1 The light cone gauge

We have seen in the previous set of notes how to make physical states at level 2. After writing down all possible oscillator excitations at this level, we had to impose the physical constraints

$$L_n|\psi\rangle = 0, \quad n > 0 \tag{1}$$

$$(L_0 - 1)|\psi\rangle = 0 \tag{2}$$

and then also remove pure gauge states which were physical but also had the special form

$$|\psi\rangle = L_{-n_k} \dots L_{-n_1}|\lambda\rangle \tag{3}$$

Only then did we get a correct count of the physical degrees of freedom.

This procedure was cumbersome, and it would be good if we had a way to get all the relevant physical degrees of freedom without having to worry about the constraints. We will see now that this can be done in the light cone gauge formalism, but the price that we have to pay is that we will lose manifest Lorentz invariance.

2 Conformal transformations

We have chosen coordinates of the worldsheet so that the metric had the form

$$g_{ab} = \rho(\xi)\eta_{ab} \tag{4}$$

Can we make a further coordinate transformation and still keep this form of the metric? There is indeed such a set of residual transformations. To see this, it may be easier to go to the Euclidean metric where we have coordinates

$$z = \sigma + i\tau, \bar{z} = \sigma - i\tau \tag{5}$$

and the metric (4) has

$$g_{zz} = 0, \quad g_{\bar{z}\bar{z}} = 0 \tag{6}$$

We preserve the form of this metric under any transformation of the form

$$z' = f(z) \tag{7}$$

where f is a holomorphic function of z. Recall that the real and imaginary parts of a holomorphic function are harmonic functions. Further, suppose we chose one of these harmonic functions in any way that we like. Then a holomorphic f can be found by a suitable choice of the other harmonic function.

These notions can be continued back to Lorentzian signature, where we can chose τ' to be any functions satisfying the wave equation

$$\Box\tau' = 0 \tag{8}$$

and then there will be a suitable σ' which will make the metric have the form (4). We will therefore use the word 'harmonic' to talk of functions satisfying (8) in what follows below.

Which harmonic function should we chose for τ'? Recall that the spacetime coordinates are harmonic functions on the worldsheet. We had the expansion

$$\begin{aligned} X^\mu(\tau,\sigma) &= [\frac{1}{2}x^\mu + \alpha' p_L^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in(\tau+\sigma)}] \\ &\quad +[\frac{1}{2}x^\mu + \alpha' p_R^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n}e^{-in(\tau-\sigma)}] \end{aligned} \tag{9}$$

Let us chose one space direction X^1 and define

$$X^+ = X^0 + X^1, \quad X^- = X^0 - X^1 \tag{10}$$

Then $X^+(\tau,\sigma)$ is also harmonic. We can set

$$\tau' = a + bX^+(\tau,\sigma) \tag{11}$$

and let σ' be determined by the requirement that the metric be in conformal gauge. Here a, b are constants that we will fix presently. From now on we will call τ', σ' as just τ, σ, since we will stay with this gauge choice in what follows.

Thus we have used our freedom to chose coordinates on the worldsheet in such a way that we have simplified at least one of the spacetime coordinates. But there will be an additional advantage that we will see now. The main complication is from the constraints, which are

$$\frac{\partial X^\mu}{\partial\xi^+}\frac{\partial X_\mu}{\partial\xi^+} = 0, \qquad \frac{\partial X^\mu}{\partial\xi^-}\frac{\partial X_\mu}{\partial\xi^-} = 0 \tag{12}$$

The first of these can be written as

$$-\frac{\partial X^+}{\partial\xi^+}\frac{\partial X^-}{\partial\xi^+} + \frac{\partial X^i}{\partial\xi^+}\frac{\partial X^i}{\partial\xi^+} = 0 \tag{13}$$

From the mode expansion we see that

$$X^+ = x^+ + \alpha' p^+\tau + \ (oscillators\ modes) \tag{14}$$

Thus we see that

$$a = x^+, \quad b = \alpha' p^+ \tag{15}$$

and we have

$$\frac{\partial X^+}{\partial\xi^+} = \frac{\partial X^+}{\partial\tau}\frac{\tau}{\xi^+} = \frac{1}{2}\alpha' p^+ \tag{16}$$

Thus the constraint is

$$-\frac{1}{2}\alpha' p^+\partial_+X^- + \partial_+X^i\partial_+X^i = 0 \tag{17}$$

The remarkable thing now is that we can solve for ∂_+X^-

$$\partial_+X^- = \frac{2}{\alpha' p^+}\partial_+X^i\partial_+X^i \tag{18}$$

so that we can write

$$X^- = \frac{2}{\alpha' p^+} \int d\xi^+ \partial_+ X^i \partial_+ X^i + F(\xi^-) \tag{19}$$

