_{GUT} \tag{44}$$

Let us now see what the value of t_{GUT} is. The GUT energy scale is 10^{15} Gev, 4 orders below the planck scale of 10^{19} GeV. Thus the temperature at the GUT scale is $10^{15} Gev$, and the wavelength of typical quanta is

$$\lambda_{GUT} \sim 10^4 l_p \sim 10^4 \times 10^{-33} cm \sim 10^{-29} cm \tag{45}$$

What is the time t_{GUT}? We have for an radiation filled Universe

$$\rho \sim T^4, \quad (\frac{\dot{a}}{a})^2 \sim \frac{1}{t^2} \sim G\rho \sim GT^4 \tag{46}$$

Thus

$$T \sim \frac{1}{(Gt^2)^{\frac{1}{4}}} \tag{47}$$

In units where $c = 1, \hbar = 1$ we have $G = l_p^2$, and we see that at $t = t_p$ we have $T = \frac{1}{l_p}$. Thus at planck time we have planck temperature. But thereafter the temperature falls as $\frac{1}{t^{\frac{1}{2}}}$, so it drops by the factor 10^4 needed to reach the GUT scale after at

$$t_{GUT} \sim t_p(10^4)^2 \sim 10^{-44} \times 10^8 sec \sim 10^{-36} sec \tag{48}$$

The horizon scale is then

$$d_{GUT} \sim 2ct_{GUT} \sim 2 \times 3 \times 10^{10} \times 10^{-36} cm \sim 10^{-25} cm \tag{49}$$

5.2 The inflationary phase

Next we assume that the Universe has a scalar field which leads to inflation. In this phase we have seen that the scale factor will expand like

$$a = \tilde{a}_0 e^{H_0 t} \tag{50}$$

Each time the coordinate t increases by H_0^{-1}, the Universe grows in size by a factor e. We call this one 'e-fold'. Suppose we have n e-folds. Then the distance d_{GUT} expands to a length

$$d_{end} = e^n d_{GUT} \tag{51}$$

at the end of inflation. During inflation, the temperature of the Universe remains low because all the energy is bound up in the scalar field vacuum energy. But when inflation ends, this energy is released into heat. The resulting temperature is again t_{GUT}, the temperature before inflation began. The reason for this is simple. The temperature $TGUT$ allowed the field to achieve a vacuum state where the energy was also of order T_{GUT}. During inflation the vacuum energy density does not change. Thus when inflation ends, we get a thermal bath with temperature T_{GUT} again.

5.3 Expansion after inflation

After the end of inflation, the Universe expands again, to reach its present size. The size of the visible Universe today is 3000 Mpc, where 1 pc is 3×10^{18} cm. This gives our present horizon a size

$$d = 10^{28}\ cm \tag{52}$$

How much has the Universe expanded since the end of inflation? Part of this expansion has been radiation dominated and part matter dominated, but there is a simple way to get the overall expansion factor. The temperature at the GUT era was 10^{15} GeV. The relics of this temperature are the microwave photons today, with a temperature $2.7^0 K \sim 10^{-4}\ eV$. We have seen that the wavelength of the photons increases linearly with the scale factor a. Thus the scale factor has increased since the end of inflation by a factor

$$\frac{a(t)}{a(t_{end})} \sim \frac{10^{15} \times 10^9}{10^{-4}} = 10^{28} \tag{53}$$

5.4 Inflation and the horizon problem

Now we come to the effect of inflation on the horizon problem. We have assumed that at time t_{GUT} we have a horizon size given by the distance that light could travel from $t = 0$ to t_{GUT} in a radiation dominated expansion. This gave a horizon distance d_{GUT}. We will not look for any further effects of communication between different regions, but ask that this region of size t_{GUT} expand to the present day horizon distance. We will get a factor e^f from inflation, and a factor 10^{28} from the time since the end of inflation. Thus the overall expansion factor is $e^f \times 10^{28}$. This expansion should convert d_{GUT} to d, which is a ratio of

$$\frac{d}{d_{GUT}} = \frac{10^{28}\ cm}{10^{-25}\ cm} \sim 10^{53} \tag{54}$$

Thus we need

$$e^f \times 10^{28} \sim 10^{53}, \quad e^f \sim 10^{25}, \quad f \sim 58 \tag{55}$$

If we had been more careful with our factors of 2 etc, we would actually find $f \sim 60$. Thus if we had more than 60 e-folds of inflation, then we would solve the horizon problem; i.e., the Universe can be expected to look homogeneous across the $3000 Mpc$ that we are able to see in the sky today.

