Econ 640 F 08
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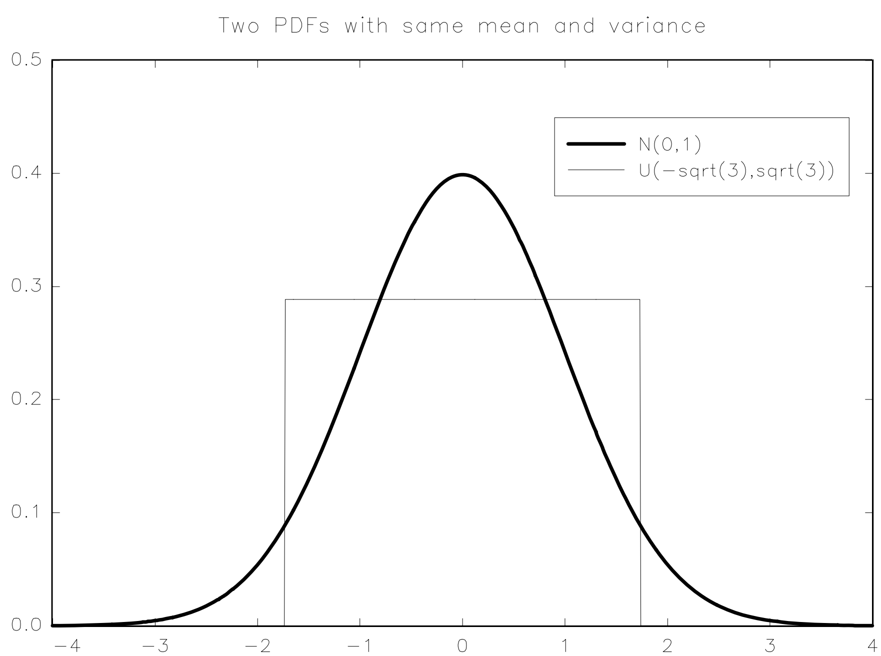
Mean-Scale Indifference Curves

For gambles with small variance, a second order Taylor's series expansion enables us to approximately quantify the risk premium on a risky prospect in terms of its variance. For gambles with large variance, however, it is conceivable that two gambles with the same mean and variance might not have exactly the same expected utility.

Figure 1, for example, shows two distributions with mean 0 and unit variance. The first is a Gaussian distribution N(0,1), while the second is uniform $U(-\sqrt{3}, \sqrt{3})$. It can easily be shown that a uniform distribution U(a,b) has mean (a+b)/2 and variance $(a-b)^2/12$, so that the second distribution indeed has 0 mean and unit variance.

Figure 1



An expected utility function over wealth can easily be found for which these two prospects would yield different expected utility.

However, if all gambles are drawn from the same location-scale family of distributions, the location and scale will completely describe the distribution of any given gamble, and so expected utility (if it exists) will be an exact function of location and scale, or equivalently of mean and standard deviation if the mean and variance exist. This insight and the following theorems are due to James Tobin, in his article "Liquidity Preference as Behavior towards Risk," which appeared in the *Review of Economic Studies* in 1958.

Let random future real wealth W be determined by

$$W = \mu + \sigma Z,$$

where Z is a standardized random variable with standardized density $f_Z(z)$. Then μ and σ completely determine the distribution of W. If Z has finite mean, we may w.l.g. (without loss of generality) set EZ = 0 so that EW = μ. If Z has finite variance, we may w.l.g. set Var(Z) = 1, so that s.d.(W) = σ. It is OK if Z has infinite variance (e.g. for stable distributions and some Student t distributions), however, in which case σ is just a scale parameter that cannot be interpreted as a standard deviation.

Expected utility (if it exists) may now be expressed in terms of an *indirect utility function* V(μ, σ):

$$EU(W) = V(\mu, \sigma) = \int_{-\infty}^{\infty} U(\mu + \sigma z) f(z) dz \, .$$

Theorem 1 (Tobin 1958): As long as utility is increasing (i.e. $U'(w) > 0$) and all gambles are drawn from a common location-scale family, more mean is better than less, holding risk constant.

Proof: Assuming we may pass the differentiation operator inside integral sign (cp. Thm 2.4.2 in Casella and Berger),

$$\partial V(\mu,\sigma)/\partial\mu = \int_{-\infty}^{\infty} U'(\mu+\sigma z)\, f(z)dz > 0$$

□

(The little box means QED, also written ///.)

It follows from Theorem 1 that if we place the location μ on the vertical axis and the scale σ on the horizontal axis, and plot out "Indifference Curves" that show combinations of μ and σ for which V(μ, σ) takes on a constant value, these indifference curves will pass from left to right without doubling back on themselves. Furthermore, higher curves (in the μ direction) will represent higher expected utility.

