1 Green's functions

The harmonic oscillator equation is

$$m\ddot{x} + kx = 0 \tag{1}$$

This has the solution

$$x = A\sin(\omega t) + B\cos(\omega t), \quad \omega = \sqrt{\frac{k}{m}} \tag{2}$$

where A, B are arbitrary constants reflecting the fact that we have two arbitrary initial conditions (position and velocity).

Suppose we have a forced harmonic oscillator

$$m\ddot{x} + kx = F(t) \tag{3}$$

How do we obtain the solution, if we are given $F(t)$?

First we note that suppose someone did give us one solution of this equation

$$m\ddot{x}_p(t) + kx_p(t) = F(t) \tag{4}$$

then we could find other solutions by adding solutions of the free equation

$$x(t) = x_p(t) + A\sin(\omega t) + B\cos(\omega t) \tag{5}$$

This reflects the fact that we still have two initial conditions that we can chose for the problem.

But how do we find $x_p(t)$? This will depend on $F(t)$, and since there can be an infinite number of choices for $F(t)$ it may look hopeless to write a solution to this problem. But there is a simplification because the LHS of (3) is *linear* in x. Thus suppose we have

$$m\ddot{x}_1(t) + kx_1(t) = F_1(t) \tag{6}$$

$$m\ddot{x}_2(t) + kx_2(t) = F_2(t) \tag{7}$$

Then

$$m\frac{d^2}{dt^2}[x_1(t) + x_2(t)] + k[x_1(t) + x_2(t)] = F_1(t) + F_2(t) \tag{8}$$

so we can find an answer to the problem with forcing function $F_1 + F_2$ if we knew the solutions to the problems with forcing functions F_1 and F_2 separately.

This suggests that we choose a simple set of forcing functions F, and solve the problem for these forcing functions. Then by adding the results with various proportionality constants we can get the solution to the problem for arbitrary F.

2 Forcing functions of the form $\delta(t-t')$

What is the simplest $F(t)$? We look for an F which is zero everywhere except at one point of time

$$F(t) = \delta(t-t') \tag{9}$$

Even though this function is nonzero only at one point of time, its integral is nonzero

$$\int_{t=-\infty}^{\infty} F(t)dt = 1 \tag{10}$$

Let us look for the solution to the equation

$$m\ddot{x} + kx = \delta(t) \tag{11}$$

We have to choose initial conditions to specify a solution, but we can take any conditions we please, since we can later get any other solution by using (5). Thus let us assume that

$$x(t) = 0, \quad t < 0 \tag{12}$$

This is certainly an allowed solution, since before we apply any force we can imagine that the oscillator is at rest; one may also think of it as a natural condition to take from a physical point of view (this choice of condition will give us a Green's function that will be called the 'retarded Green's function', reflecting the fact that any effects of the force F appear only *after* the force is applied.)

What is $x(t)$ for $t > 0$? There is again no force after $t = 0$, so we will have a solution of the form

$$x = A\sin(\omega t) + B\cos(\omega t), \quad t > 0 \tag{13}$$

where now A, B will be determined by the F that is applied at $t = 0$.

Thus we need 'junction conditions' that will connect the solution at $t < 0$ to the solution at $t > 0$. Such conditions are found by looking at the equation for x

$$m\ddot{x} + kx = \delta(t) \tag{14}$$

Suppose we integrate both ides over a small interval containing the origin. Then we get

$$\int_{t=-\epsilon}^{\epsilon} m\ddot{x}(t)dt + \int_{t=-\epsilon}^{\epsilon} kx(t)dt = \int_{t=-\epsilon}^{\epsilon} \delta(t)dt \tag{15}$$

which gives

$$m\dot{x}|_{t=-\epsilon}^{t=\epsilon} + O(\epsilon) = 1 \tag{16}$$

where the second term on the LHS will be small because

$$|\int_{t=-\epsilon}^{\epsilon} kx(t)| \le \max|x(t)|2\epsilon \tag{17}$$

We take the limit $\epsilon \to 0$, getting

$$m\dot{x}(t = 0^+) - m\dot{x}(t = 0^-) = 1 \tag{18}$$

Since $\dot{x}(t = 0^-) = 0$, we find

$$\dot{x}(t = 0^+) = \frac{1}{m} \tag{19}$$

The variable x itself is expected to be continuous at $x = 0$, since

$$\int_{t=-\epsilon}^{\epsilon} \dot{x}(t)dt = x(t = \epsilon) - x(t = -\epsilon) \tag{20}$$

