1 Rotation around an axis

The simplest case of rotation is ‘rotation about an axis’. Imagine a body, through which we have drilled a hole and passed a frictionless rod. This rod is fixed; let it point along the z direction. The body can rotate around the z axis. This rotation will be described by an angular velocity ω. Any point on the body will rotate in a circle around the z axis. If the distance of the point from the z axis is ρ, then the velocity of the point will be

$$v = \omega\rho \tag{1}$$

Let the body be made up of point masses m_i with

$$M = \sum_i m_i \tag{2}$$

The angular momentum about the z axis of the point mass m_i is

$$L_i = m_i \rho_i v_i = m_i \rho_i^2 \omega \tag{3}$$

and the total angular momentum is

$$L = [\sum_i m_i \rho_i^2]\omega \equiv I\omega \tag{4}$$

The quantity

$$I \equiv \sum_i m_i \rho_i^2 \tag{5}$$

is called the *moment of inertia* of the body around this axis of rotation. The kinetic energy of the body is

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2}[\sum_i m_i \rho_i^2]\omega^2 = \frac{1}{2} I\omega^2 \tag{6}$$

If there is a torque τ applied to the body around the z-axis, then we will have

$$\tau = \frac{dL}{dt} \tag{7}$$

1.1 Rotation around a point

Now let us assume that a point O on the body is fixed, but the body can rotate in any direction around this point. The angular velocity is now a vector. We can describe this vector by giving its components in any frame; for example fixing a standard orthonormal frame in space we can write

$$\vec{\omega} = \{\omega_x, \omega_y, \omega_z\} \tag{8}$$

Consider the point mass m_i. Let its position from the origin O be given by the vector $\vec{r}_i$. The velocity of the mass is

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{9}$$

The total angular momentum will be

$$\vec{L} = \sum_i m_i \vec{r}_i \times v_i = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \sum_i m_i[(r_i)^2 - (r_i \cdot \omega)\vec{r}_i] \tag{10}$$

We can write this in components. Let $a = 1, 2, 3$ denote the three components of a vector V_a. Then we have

$$L_a = \sum_b I_{ab} \omega_b \tag{11}$$

where

$$I_{ab} = \sum_i m_i[(r_i)^2 \delta_{ab} - r_{ia} r_{ib}] \tag{12}$$

More explicitly,

$$\begin{aligned}
I_{xx} &= \sum_i m_i (y_i^2 + z_i^2) \\
I_{yy} &= \sum_i m_i (z_i^2 + x_i^2) \\
I_{zz} &= \sum_i m_i (x_i^2 + y_i^2) \\
I_{xy} &= -\sum_i m_i x_i y_i \\
I_{yz} &= -\sum_i m_i y_i x_i \\
I_{zx} &= -\sum_i m_i z_i x_i
\end{aligned} \tag{13}$$

We also write

$$\vec{L} = I \cdot \vec{\omega} \tag{14}$$

where we note that I is a matrix (which happens to be symmetric), and $\vec{\omega}, \vec{L}$ are vectors.

The kinetic energy is

$$\begin{aligned}
T = \frac{1}{2} \sum_i m_i v_i^2 &= \frac{1}{2} \sum_i m_i \vec{v}_i \cdot (\vec{\omega} \times \vec{r}_i) \\
&= \frac{1}{2} \sum_i m_i \vec{\omega} \cdot (\vec{r}_i \times \vec{v}_i) \\
&= \frac{1}{2} \vec{\omega} \cdot \sum_i m_i (\vec{r}_i \times \vec{v}_i) \\
&= \frac{1}{2} \vec{\omega} \cdot \vec{L}
\end{aligned} \tag{15}$$

If there is no torque on the body (i.e. $\vec{\tau} = 0$) then $\vec{L}$ is constant

$$\frac{d\vec{L}}{dt} = 0 \tag{16}$$

But this does not mean that $\vec{\omega}$ is constant. So we have an interesting class of rotations to study: *torque free rotations.*