If we now add the assumption of risk aversion (so that the expected utility function is concave), Jensen's inequality tells us that some risk will be worse than no risk, holding the mean constant. The following theorem tells us even more:

Theorem 2 (Tobin 1958): As long as utility is concave (i.e. $U''(w) < 0$) and all gambles are drawn from a common location-scale family with finite mean, more risk is always worse than less risk, holding mean constant.

Proof: Since Z has finite mean, we may w.l.g. set EZ = 0. The trick of the proof is to break the required integrals in half at z = EZ = 0, and to consider the two halves separately. Again assuming we may pass the differentiation operator inside the integral,

$$\begin{aligned}\partial V(\mu,\sigma)/\partial\sigma &= \int_{-\infty}^{\infty} zU'(\mu+\sigma z)\,f(z)dz \\ &= \int_{-\infty}^{0} zU'(\mu+\sigma z)\,f(z)dz + \int_{0}^{\infty} zU'(\mu+\sigma z)\,f(z)dz \\ &= \qquad\qquad A \qquad\qquad + \qquad\qquad B\end{aligned}$$

Furthermore, since EZ = 0 and $U'(\mu)$ is just a scalar, $E(ZU'(\mu)) = 0$. Therefore,

$$\begin{aligned}0 &= \int_{-\infty}^{\infty} zU'(\mu)\,f(z)dz \\ &= \int_{-\infty}^{0} zU'(\mu)\,f(z)dz + \int_{0}^{\infty} zU'(\mu)\,f(z)dz \\ &= \qquad C \qquad + \qquad D\end{aligned}$$

By Diminishing Marginal Utility ($U''(w) < 0$), $U'(\mu) > U'(\mu + \sigma z)$ in integrals B and D, where $z > 0$. It follows that D > B > 0. Furthermore, $U'(\mu) < U'(\mu + \sigma z)$ in integrals A and C, where $z < 0$. However, since $z < 0$ in these integrals, $0 > zU'(\mu) > zU''(\mu + \sigma z)$, so that 0 > C > A. It follows that

$$\partial V(\mu,\sigma)/\partial\sigma = A + B < C + D = 0\,.$$

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Theorems 1 and 2 together tell us that our μ-σ indifference curves slope upward, but still don't tell us how they may curve.

Theorem 3 (Tobin 1958 "The non-existence of plungers"): If U(w) is strictly concave, then for any mean-scale family of distributions, the mean is a strictly convex function of scale along expected utility indifference curves.

Proof: Consider any two points on a common expected utility indifference curve:

$$V(\mu_1,\sigma_1) = V(\mu_2,\sigma_2) = V_0\,,$$

and define

$$\mu_\theta = \theta\mu_1 + (1-\theta)\mu_2,$$
$$\sigma_\theta = \theta\sigma_1 + (1-\theta)\sigma_2,$$

for any $\theta \in (0, 1)$. (See Figure 2 below.) Then for any real number z,

$$\mu_\theta + z\sigma_\theta = \theta(\mu_1 + \sigma_1 z) + (1-\theta)(\mu_2 + \sigma_2 z),$$

whence, by the strict concavity of U(w),

$$U(\mu_\theta + \sigma_\theta z) > \theta U(\mu_1 + \sigma_1 z) + (1-\theta)U(\mu_2 + \sigma_2 z)$$

It follows that

$$\begin{aligned} V(\mu_\theta, \sigma_\theta) &= \int_{-\infty}^{\infty} U(\mu_\theta + \sigma_\theta z) f_Z(z)dz \\ &> \int_{-\infty}^{\infty} \left(\theta U(\mu_1 + \sigma_1 z) + (1-\theta)U(\mu_2 + \sigma_2 z)\right) f_Z(z)dz \\ &= \theta V(\mu_1, \sigma_1) + (1-\theta)V(\mu_2, \sigma_2) \\ &= V_0 \end{aligned}$$

Along the V_0 expected utility indifference curve, the mean is a function $\mu(\sigma)$ of scale, defined implicitly by