6 The flatness problem

Inflation also solves the flatness problem, which we discuss now.

6.1 Curvature in the Universe

In our discussions we have used the ansatz for the metric which has flat spatial section. The reason for this is that the Universe we observe around us indeed appears to be flat, or at least so close to flat that we cannot tell it apart from a flat Universe. But we could have had instead one of the other two Cosmologies; one with positively curved spheres as spatial slices, or one with negatively curved hyperboloids. Let us take the positive curvature Universe as an illustration and see what the flatness problem is.

For the closed Universe the equation $G_{tt} = 8\pi G T_{tt}$ gives

$$(\frac{\dot{a}}{a})^2 + \frac{1}{a^2} = \frac{8\pi G}{3}\rho \tag{56}$$

which we can write as

$$\dot{a}^2 + 1 = \frac{8\pi G}{3}\rho a^2 \tag{57}$$

Let us get a rough physical interpretation of the terms in this equation. Write the RHS as

$$\frac{8\pi G}{3}\frac{\rho a^3}{a} \sim \frac{GM}{a} \tag{58}$$

where $\rho a^3 \sim M$ is the mass in the ball. Recall that $\frac{GM}{r}$ gives the potential energy per unit mass. (Note that this quantity has no unit, since $c = 1$.) Thus the RHS is like a potential energy term. The first term on the LHS is now seen to be like a kinetic energy per unit mass, since $\dot{a}$ gives the 'velocity of expansion' of the matter in the Universe.

Now suppose we have a radiation dominated Universe with $\rho = \frac{A}{a^4}$. Then we have $\frac{8\pi G}{3}\rho a^2 = \frac{8\pi GA}{3a^2}$. At very small a we see that this goes to infinity. To balance (57) we need $\dot{a}^2$ to diverge as well. Thus for small a we see that the second term on the LHS of (57) becomes irrelevant, and we get

$$\dot{a}^2 \approx \frac{8\pi G}{3}\rho a^2 \tag{59}$$

which is the equation for a *flat* Universe.

On the other hand for larger a the closed Universe does not behave like a flat one. At some point we reach a value $a = a_{max}$ where

$$\frac{8\pi GA}{3a_{max}^2} = 1 \tag{60}$$

At this point we see that $\dot{a}$ must vanish, and the Universe turns back and starts contracting towards smaller a. We now ask: What should be the value of a_{max}?

Note that a has the units of length. From the constants G, c available in our physical set up, we cannot make a quantity with the units of length, so we cannot guess at a typical turning lengthscale a_{max} from these constants. If we include $\hbar$ however, we can make a quantity with the units of length – planck length l_p

$$l_p \sim \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33} cm \tag{61}$$

Thus we may expect, on the grounds of naturalness, that the Universe will look flat for a much smaller than the turning point, but at around $a = l_p$ the Universe will turn over and recollapse.

But of course the Universe around us is very large ($10^{28} cm$ as we saw above). How should we explain this?

6.2 The GUT era

Let us assume that in some way we manage to extend the life of out Universe to to time t_{GUT}. We have

$$t_{GUT} = 10^8 t_p \sim 10^{-36} sec \tag{62}$$

At this point since the Universe is recollapsing, all terms in (56) are comparable; in particular

$$(\frac{\dot{a}}{a})^2 \sim \frac{1}{t_{GUT}^2} \sim \frac{1}{a^2} \tag{63}$$

so that the curvature radius of the Universe is

$$a_{GUT} \sim ct_{GUT} \sim 10^8 ct_p cm \sim 3 \times 10^{-26} cm \tag{64}$$

where we have restored the factor c in our equation. Thus at t_{GUT} the spatial slices of the Universe look flat if we look at distances much less than a_{GUT}, but we do see significant curvature over distances of order a_{GUT}. Note that