Before we proceed, we note a complication. The way we have defined the matrix I, we see that its components will change with time. This will happen because as the body rotates, the location $\vec{r}_i$ of each mass point m_i will change, and these locations go into determining I. Computing I is hard, since we have to sum over all the masses, and if we have to do a fresh computation at each point time, then things will become quite impossible. So we adopt a different strategy: we choose a set of orthonormal axes that are fixed in the body; these will be called *body-fixed axes.* As the body rotates, the components of I will not change in this frame. We get a second advantage; we can choose this orthonormal frame fixed to the body in such a way that I becomes very simple. Note that I is a symmetric matrix, so it can be diagonalized by an orthonormal transformation. This means that we can choose the body-fixed axes in such a way that we will have

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{17}$$

The quantities I_1, I_2, I_3 are called the *principal moments of inertia.* The disadvantage of using body-fixed axes of course is that these axes will rotate with time, so even after we have solved the problem in the body-fixed axes we will have an additional step to do: we will have to find the orientation of the body with respect to the space-fixed axes.

Let us label the body-fixed axes as $1, 2, 3$, to distinguish them from the space-fixed axes which we have called x, y, z. Then using the body-fixed frame, we have

$$L_1 = I_1\omega_1, \quad L_2 = I_2\omega_2, \quad L_3 = I_3\omega_3 \tag{18}$$

The kinetic energy is

$$T = \frac{1}{2}[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2] \tag{19}$$

The relation between space-fixed and body-fixed axes is given as follows. For any vector $\vec{V}(t)$ we have

$$(\frac{dV}{dt})_s = (\frac{dV}{dt})_b + \vec{\omega} \times \vec{V} \tag{20}$$

For $\vec{\tau} = 0$ we thus have

$$0 = (\frac{d\vec{L}}{dt})_s = (\frac{d\vec{L}}{dt})_b + \omega \times \vec{L} \tag{21}$$

In components, we find for instance

$$\frac{d}{dt}(I_1\omega_1) + (\omega_2 L_3 - \omega_3 L_2) = 0 \tag{22}$$

which is

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0 \tag{23}$$

Overall we get 3 equations for 3 unknowns $\omega_1, \omega_2, \omega_3$

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= 0 \end{aligned} \tag{24}$$

1.2 The case $I_1 = I_2 = I_3$

Suppose that

$$I_1 = I_2 = I_3 \equiv I \tag{25}$$

This is the simplest possibility. Note that now we have

$$\vec{L} = I\vec{\omega} \tag{26}$$

In the absence of torques, $\vec{L}$ is constant, and we see that $\vec{\omega}$ will also be constant for the present case. The body therefore keeps turning at a fixed rate around a fixed axis, with the rate of turning and the direction of the axis being determined by the initial conditions. This motion is therefore rather trivial.

1.3 $I_1 = I_2 \neq I_3$

Let

$$I_1 = I_2 \neq I_3 \tag{27}$$

Now things will be more interesting, but still simpler than the general case where all the principal moments of inertia are different. Such bodies are called 'symmetric bodies'.

Now the torque free equations become

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_1)\omega_2\omega_3 &= 0 \\ I_1\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= 0 \\ I_3\dot{\omega}_3 &= 0 \end{aligned} \tag{28}$$

From the last equation we see that

$$\omega_3 = \text{constant} \tag{29}$$

From the first two equations we then get

$$\begin{aligned} \dot{\omega}_1 + [(\frac{I_3 - I_1}{I_1})\omega_3]\omega_2 &= 0 \\ \dot{\omega}_2 - [(\frac{I_3 - I_1}{I_1})\omega_3]\omega_1 &= 0 \end{aligned} \tag{30}$$

so we see that ω_1, ω_2 describe a circle with angular frequency

$$\Omega = [(\frac{I_3 - I_1}{I_1})\omega_3] \tag{31}$$

The radius of the circle is fixed, so we see that $\omega_1^2 + \omega_2^2$ is constant. This could also have been seen from the fact that the kinetic energy T must be constant

$$T = \frac{1}{2}[I_2(\omega_1^2 + \omega_2^2) + I_3\omega_3^2] \tag{32}$$

(Recall that ω_3 is constant.) In particular we see that the magnitude of ω is constant

$$|\vec{\omega}|^2 = (\omega_1^2 + \omega_2^2 + \omega_3^2) = \text{constant} \tag{33}$$

Since ω_3 is constant, we see that the angle between $\vec{\omega}$ and the body axis '3' is fixed; this angle is called α_b. The vector $\vec{\omega}$ will therefore describe a cone around the body axis '3'; this is called the body cone.