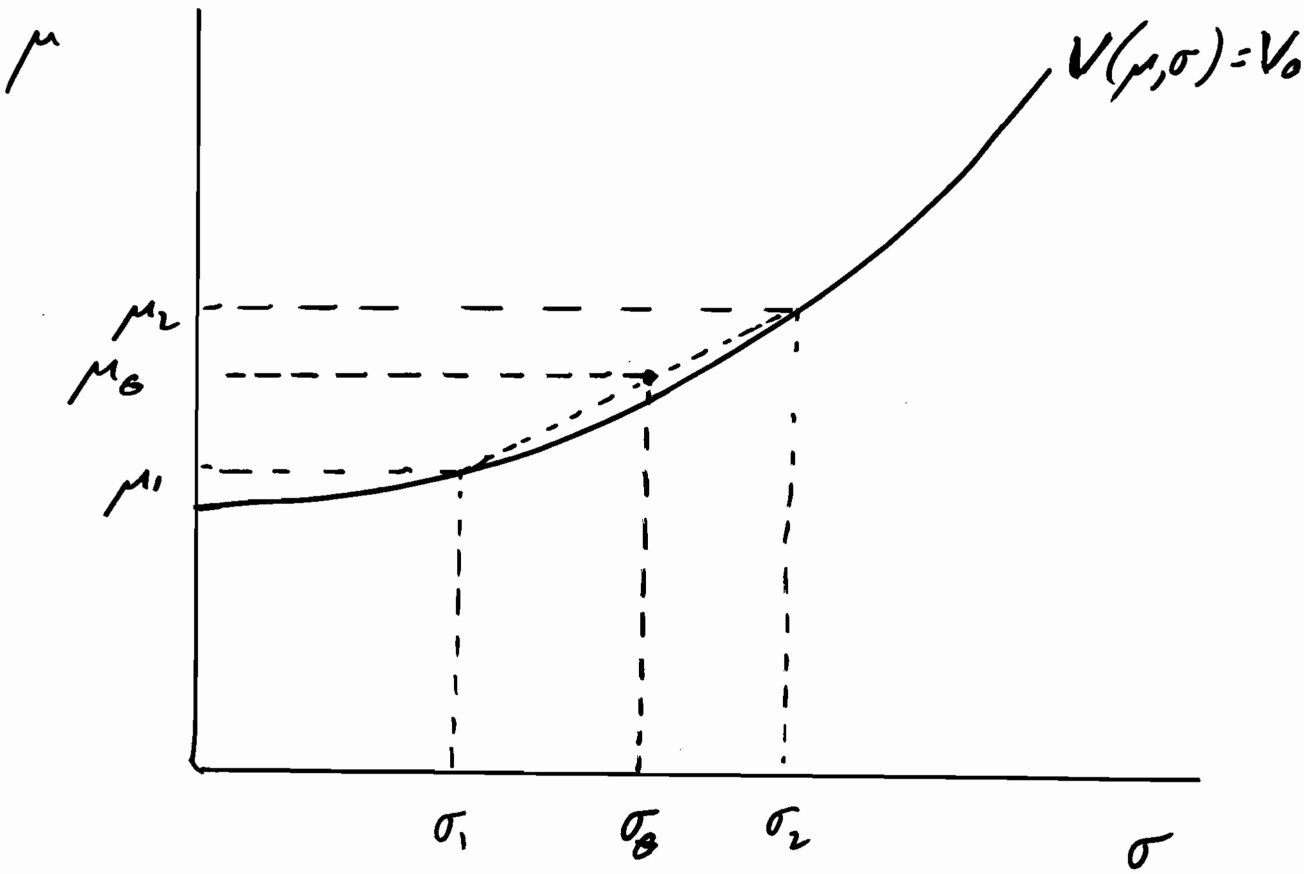
$$V(\mu(\sigma), \sigma) = V_0.$$

Since $V(\mu_\theta, \sigma_\theta) > V_0 = V(\mu(\sigma_\theta), \sigma_\theta)$, Theorem 1 requires that $\mu_\theta > \mu(\sigma_\theta)$, and hence that $\mu(\sigma)$ is indeed a convex function.

QED

Note that this convexity of the indifference curves holds only if risk is measured in terms of standard deviation (or scale) σ, and not in terms of variance σ^2. In terms of variance, the indifference curves are still upward sloping, but may have either (or alternating) curvature.

Fig. 2



Theorem 3

Quadratic Utility

An alternative approach Tobin proposed for having expected utility be a well-defined function of mean and variance was to assume that utility is quadratic:

$$U(w) = w - aw^2, \; a > 0.$$

This function implies that for *any* finite variance distribution for W, regardless of shape, expected utility is a function solely of the mean and variance of W:

$$\begin{aligned} EU(W) &= EW - aE(W^2) \\ &= EW - a(\sigma_W^2 + (EW)^2) \\ &= \mu_W - a\,\sigma_W^2 - a\,\mu_W^2 \end{aligned}$$

One immediately apparent drawback of this function is that utility hits a peak at $w = 1/(2a)$, after which it declines. However, if most of the probability density is in the region $w < 1/(2a)$, this in itself is not necessarily a big problem.

A more serious problem with quadratic utility is that it implies increasing absolute risk aversion when $w < 1/(2a)$. This in turn implies that risky assets are actually *inferior*: As people get wealthier, they will not just hold constant quantities of risky assets as with CARA exponential utility. Rather they will actually hold fewer risky assets the more wealthy they become. This implausible property makes quadratic utility highly unsatisfactory for most purposes.