$$a_{GUT} \sim d_{GUT} \tag{65}$$

6.3 The effect of inflation

At this GUT era though we can enter into a phase of inflation. This expands the radius a_{GUT} by a factor e^f. The remaining expansion to the present day supplies a factor 10^{28}, as we saw above. At the end of this expansion, all we ask is that the scale on which we could see curvature equal the scale of our present visible Universe (or greater), since we see no clear evidence for curvature across our present horizon. Since the present horizon radius is $10^{28} cm$, we ask for

$$a_{GUT} e^f 10^{28} \sim 10^{28} cm, \quad e^f \sim \frac{1}{3 \times 10^{-26}}, \quad f \sim 59 \tag{66}$$

so that we again get a requirement $f \gtrsim 60$.

7 Restoring factors of c

Let us begin with the metric. In flat space we write

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{67}$$

Thus ds has units of length (L). The volume element $\sqrt{-g} d^4 x$ will have units of L^4. The action has units $\hbar = ET$ where E is energy and T is time. The action for gravity is

$$\frac{c^4}{16\pi G} \int \frac{1}{c} \sqrt{-g} d^4 \xi R \tag{68}$$

where we have used that R has units $\frac{1}{L^2}$. For matter we study the units of action by looking at the case of dust

$$S = \int \frac{1}{2} m v^2 dt \to \int \frac{1}{2} \rho v^2 \frac{1}{c} \sqrt{-g} d^4 x \tag{69}$$

The stress tensor is given by

$$\delta S = -\int \frac{1}{c} d^4 \xi \sqrt{-g} \frac{1}{2} T_{ab} \delta g^{ab} \tag{70}$$

T_{tt} for the flat metric above has units (note that g^{tt} has units $\frac{1}{c^2}$)

$$T_{tt} \sim (ET) \frac{1}{c^2} \frac{c}{L^4} = \frac{M}{L^3} c^4 \sim \rho c^4 \tag{71}$$

Note that now we will have $T_t{}^t = \rho c^2$, which gives a traceless tensor for radiation with $T_x{}^x = p = \frac{1}{3} \rho c^2$. The Einstein equation now becomes

$$G_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab} \tag{72}$$

8 Amplitude of the perturbation

The action for the scalar field is

$$S = \int d^4 x \sqrt{-g} [\frac{1}{2} \phi_{,a} \phi^{,a} - V(\phi)] \tag{73}$$

Let $\phi = \phi_0$ be the minimum of V, and look at small fluctuations around $\phi = \phi_0$. We write

$$\phi = \phi_0 + f(t) e^{ikx} \equiv \phi_0 + \phi_1 \tag{74}$$

The leading order solution $\phi = \phi_0$ causes the Universe to expand as

$$a = a_0 e^{Ht} \tag{75}$$

where

$$H = \sqrt{\frac{8\pi G V(\phi_0)}{3}} \tag{76}$$

We ignore the backreaction of the small perturbation ϕ_1 and examine the evolution of ϕ_1 on the inflating metric.

8.1 Estimating the amplitude of fluctuations

Let us assume that V is quite flat around its minimum, so we ignore the effect of V on the evolution of ϕ_1. The waveequation for ϕ_1 gives

$$\ddot{f} + 3H\dot{f} + \frac{k^2}{a^2}f = 0 \tag{77}$$

First assume that $\frac{k}{a} \gg H^{\frac{1}{2}}$. Then we have to solve the equation

$$\ddot{f} + \frac{k^2}{a^2}f = 0 \quad \rightarrow \quad f = e^{-i\frac{k}{a}t} \tag{78}$$

Thus

$$\phi_1 = Ae^{ikx}e^{-i\frac{k}{a}t} \tag{79}$$

Let us estimate the amplitude A arising from vacuum fluctuations. The wavelength is $\Delta x = k^{-1}$. Consider a region with size $\Delta x = \frac{k}{a}$ on each side. This has energy

$$E \sim \frac{A^2}{\lambda^2}V \sim \frac{A^2}{(\frac{a}{k})^2}(\frac{a}{k})^3 \sim A^2\frac{a}{k} \tag{80}$$

