The black hole geometry

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September 5, 2009

1 The Schwarzschild hole

Let us start with the Schwarzschild metric of the 3+1 dimensional black hole

$$ds^2 = -(1 - \frac{2GM}{c^2 r})dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

We will set

$$G = c = \hbar = 1 \quad (2)$$

and write $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ for the metric on the unit 2-sphere S^2. Then (1) becomes

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2 \quad (3)$$

Consider the line

$$r = r_0, \quad \theta = \theta_0, \quad \phi = \phi_0 \quad (4)$$

so that only t changes along this line

(i) For $r > 2M$ the metric along this line gives

$$ds^2 = -(1 - \frac{2M}{r})dt^2 < 0 \quad (5)$$

so this is a timelike line, and can be the worldline of an actual particle.

(ii) For $r < 2M$ we get

$$ds^2 = -(1 - \frac{2M}{r})dt^2 > 0 \quad (6)$$

so this is a spacelike line, and *cannot* be the path of a particle. In other words, a particle cannot sit at constant r, θ, ϕ for $r < 2M$.

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Problem 1: Prove that even if we allow θ, ϕ to change, it remains true that any particle at $r < 2M$ will have to move towards smaller r, and will thus end up at $r = 0$.

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The surface $r = 2M$ is called the *horizon*. Classically (i.e. without quantum effects) no particle can emerge from inside the horizon to the outside.

2 Kruskal coordinates

In the metric (3) we see that there is a problem when $1 - \frac{2M}{r} = 0$, since the coefficient of dt^2 vanishes and the coefficient of dr^2 diverges. Before we know any more, we cannot be sure if this means that the coordinates are bad at this location $r = 2M$ or if the metric has a geometrical singularity of some kind. It will turn out that the singularity at the horizon is only a coordinate singularity. To show this, we need to use coordinates that are well behaved at the horizon. Let us find such coordinates.

(i) First we look at the t, r part of the metric and write

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} = (1 - \frac{2M}{r})[-dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})^2}] \tag{7}$$

We would now like to find a coordinate r^* such that

$$dr^{*2} = \frac{dr^2}{(1 - \frac{2M}{r})^2} \tag{8}$$

This gives the equation $dr^* = \frac{dr}{1 - \frac{2M}{r}}$ which has the solution

$$r^* = \int^r \frac{dr}{1 - \frac{2M}{r}} = \int^r \frac{r}{r - 2M} = \int^r dr[1 + \frac{1}{\frac{r}{2M} - 1}] = r + 2M \ln(\frac{r}{2M} - 1) \tag{9}$$

where we have set the arbitrary additive constant to zero. The metric (3) becomes

$$ds^2 = (1 - \frac{2M}{r})[-dt^2 + dr^{*2}] + r^2 d\Omega_2^2 \tag{10}$$

(ii) Now we move to null coordinates by writing

$$u = t + r^*, \quad v = t - r^* \tag{11}$$

This gives

$$ds^2 = (1 - \frac{2M}{r})[-du dv] + r^2 d\Omega_2^2 \tag{12}$$

(iii) Let us now look at the ranges of these coordinates. Note that the range $r = (2M, \infty)$ maps to $r^* = (-\infty, \infty)$. Now consider a null geodesic falling radially into the hole. Thus θ, ϕ are constant, and the worldline will be given by solving $ds^2 = 0$ in the (t, r^*) space. At infinity where the metric is flat the ingoing geodesic is $t + r = const.$. From (12) we see that taking into account the metric of the hole changes this to

$$t + r^* = u = u_0 \tag{13}$$

By taking geodesics starting from a given r^* with different values of t we see that we can cover the full range $-\infty < u_0 < \infty$ for points outside the horizon. Similarly, $v = t - r^*$ can cover this full range. But note in addition that as the ingoing null geodesic approaches the horizon we get

$$v = t - r^* = u_0 - 2r^* \to \infty \tag{14}$$

In short, the 'future horizon' (i.e. the horizon which is crossed in the future by an observer who decides to fall into the black hole) is given by

$$-\infty < u < \infty, \quad v = \infty \tag{15}$$

(iv) From (15) we see that our coordinates (u, v) 'end' at the horizon. If we wish to see the horizon as a regular region of our manifold, then we would like to have coordinates that smoothly take us across the horizon. Thus we need the horizon to be at *finite* values of our coordinates, unlike (15). Let we write

$$U = e^{\alpha u}, \quad V = -e^{-\alpha v} \tag{16}$$

where we will choose the constant α later. Assuming $\alpha > 0$, we see that the region outside the horizon is

$$U > 0, \quad V < 0 \tag{17}$$

and the horizon itself is

$$0 < U < \infty, \quad V = 0 \tag{18}$$

Thus we have brought the horizon to a finite position in our new coordinates U, V, and if the metric is smooth at $U = V = 0$ then we can continue the spacetime past the region (17).