The LHS goes to zero in the limit $\epsilon \to 0$ since

$$|\int_{t=-\epsilon}^{\epsilon} \dot{x}(t)dt| \leq \max|\dot{x}|(2\epsilon) \tag{21}$$

Thus

$$x(t = 0^+) = x(t = 0^-) = 0 \tag{22}$$

We can now find A, B. Consider the solution (13) which is valid at $t = 0^+$. From (22) we get at $t = 0$

$$B = 0 \tag{23}$$

From (19) we get at $t = 0$

$$A\omega = \frac{1}{m}, \quad \to \quad A = \frac{1}{m\omega} \tag{24}$$

Thus

$$x(t) = \frac{1}{m\omega}\sin(\omega t), \quad t > 0 \tag{25}$$

To summarize, a forcing function $F = \delta(t)$ acting on an oscillator at rest converts the oscillator motion to $x(t) = \frac{1}{m\omega}\sin(\omega t)$. More generally, a forcing function $F = \delta(t - t')$ acting on an oscillator at rest converts the oscillator motion to

$$x(t) = \frac{1}{m\omega}\sin(\omega(t - t')) \tag{26}$$

3 Putting together simple forcing functions

We can now guess what we should do for an arbitrary forcing function $F(t)$. We can imagine that any function is made of delta functions with appropriate weight. Around a point t', imagine a delta function of strength $F(t')$. If we add up such delta functions, then we should get the function F. Thus

$$F(t) = \int_{t'=-\infty}^{\infty} dt' F(t')\delta(t - t') \tag{27}$$

We can now guess that we should take the solution $x(t)$ generated by each such delta function, and add them up. There are two things to note:

(a) Since the delta function at t' has strength $F(t')$ instead of strength unity, we should multiply the solution (26) by $F(t')$ before adding it to the mix.

(b) Suppose we want to find $x(t)$. Then we should take into account the effect of all delta functions at $t' < t$, but not those at $t' > t$.

Thus we should write

$$x_p(t) = \int_{t'=-\infty}^{t} dt' F(t') \frac{1}{m\omega} \sin(\omega(t-t')) \tag{28}$$

This is indeed the solution to the problem, since we can check that it satisfies (3). To check this, note that

$$\begin{aligned} \frac{d}{dt}\int_{t'=-\infty}^{t} dt' F(t') \frac{1}{m\omega} \sin(\omega(t-t')) &= [F(t')\frac{1}{m\omega}\sin(\omega(t-t'))]_{t'=t} + \int_{t'=-\infty}^{t} dt' F(t') \frac{1}{m\omega}\frac{d}{dt}\sin(\omega(t-t')) \\ &= \int_{t'=-\infty}^{t} dt' F(t') \frac{1}{m}\cos(\omega(t-t')) \end{aligned} \tag{29}$$

$$\begin{aligned} \frac{d^2}{dt^2}\int_{t'=-\infty}^{t} dt' F(t') \frac{1}{m\omega} \sin(\omega(t-t')) &= \frac{d}{dt}\int_{t'=-\infty}^{t} dt' F(t') \frac{1}{m}\cos(\omega(t-t')) \\ &= [F(t')\frac{1}{m}\cos(\omega(t-t'))]_{t'=t} + \int_{t'=-\infty}^{t} dt' F(t') \frac{1}{m}\frac{d}{dt}\cos(\omega(t-t')) \\ &= \frac{1}{m}F(t) - \frac{\omega}{m}\int_{t'=-\infty}^{t} dt' F(t')\sin(\omega(t-t')) \end{aligned} \tag{30}$$

$$kx_p(t) = \omega^2 m \int_{t'=-\infty}^{t} dt' F(t') \frac{1}{m\omega} \sin(\omega(t-t')) \tag{31}$$

$$= \omega \int_{t'=-\infty}^{t} dt' F(t') \sin(\omega(t-t')) \tag{32}$$

and we find that

$$m\frac{d^2x_p(t)}{dt^2} + kx_p(t) = F(t) \tag{33}$$

The general solution of the problem is then found by using (5).