Thus the ξ^+ dependence of X^- has been determined from the constraint, and with this determination we do not need to worry about the this constraint any more; it has already been solved for. In a similar manner we can fix the $F(\xi^-)$ in X^- from the other constraint, and then both constrains have been solved for. Thus we have used the constraints to determine X^+, X^-, and so we do not have to worry about constraints. The X^i can be independently chosen as any harmonic functions that we wish, and we will obtain a solution to the string equations of motion. Note that there were two constraints, and they removed two of the D spacetime degrees of freedom, leaving the other $D-2$ as arbitrary functions of the world sheet.

The worldsheet expansion of X^- should have the general form

$$X^- = \frac{1}{2}x^- + \frac{\alpha' p^-}{2}(\tau + \sigma) + \ (oscillator\ modes) \tag{20}$$

where we have written only the ξ^+ part. Thus

$$\partial_+ X^- = \frac{p^-}{2} + \ (oscillator\ terms) \tag{21}$$

If we integrate over σ, the oscillator terms will be killed, and we will just get the zero mode contribution

$$\int_0^{2\pi} d\sigma \partial_+ X^- = \frac{\alpha' p^-}{2}(2\pi) = \alpha' \pi p^- \tag{22}$$

Doing such an integral on (18) we get

$$\alpha' \pi p^- = \frac{2}{\alpha' p^+} \int_0^{2\pi} d\sigma \partial_+ X^i \partial_+ X^i \tag{23}$$

From the mode expansion (9) we find that

$$\int_0^{2\pi} d\sigma \partial_+ X^i \partial_+ X^i = (2\pi)[\alpha'^2 \frac{p^i p^i}{4} + 2\sum_{n>0} \frac{\alpha'}{2} \alpha^i_{-n} \alpha^i_n] \tag{24}$$

Thus (23) gives

$$(\alpha' \pi p^-)(\frac{\alpha' p^+}{2}) - 2\pi\alpha'^2 \frac{p^i p^i}{4} = (2\pi)\alpha' \sum_{n>0} \alpha^i_{-n} \alpha^i_n \tag{25}$$

or

$$-p^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha^i_{-n} \alpha^i_n \tag{26}$$

Note that

$$m^2 = -p^2 \tag{27}$$

and that the oscillator sum on the RHS is just the level N of oscillator excitations. In this sum we should do the usual shift from the normal ordering constant

$$N \to N - 1 \tag{28}$$

and then we get

$$m^2 = \frac{4}{\alpha'}(N-1) \tag{29}$$

Thus we get the same mass relation from the light cone gauge method as we had obtained from covariant quantization. But the advantage now is that the oscillators α^i_{-n} can be applied in all possible ways to generate states without having to worry about the constraints.

3 States at level 2

Let us use the light cone gauge to compute the allowed states at level 2. The oscillators α^i_{-n} are called 'transverse', since they are orthogonal to the directions X^+, X^-. The range of i is $i = 1, \dots D-2$. The allowed states are

$$[A_{ij}\alpha^i_{-1}\alpha^j_{-1} + B_i\alpha^i_{-2}]|\rangle \tag{30}$$

Since A_{ij} is symmetric it gives

$$\frac{(D-2)(D-1)}{2} \tag{31}$$

possibilities, while B_i gives

$$(D-2) \tag{32}$$

possibilities. Thus overall we get

$$\frac{(D-2)(D-1)}{2} + (D-2) = \frac{(D-2)(D+1)}{2} = \frac{1}{2}(D^2 - D - 2) \tag{33}$$

physical states.

Let us compare this to the counting in the covariant method. There we had showed that the degrees of freedom made a traceless symmetric tensor C_{ij}, but where $i, j = 1, \dots D-1$. This has

$$\frac{(D-1)D}{2} - 1 = \frac{1}{2}(D^2 - D - 2) \tag{34}$$

states. So we see that we get the same number of states each way. The counting is much simpler in the light cone method, but we have lost Lorentz symmetry. The massive particle that we are looking at can be brought to its rest frame, and then we see that all polarizations of indices should have $D-1$ allowed values from the $D-1$ space directions which are all on the same footing. This does happen in the covariant method, but in the light cone method the space direction X^1 is split off from the other $d-2$ space directions, and so the states appeared in a symmetric tensor A_{ij} and a vector B_i. We have to put these two together to get a representation of the rotation group in $D-1$ space dimensions.