This should equal the energy $\frac{k}{a}$ that we get from the frequency of the mode. This gives

$$A^2\frac{a}{k} \sim \frac{k}{a}, \quad A \sim \frac{k}{a} \tag{81}$$

We see that a a increases, the amplitude keeps dropping. But when the wavelength $\lambda \sim \frac{a}{k}$ reaches the the scale H^{-1}, we can ignore the term $\frac{k^2}{a^2}f$ and keep the term $3H\dot{f}$ instead. Then the equation becomes

$$\ddot{f} + 3H\dot{f} = 0 \tag{82}$$

which has the solution

$$f = A + Be^{-3Ht} \tag{83}$$

The decaying part quickly dies, and we are left with $f = A$. Thus the amplitude stops decreasing and becomes a constant. The value of this a at the point is given by

$$\frac{k}{a} \sim H, \quad a \sim kH^{-1} \tag{84}$$

which gives

$$A \sim \frac{k}{a} \sim H \tag{85}$$

Thus all modes freeze at an amplitude $A \sim H$ after they stretch to a size

$$\lambda \sim a\Delta x \sim kH^{-1}k^{-1} \sim H^{-1} \tag{86}$$

We say that all modes stretch to the Hubble length and then freeze in amplitude.

Let us compute the energy in the mode when it has wavelength $\lambda' \gg H^{-1}$. The amplitude is fixed to H, so the energy is

$$E \sim \frac{A^2}{\lambda'^2}\lambda'^3 \sim H^2\lambda' \tag{87}$$

Thus as the mode stretches, its energy increases.

We can solve the following equation, which corresponds to inflation with flat potential

$$\ddot{f} + 3H\dot{f} + k^2 e^{-2Ht} f = 0 \tag{88}$$

The solution is

$$f = A[-\frac{k}{a}\cos\frac{k}{aH} + H\sin\frac{k}{aH}] + B[\frac{k}{a}\sin\frac{k}{aH} + H\cos\frac{k}{aH}] \tag{89}$$

For large k we first see the expected $\frac{1}{a}$ falloff, and then the freeze out. Since we expect the amplitude $\frac{k}{a}$ at small wavelengths, we see that $A, B \sim 1$, and then we see that the freeze out happens at $\phi \sim H$.

9 Slow roll inflation

Now assume that the potential $V(\phi)$ is nontrivial, and causes ϕ to roll to its minimum from a value that is not at the minimum. The equation for ϕ is

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0 \tag{90}$$

Let us see when the first term can be ignored. In this situation, we have

$$3H\dot{\phi} = -V' \tag{91}$$

Thus assuming that H does not change,

$$3H\ddot{\phi} = -V''\dot{\phi} = V''\frac{V'}{3H}, \quad \ddot{\phi} = \frac{V''V'}{9H^2} \tag{92}$$

Thus we can ignore the first term if

$$\frac{V''\dot{\phi}}{3H} < 3H\dot{\phi}, \quad V'' < 9H^2 \tag{93}$$

Let us see how this would work for $V = \lambda\phi^4$. Then

$$V'' \sim \lambda\phi^2 < H^2 \sim GV \sim G\lambda\phi^4 \tag{94}$$

Thus to be able to ignore the first term we need

$$G\phi^2 > 1 \tag{95}$$

This means that $\phi > m_p$, the planck scale, so it is larger than the GUT scale.

10 The number N of e-folds

We have

$$N = \int d(\ln a) = \int dt \frac{d \ln a}{dt} = \int dt \frac{\dot{a}}{a} = \int dt \sqrt{\frac{8\pi G V}{3}} \tag{96}$$

We can write

$$dt = \frac{dt}{d\phi} d\phi = \frac{d\phi}{\dot{\phi}} \tag{97}$$

so that we get

$$N = \int d\phi \sqrt{\frac{8\pi G V}{3}} \frac{1}{\dot{\phi}} \tag{98}$$

In the slow roll approximation with a large H we have

$$3H\dot{\phi} + V' = 0, \quad \dot{\phi} = -\frac{V'}{3H} \tag{99}$$

Thus

$$N \sim \int \frac{\sqrt{GV}}{\dot{\phi}} d\phi \sim \int \frac{\sqrt{GV} H}{V'} d\phi \sim \int \frac{GV}{V'} d\phi \tag{100}$$

11 Putting it all together

Let us take a potential

$$V = \lambda \phi^4 \tag{101}$$

We see that λ has no units, so a natural value would be $\lambda \sim 1$. As it will turn out, we will need to take $\lambda \ll 1$, which will be a 'fine-tuning' problem associated with inflation.