4 Defining Green's functions

To make this solution more formal, we define a function

$$G(t,t') \;=\; 0, \quad t < t'$$

$$G(t,t') = \frac{1}{m\omega}\sin(\omega(t-t')) \tag{34}$$

Then we have

$$m\frac{d^2G(t,t')}{dt^2} + kG(t,t') = \delta(t-t') \tag{35}$$

Now suppose we have an arbitrary forcing function $F(t)$. Then we write

$$x_p(t) = \int_{t'=-\infty}^{\infty} G(t,t')F(t')dt' \tag{36}$$

We then check that

$$m\ddot{x}_p(t)+kx_p(t) = m\int_{t'=-\infty}^{\infty}\frac{d^2G(t,t')}{dt^2}F(t')dt'+k\int_{t'=-\infty}^{\infty}G(t,t')F(t')dt' = \int_{t'=-\infty}^{\infty}\delta(t-t')F(t')dt' = F(t) \tag{37}$$

so that $x_p(t)$ is a solution of (3). The general solution will then be given by (5).

5 Perturbation theory

Above we considered a harmonic oscillator that was subject to an external force $F(t)$. More often we have an oscillator that is not subject to an external force, but where the Lagrangian differs by a small amount from that of a harmonic oscillator. Thus consider

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - \epsilon x^3 \tag{38}$$

where ϵ is small. How do we solve this problem?

The equation of motion is

$$m\ddot{x} + kx + 3\epsilon x^2 = 0 \tag{39}$$

We write it with the new term on the RHS

$$m\ddot{x} + kx = -3\epsilon x^2 \tag{40}$$

If we knew the RHS, then we could solve it by the method of Green's functions shown above. Of course we do not know the RHS until we solve for $x(t)$. But if ϵ is small, then we *almost* know the RHS. We first ignore the perturbation and solve the equation

$$m\ddot{x} + kx = 0 \tag{41}$$

This is not the full equation of course, but since ϵ is small it gives a good approximation. We will call the solution $x_0(t)$, to denote the fact that this solution is the zeroth order approximation, and later corrections will be added later. Thus we will have

$$x_0(t) = A\sin(\omega t) + B\cos(\omega t) \tag{42}$$

where A, B are determined from our initial conditions, which we assume are given at $t = t_i$. Thus

$$x(t_i) = A\sin(\omega t_i) + B\cos(\omega t_i), \quad \dot{x}(t_i) = A\omega\cos(\omega t_i) - B\omega\sin(\omega t_i) \tag{43}$$

The full solution $x(t)$ will be written as

$$x(t) = x_0(t) + x_1(t) + x_2(t) + \ldots \tag{44}$$

where the term $x_k(t)$ is order ϵ^k.

Let us now find the first correction $x_1(t)$. In (40) the RHS is

$$-3\epsilon x^2 = -3\epsilon(x_0 + x_1 + x_2 + \ldots)^2 = -3\epsilon[x_0^2 + 2x_0x_1 + (x_1^2 + 2x_0x_2) + \ldots] \tag{45}$$

where we have grouped together terms of different order in ϵ. To find $x_1(t)$ we just keep the lowest term $-3\epsilon x_0^2$ on the RHS. But this is known, since we have chosen $x_0(t)$ above. Since the initial conditions were given at $t = t_i$, we can let the perturbation term act at times after that to determine the solution for $t > t_i$. We then get

$$x_1(t) = \int_{t'=t_i}^{t} G(t, t')[-3\epsilon x_0^2(t)] \tag{46}$$

and the solution to this order is

$$x(t) = x_0(t) + x_1(t) = A\sin(\omega t) + B\cos(\omega t) + \int_{t'=t_i}^{t} G(t, t')[-3\epsilon(A\sin(\omega t') + B\cos(\omega t'))^2] \tag{47}$$

Now that we know $x_0(t), x_1(t)$ we can find out $x_2(t)$, since this needs the forcing function $-6\epsilon x_0 x_1$. Continuing in this way, we can find the answer to any desired order in perturbation theory.