The vector $\vec{L}$ is fixed in space since $\vec{\tau} = 0$. Note that

$$\vec{L} \cdot \vec{\omega} = L_1\omega_1 + L_2\omega_2 + L_3\omega_3 = I_1(\omega_1^2 + I_2\omega_2^2) + I_3\omega_3^2 = \text{constant} \tag{34}$$

Given that $\vec{L}$ and $|\vec{\omega}$ are constant, we find that the angle between $\vec{L}$ and $\vec{\omega}$ is constant. Thus $\vec{\omega}$ will describe a cone around $\vec{L}$; this is called the 'space cone' and the angle between $\vec{L}$ and $\vec{\omega}$ is called α_s.

The next thing we observe is that $\vec{L}, \vec{\omega}$ and the '3' axis of the body all lie in one plane. To see this, suppose that any instant the body axes are set up so that

$$\vec{\omega} = \omega_1\hat{1} + \omega_3\hat{3} \tag{35}$$

i.e., there is no component ω_2. We see that $\vec{\omega}$ and the '3' axis of the body define the $1-3$ plane. But now we observe that

$$\vec{L} = I_1\omega_1\hat{1} + I_3\omega_3\hat{3} \tag{36}$$

so it also lies on the $1-3$ plane.

Note that if $I_3 > I_1$ then $\vec{L}$ will lie between $\vec{\omega}$ and the '3' axis, while if $I_3 < I_1$ then $\vec{L}$ will lie outside the angle made by $\vec{\omega}$ and the '3' axis.

The last thing that we need to know is that the body cone rolls on the space cone 'without slipping'. This follows because at any instant of time the points that lie on the line through $\vec{\omega}$ are stationary. But this line is just the line of contact between the two cones.

Let us use this fact to find the rate at which $\vec{\omega}$ precesses in space. To do this, imagine marking the point on the body which lies at the tip of the vector $\vec{\omega}$ at $t = 0$. At a slightly later time dt a different point on the body will lie at the tip of ω. We ask how the distance ds between these points changes with t. The answer is

$$\frac{ds}{dt} = \Omega\sqrt{\omega_1^2 + \omega_2^2} \tag{37}$$

We now ask how far the tip of $\vec{\omega}$ has travelled on the space cone. This distance should equal the distance ds, so we will have

$$\begin{aligned}
\frac{ds}{dt} &= \Omega_s\sqrt{\omega^2 - \frac{(\vec{\omega}\cdot\vec{L})^2}{L^2}} \\
&= \Omega_s\sqrt{[(\omega_1^2 + \omega_2^2) + \omega_3^2] - \frac{[I_1(\omega_1^2 + \omega_2^2) + I_3\omega_3^2]^2}{I_1^2(\omega_1^2 + \omega_2^2) + I_3^2\omega_3^2}} \\
&= \Omega_s(I_3 - I_1)\frac{\sqrt{(\omega_1^2 + \omega_2^2)}\omega_3}{\sqrt{I_1^2(\omega_1^2 + \omega_2^2) + I_3^2\omega_3^2}}
\end{aligned} \tag{38}$$

We thus get

$$\Omega_s = \Omega\frac{\sqrt{I_1^2(\omega_1^2 + \omega_2^2) + I_3^2\omega_3^2}}{(I_3 - I_1)\omega_3} \tag{39}$$

Note that if

$$\omega_1^2 + \omega_2^2 << \omega_3^2 \tag{40}$$

then we get

$$\Omega_s = \Omega \frac{I_3}{I_1} \tag{41}$$