(v) From (16) we get

$$dU = \alpha e^{\alpha u} du, \quad dV = \alpha e^{-\alpha v} dv \tag{19}$$

Thus the metric (12) becomes

$$ds^2 = -(1 - \frac{2M}{r})\frac{e^{-\alpha(u-v)}}{\alpha^2} dU dV + r^2 d\Omega_2^2 = -\frac{(r - 2M)}{r}\frac{e^{-\alpha(u-v)}}{\alpha^2} dU dV + r^2 d\Omega_2^2 \tag{20}$$

Now note that

$$e^{-\alpha(u-v)} = e^{-2\alpha r^*} = e^{-2\alpha[r+2M\ln(\frac{r}{2M}-1)]} = e^{-2\alpha r}(\frac{r}{2M}-1)^{-4\alpha M} = e^{-2\alpha r}(2M)^{4\alpha M}(r-2M)^{-4\alpha M} \tag{21}$$

We now see that if we choose

$$\alpha = \frac{1}{4M} \tag{22}$$

then we cancel the factor $r - 2M$ in (20), getting

$$ds^2 = -\frac{32M^3}{r}e^{-\frac{r}{2M}}dUdV + r^2 d\Omega^2 \tag{23}$$

This metric is now written in coordinates U, V, θ, ϕ, with

$$\begin{aligned} U &= (\frac{r}{2M} - 1)^{\frac{1}{2}} e^{\frac{r}{4M}} e^{\frac{t}{4M}} \\ V &= -(\frac{r}{2M} - 1)^{\frac{1}{2}} e^{\frac{r}{4M}} e^{-\frac{t}{4M}} \end{aligned} \tag{24}$$

Note that

$$UV = -(\frac{r}{2M} - 1)e^{\frac{r}{2M}} \tag{25}$$

and we should understand the symbol r in (23) as the function $r(U, V)$ given through the transcendental equation (25). Since we do not need the explicit form of this function for our analysis, we leave it as the symbol r. All we note for now is that at the horizon (where we are trying to get smooth coordinates) the function r is smooth function on the manifold, with $r \approx 2M(1 - UV)$.

3 Extending past the horizon

The region outside the horizon was given by the coordinate range (17). Let us now see how we would extend the spacetime past the horizon, to reach the interior of the black hole. We let the metric continue to have the form (23), where $r(U, V)$ will continue to be given through (25). There is no problem with either equation at $r = 2M$. There will be a singularity at $r = 0$, which is a real singularity: the curvature diverges there, and we cannot remove this singularity with a coordinate transformation. From (25) we see that

$$r = 0 \ \leftrightarrow \ UV = 1 \tag{26}$$

We see that we can extend the coordinate range from the initial range (17) to all values of U, V satisfying $UV < 1$. This spacetime is called the 'extended black hole spacetime', and we depict it in fig.1. There is a 'future singularity' at $U > 0, V > 0, UV = 1$; if an observer decides to fall into the black hole then he will hit this singularity sometime in his future. But there is another singularity – the 'past singularity' at $U < 0, V < 0, UV = 1$. We will discuss the structure of this spacetime in more detail after drawing the Penrose diagram.