11.1 The constraint from the number of e-folds

We have

$$N \sim \int \frac{GV}{V'} d\phi \sim G\phi^2 \tag{102}$$

where we have taken the limits of the range of the integral to start at some value ϕ typical of the inflation process and to end at $\phi = 0$. Noting that $G \sim m_p^{-2}$, and recalling that ϕ has units of mass, we learn that

$$\frac{\phi}{m_p} \sim N^{\frac{1}{2}} > \sqrt{60} \tag{103}$$

Note that the condition $\phi \gg m_p$ is also the condition needed to ignore the $\ddot{\phi}$ term in the evolution equation for ϕ.

Thus while we may have expected the value of ϕ to be near the GUTS scale, we find that it actually has to be much higher than the planck scale. This large value of ϕ does not necessarily mean that the potential V caused by ϕ is larger than the planck scale; we can make it of order the GUTS scale if we choose λ small enough.

11.2 The constraint from the magnitude of fluctuations

The fractional perturbations are given as follows. At the time that a mode leaves the horizon and gets frozen in amplitude, we have seen that its amplitude has fluctuations $\delta\phi \sim H$. Because we start with a slightly different value of ϕ at different places, we will fall to the bottom of the potential and cease inflating after slightly different times at different places. This in turn means that we will inflate by different amounts at different places. Let us compute these differences.

We have

$$\delta t \sim \frac{\delta\phi}{\dot{\phi}} \sim \frac{H}{\dot{\phi}} \sim \frac{H^2}{V'} \tag{104}$$

where $\dot{\phi}$ is computed at the time that the mode freezes. The scale factor increases as $a = a_0 e^{Ht}$, so a small change δt in the total inflation time causes a change in volume

$$\frac{\delta v}{v} \sim \frac{\delta a}{a} \sim \frac{e^{H(t+\delta t)} - e^{Ht}}{e^{Ht}} \sim H\delta t \tag{105}$$

Thus we get for the fractional density change caused by quantum fluctuations

$$\delta \sim H\delta t \sim \frac{H^3}{V'} \tag{106}$$

Observations show that $\delta \sim 10^{-5}$. Applying this to our potential, we have

$$\delta \sim \frac{(GV)^{\frac{3}{2}}}{V'} \sim \frac{\lambda^{\frac{1}{2}}\phi^3}{m_p^3} \tag{107}$$

Thus we see that

$$\lambda \sim (\frac{m_p}{\phi})^6\delta^2 \sim N^{-3}\delta^2 < (60)^{-3}(10^{-5})^2 \sim 10^{-15} \tag{108}$$

12 Tensor perturbations

These are perturbations of gravity, of the form

$$g_{ab} = \bar{g}_{ab} + h_{ab} \tag{109}$$

The action for gravity has the form

$$\frac{1}{16\pi G}R\sqrt{-g}d^4x \approx m_p^2\int d^4x\sqrt{-g}\frac{1}{2}\partial_a h_{bc}\partial^a h_{b'c'}g^{bb'}g^{cc'} \tag{110}$$

This is like the action of a scalar field, if we write

$$m_p {h_a}^b \sim \phi \tag{111}$$

Since ${g_a}^b = \delta_a^b$, we see that ${h_a}^b$ give the fractional fluctuations of the metric. We have learnt that the quantum fluctuations of a scalar field like ϕ get frozen to a value $\delta\phi \sim H^{-1}$ during inflation. Thus we expect

$$\delta_h \sim \frac{H}{m_p} \tag{112}$$

But

$$H \sim \sqrt{GV} \sim \frac{V^{\frac{1}{2}}}{m_p} \tag{113}$$

Thus

$$\delta_h \sim \frac{V^{\frac{1}{2}}}{m_p^2} \tag{114}$$

Since we expect that $V \sim m_{GUT}^4$, we get

$$\delta_h \sim (\frac{m_{GUT}}{m_p})^2 \sim 10^{-8} \tag{115}$$

Thus these perturbations are very hard to detect.