1 The Virasoro constraints

Consider the constraint

$$X^\mu_{,+} X_{\mu,+} = 0 \tag{1}$$

From the mode expansion we find this constraint can be written as

$$L_n \equiv \frac{1}{2} \sum_{n'} \alpha_{n-n'} \alpha_{n'} = 0 \tag{2}$$

where we have chosen the normalization constant for later convenience.

Classically we would like to impose these constraints in the form

$$L_n = 0 \quad \text{for all } n \tag{3}$$

Quantum mechanically we will find that such a condition is not consistent, since we will find that

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{4}$$

where c is a constant, called the 'central charge', to be determined later. Thus we cannot set both L_n and L_{-n} to vanish. We will see that

$$L_{-n} = (L_n)^\dagger \tag{5}$$

We will require only the weaker physical condition that the expectation value of the constraints vanish between the physical states. Thus we will require that

$$\langle \psi | L_n \chi \rangle = 0 \tag{6}$$

Suppose we impose that

$$L_n |\psi\rangle = 0 \quad \text{for } n > 0 \tag{7}$$

Then for $n > 0$ we get $\langle \psi | L_n \chi \rangle = 0$ since $L_n|\chi\rangle = 0$ and for $n < 0$ we get $\langle \psi | L_n \chi \rangle = 0$ since $\langle \psi | L_n = \langle \psi | (L_{-n})^\dagger = 0$.

We now wish to find the above mentioned algebra of the L_n.

2 The Virasoro algebra

First look at the commutation relation

$$\begin{aligned}
[AB, CD] &= ABCD - CDAB \\
&= A[B,C]D + ACBD + C[A,D]B - CADB \\
&= A[B,C]D + C[A,D]B + [A,C]BD + CABD - CADB \\
&= A[B,C]D + C[A,D]B + [A,C]BD + CA[B,D]
\end{aligned} \tag{8}$$

Now note that

$$L_n = \frac{1}{2}\alpha_{n-n'}\alpha_{n'} \tag{9}$$

We find that

$$\begin{aligned}
[L_n, L_m] &= \frac{1}{4}[\alpha_{n-n'}\alpha_{n'}, \alpha_{m-m'}\alpha_{m'}] \\
&= \frac{1}{4}(\alpha_{n-n'}n'\delta_{n'+m-m'}\alpha_{m'} \\
&\qquad + \alpha_{m-m'}(n-n')\delta_{n-n'+m'}\alpha_{m'} \\
&\qquad + (n-n')\delta_{n'-n+m'-m}\alpha_{n'}\alpha_{m'} \\
&\qquad + \alpha_{m-m'}\alpha_{n-n'}n'\delta_{n'+m'})
\end{aligned} \tag{10}$$

Working out the delta functions, these terms can be written as

$$\begin{aligned}
\frac{1}{4}(\quad & \alpha_{n+m-m'}\alpha_{m'}(m'-m) \\
& \alpha_{m+n-m'}\alpha_{m'}(n-m') \\
& \alpha_{n+m-m'}\alpha_{m'}(m'-m) \\
& \alpha_{m+n-m'}\alpha_{m'}(n-m') \quad)
\end{aligned} \tag{11}$$

These total up to

$$\frac{1}{2}(n-m)\alpha_{n+m-m'}\alpha_{m'} \tag{12}$$

Thus we get the naive Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} \tag{13}$$

We have obtained the classical commutation relation, but we did not get the anomaly term. To obtain the anomaly, we have to do the computation more carefully, noting that terms in L_n may need to be normal ordered.

3 Normal ordering the operators

First consider the commutator

$$[L_m, L_n], \quad m+n \neq 0 \tag{14}$$

In this case we see that there can be no operator ordering problem at the end. Each time we compute a commutator, we are left with a product of the form

$$\alpha_p\alpha_q, \quad p+q = m+n \neq 0 \tag{15}$$

So these two oscillators commute with each other, and we can order them anyway we wish. Thus the only time we will have a problem with normal ordering will be when we compute

$$[L_m, L_{-m}] \tag{16}$$

where we will assume without loss of generality that $m > 0$. How should we proceed to compute the commutator?

First consider $L_m, m > 0$. This can be written in the form

$$\begin{aligned} L_m &= \frac{1}{2}\sum_{k=-\infty}^{\infty} \alpha_{-k}\alpha_{m+k} \\ &= \sum_{k=1}^{\infty} \alpha_{-k}\alpha_{m+k} + \frac{1}{2}[\alpha_0\alpha_m + \alpha_1\alpha_{m-1} + \ldots \alpha_m\alpha_0] \end{aligned} \tag{17}$$

Now consider $L_{-m}, m > 0$. This can be written as

$$\begin{aligned} L_{-m} &= \frac{1}{2}\sum_{k=-\infty}^{\infty} \alpha_{-m-k'}\alpha_{k'} \\ &= \sum_{k'=1}^{\infty} \alpha_{-m-k'}\alpha_{k'} + \frac{1}{2}[\alpha_{-m}\alpha_0 + \alpha_{-m+1}\alpha_1 + \ldots \alpha_0\alpha_{-m}] \end{aligned} \tag{18}$$

In each of these cases we have first written out the terms where the operators are manifestly normal ordered. We have then written a finite number of terms which make up the rest of the operator. The main part of the correlator is

$$[L_m, L_{-m}] \to \sum_{k=1}^{\infty}\sum_{k'=1}^{\infty}[\alpha_{-k}\alpha_{m+k}, \alpha_{-m-k'}\alpha_{k'}] \tag{19}$$

Here it is clear that the subscripts $m+k, k'$ are positive and $-k, -m-k'$ are negative. Since we have a nontrivial commutator only between a positive and a negative index, the surviving terms are

$$\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty}[(m+k)\alpha_{-k}\alpha_k + (-k)\alpha_{-m-k}\alpha_{m+k}] \tag{20}$$

where we have noted that in each case we will have to get $k = k'$ for the commutator to work. But each of these terms is correctly normal ordered already.

Now let us look at the remaining terms. We can see that there are no cross terms between one of the inifinite sums and one of the finite sums, since we cannot find oscillators of the form α_k, α_{-k} in such a set. Now look at the commutator of the two finite sums. These are of the form

$$\frac{1}{4}\sum_{k=0}^{m}\sum_{k'=0}^{m}[\alpha_k\alpha_{m-k}, \alpha_{-m+k'}\alpha_{-k'}] \tag{21}$$

Let us assume that $m = 2n+1$ is odd; the answer will turn out to be similar for m even. Then we can write for the above sum

$$\sum_{k=0}^{n}\sum_{k'=0}^{n}[\alpha_k\alpha_{m-k}, \alpha_{-m+k'}\alpha_{-k'}] \tag{22}$$

Now we see that α_k can only have a commutator with $\alpha_{-k'}$ (when $k = k'$) and similarly we get a commutator between α_{m-k} and $\alpha_{-m+k'}$ (when $k = k'$). Thus we get

$$\sum_{k=0}^{n}\sum_{k'=0}^{n}(k\alpha_{-m+k}\alpha_{m-k} + (m-k)\alpha_k\alpha_{-k}) \tag{23}$$

The first part is already normal ordered, but the second is not. If we normal order it, then we get an extra contribution

$$\sum_{k=0}^{n}(m-k)k = m\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} = \frac{mn(n+1)}{3} = \frac{m^3-m}{12} \tag{24}$$

Thus the full algebra becomes

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0} \tag{25}$$