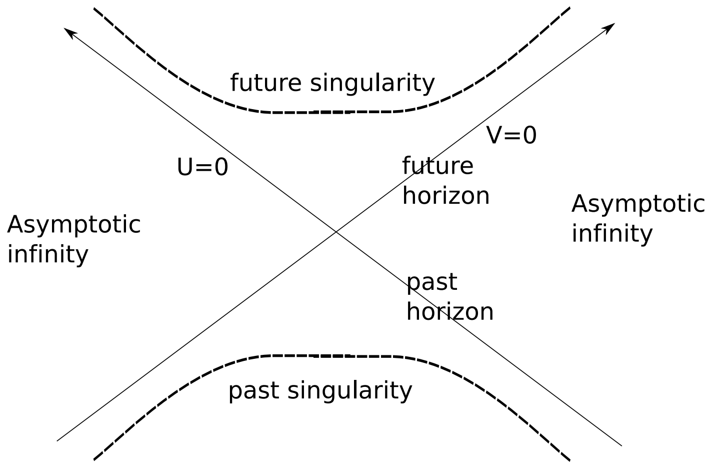

Figure 1: The fully extended Schwarzschild geometry

4 The Penrose diagram

The U, V coordinates cover all of our spacetime, but these coordinates do not have a bounded range. Thus if we try to draw the U, V space on a sheet of paper, we have to stop at a finite value of U, V, and we do not explicitly see the picture of how the 'points at infinity' border our spacetime. To bring these 'points at infinity' to a finite coordinate distance from the points in the interior of our spacetime, we make a conformal rescaling of the metric. Here the word 'conformal' means that at each point the metric is scaled by a number $g_{ab}(x) \to \Omega^2(x) g_{ab}(x)$, so that the angles between different directions at the point x do not change, and in particular null directions remain null directions. Such a rescaling will help us to understand the causal structure of the spacetime, including the behavior of 'infinity'.

Let us first carry out this process for Minkowski spacetime; we will need this result anyway to describe part of the black hole spacetime when the black hole is made by 'collapse' of a shell. Minkowski spacetime is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 \tag{27}$$

Let us write

$$U = t + r, \quad V = t - r \tag{28}$$

getting

$$ds^2 = -dUdV + r^2 d\Omega^2 \tag{29}$$

where now the coordinates are U, V, θ, ϕ, and $r = \frac{1}{2}(U - V)$. Since

$$r = \frac{1}{2}(U - V) \geq 0 \tag{30}$$

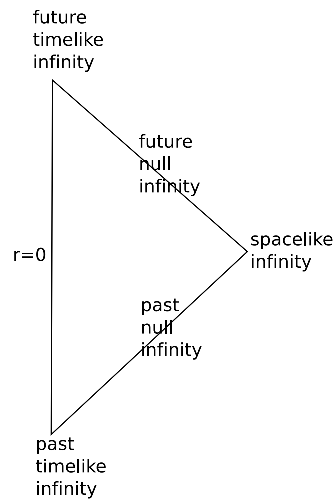

Figure 2: Penrose diagram of Minkowski space

we have the alllowed range

$$-\infty < U < \infty, \quad -\infty < V < \infty, \quad U \geq V \tag{31}$$

so we have an infinite coordinate range. Let us write

$$\tilde{U} = \tanh U, \quad \tilde{V} = \tanh V \tag{32}$$

so that

$$-1 < \tilde{U} < 1, \quad -1 < \tilde{V} < 1, \quad \tilde{U} \geq \tilde{V} \tag{33}$$

and the metric is

$$ds^2 = -[\frac{dU}{d\tilde{U}}\frac{dV}{d\tilde{V}}]d\tilde{U}\tilde{V} + r^2 d\Omega^2 \tag{34}$$

But

$$\frac{dU}{d\tilde{U}} = \text{sech}^2 U = \frac{1}{1-\tilde{U}^2}, \quad \frac{dV}{d\tilde{V}} = \text{sech}^2 V = \frac{1}{1-\tilde{V}^2} \tag{35}$$

so that

$$ds^2 = \frac{1}{(1-\tilde{U}^2)(1-\tilde{V}^2)}[-d\tilde{U}d\tilde{V} + r^2(1-\tilde{U}^2)(1-\tilde{V}^2)d\Omega^2] \tag{36}$$

So far we have just rewritten Minkowski spacetime in new coordinates, but now let us make a conformal transformation, defining a new metric

$$g'_{ab} = (1-\tilde{U}^2)(1-\tilde{V}^2)g_{ab} \tag{37}$$

Thus the new metric is

$$ds'^2 = -d\tilde{U}d\tilde{V} + r^2(1-\tilde{U}^2)(1-\tilde{V}^2)d\Omega^2 \tag{38}$$

Let us ignore the angular directions; since we have spherical symmetry there is no nontrivial structure in these directions, and the size if the angular sphere is not relevant for the main computations we are interested in. Thus we get

$$ds'^2 = -d\tilde{U}d\tilde{V} \tag{39}$$

with the coordinate range (33). The null directions are $\tilde{U} = U_0$ and $\tilde{V} = V_0$. This gives the Penrose diagram in fig.2.

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Problem 2: Show that all points at spatial infinity $t = t_0, r \to \infty$ are represented as one point in the Penrose diagram; i.e., they are at one value $\tilde{U}, \tilde{V}$. Similarly show that all points at timelike infinity $r = r_0, t \to \infty$ are at one point in the Penose diagram. What do the different points along the boundary of the diamond describe?

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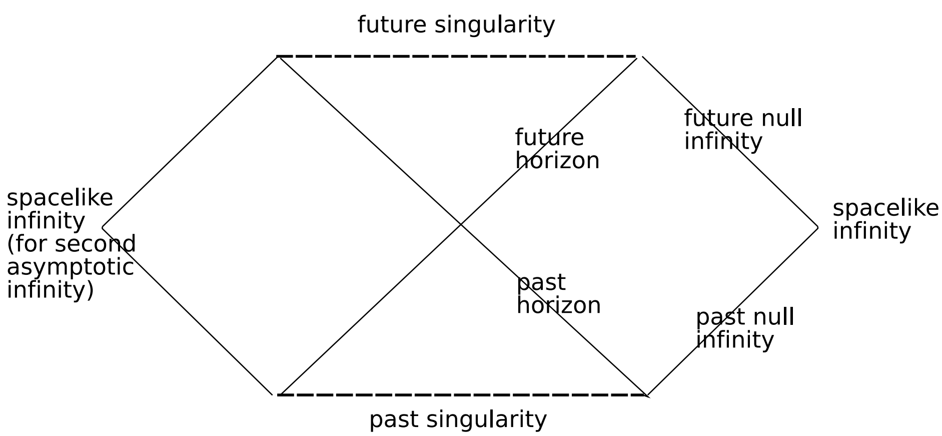

Figure 3: Penrose diagram for the 'eternal Schwarzschild hole'

We now do a similar transformation (32),(37) for the black hole metric (23), getting

$$ds'^2 = -\frac{32M^3}{r}e^{-\frac{r}{2M}}d\tilde{U}d\tilde{V} \tag{40}$$

We have to be careful about the coordinate ranges though. The spacetime again ends at $r = 0$; this time there is a singularity there instead of a 'simple origin of coordinates'. But $r = 0$ is now given by solving $UV = 1$ which is

$$\tanh^{-1}\tilde{U}\ \tanh^{-1}\tilde{V} = 1 \tag{41}$$

This is a curve in $\tilde{U}, \tilde{V}$ space, and points beyond this curve are not in the spacetime represented by the Penrose diagram, since they lie past the singularity. We draw the Penrose diagram in fig.3. The singularity runs along a curve from $\tilde{U} = 0, \tilde{V} = 1$ to $\tilde{U} = 1, \tilde{V} = 0$. Note that the causal structure of infinity is not changed by any other conformal rescaling of the metric at interior points of spacetime. Thus we can imagine a further rescaling which makes the singularity a straight line $\tilde{U} = 0, \tilde{V} = 1$ to $\tilde{U} = 1, \tilde{V} = 0$; this is easier to draw, and is typically what is done in figures. The essential property of the singularity we cannot change in the picture is that the singularity is *spacelike*; The constant r surface $r = 0$ is inside the horizon and so is spacelike instead of timelike.

5 The black hole formed by collapse

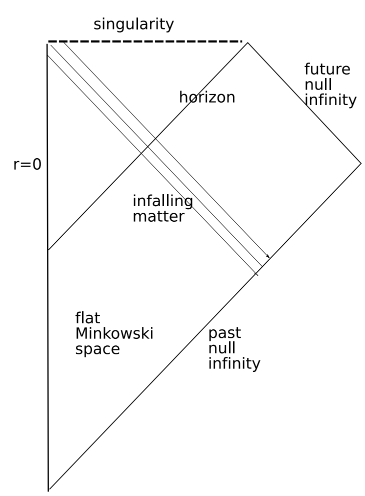

Figure 4: Penrose diagram of the black hole made by collapse of a shell

The black hole spacetime made above does not describe a realistic black hole made by collapse of a star. The spacetime we have found has a 'past singularity', which a collapsing star would not have, and also a second asymptotically flat region, which we cannot hope to produce simply by letting a star collapse in our starting spacetime. To get the correct spacetime for the collapsing star, note that the metric inside a spherical shell is *flat Minkowski spacetime*. This follows by the Birkoff theorem, which says that a spherically symmetric vacuum solution to Einstein's equations must be a piece of the Schwarzscild geometry; since we have no source inside the shell, we must choose the geometry with $M = 0$, which is just Minkowski space. Thus inside the shell we take Minkowski spacetime, and outside the shell we must glue this to the black hole spacetime (using 'Israel matching conditions' across the shell). The resulting spacetime, shown in fig.4, does not have either the past singularity or the second asymptotically flat region.