ON THE PARAMETRIZATION OF THE AFOCAL STABLE DISTRIBUTIONS

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ABSTRACT

Thanks to Hall's documentation of a situation he described as a 'Comedy of errors' in the earlier literature on stable distributions, there is now general agreement as to the parametrization of these distributions when standardized as to location and scale. However, two subtly different parametrizations of location and scale remain current in the *afocal* stable cases, defined as those with characteristic exponent $\alpha = 1$ and skewness parameter $\beta \neq 0$. The more widely used parametrization, adopted by Hall and two recent monographs, lacks the linearity property that is ordinarily expected of location and scale parameters. This note shows that the alternative parametrization has this linearity property, and compares the implications of the two parametrizations for properties of stable distributions. A third recent monograph seeks to avoid these issues by denying that the afocal stable distributions are stable at all, but it is shown that they are, in fact, integral members of the stable family.

Peter Hall, writing in this *Bulletin* [**4**], described as a 'Comedy of errors' the repeated confusion that had arisen over the sign of the skewness parameter β in the stable probability distributions. 'Scarcely an Authority,' he wrote, 'has escaped this nemesis.' Hall admitted that he himself had erred in some of his own earlier writing on the subject. Most authors now agree with Hall that the standard stable distribution, $S_{\alpha\beta}(x)$, is most usefully parametrized in terms of the log characteristic function

$$\log Ee^{ixt} = \psi_{\alpha\beta}(t) = \begin{cases} -|t|^{\alpha}[1 - i\beta\,\mathrm{sign}(t)\tan(\pi\alpha/2)], & \alpha \neq 1, \\ -|t|\,[1 + i\beta(2/\pi)\,\mathrm{sign}(t)\log|t|], & \alpha = 1, \end{cases} \tag{1}$$

where $\alpha \in (0,2]$ and $\beta \in [-1, 1]$. The minus sign before the β for $\alpha \neq 1$, introduced in 1957 by Zolotarev [**11**, p. 442n], ensures that $\beta > 0$ indicates positive skewness for all values of α. Holt and Crow [**5**], for example, following [**3**, p. 164], reverse the sign on β in (1) for $\alpha \neq 1$, with the unfortunate but easily corrected result that their 'β' > 0 indicates *negative* skewness and vice versa, unless $\alpha = 1$.

It remains to add two additional parameters to capture the location and scale of the distribution. One would think that this would be easy, but it is not! Two quite different parametrizations are current in the important special case $\alpha = 1$, $\beta \neq 0$.

DuMouchel [**2**] makes the sensible proposal that the location parameter $\delta \in (-\infty, \infty)$ should shift the distribution to the left or right, while the scale parameter $c \in (0, \infty)$ should merely expand or contract it about δ, so that the general stable c.d.f. may be written as

$$S(x; \alpha, \beta, c, \delta) = S_{\alpha\beta}((x - \delta)/c). \tag{2}$$

Now for any probability distribution defined on x, the characteristic function, Ee^{ixt}, is some function of the dummy variable t. The c.f. of c times x, $Ee^{i(cx)t} = Ee^{ix(ct)}$, is

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necessarily the same function, but now evaluated at ct. Furthermore, adding δ to cx will simply add $i\delta t$ to the log of the c.f. The general stable log c.f. implied by DuMouchel's proposal is therefore

$$\log Ee^{ixt} = i\delta t + \psi_{\alpha\beta}(ct). \tag{3}$$

DuMouchel gives explicitly the general log c.f. for $\alpha = 1$ in the form implied by (3) in equation (1.1) of [**1**].

Hall instead writes the general log c.f. as

$$\log Ee^{ixt} = i\mu t + c^{\alpha}\psi_{\alpha\beta}(t). \tag{4}$$

Samorodnitsky and Taqqu [**9**, p. 5], following Zolotarev [**11**] and, in this respect, Gnedenko and Kolmogorov [**3**, p. 164], use the same formulation. Equation (4) is equivalent to (3), with

$$\mu = \begin{cases} \delta, & \alpha \neq 1, \\ \delta - (2/\pi)\beta c \log c, & \alpha = 1. \end{cases} \tag{5}$$

For $\alpha \neq 1$, or for $\alpha = 1$ and $\beta = 0$, the DuMouchel and Hall *et al.* parametrizations are therefore identical, except for the choice of symbols. For $\alpha = 1$, however, the two location parameters δ and μ are quite different, unless $c = 1$ or $\beta = 0$.

If a random variable x has scale parameter c_0 and DuMouchel location parameter δ_0, and a is a positive constant, then ax has parameters $c_{ax} = ac_0$ and, as one might expect,

$$\delta_{ax} = a\delta_0, \quad \text{for all } \alpha. \tag{6}$$

But if the same x has Hall location parameter μ_0, then the Hall location parameter of ax is nonlinear in a for $\alpha = 1$:

$$\mu_{ax} = \begin{cases} a\mu_0, & \alpha \neq 1, \\ a\mu_0 - (2/\pi)\beta a c_0 \log a, & \alpha = 1 \end{cases} \tag{7}$$

(see [**9**, p. 11, Equation (1.2.1)]). The nonlinearity of (7) is not intrinsic to stable distributions, but is simply an artifact of choosing (4) rather than (3) as the general log c.f.

Using the linearizing parametrization (3), let $x_1 \sim S(\alpha, \beta, c_1, \delta_1)$ and $x_2 \sim S(\alpha, \beta, c_2, \delta_2)$ be independent drawings from stable distributions with a common α and β. Then their sum, $x_3 = x_1 + x_2$, has the stable distribution $S(\alpha, \beta, c_3, \delta_3)$, where

$$c_3^{\alpha} = c_1^{\alpha} + c_2^{\alpha}, \tag{8}$$

$$\delta_3 = \begin{cases} \delta_1 + \delta_2, & \alpha \neq 1, \\ \delta_1 + \delta_2 + (2/\pi)\beta(c_3 \log c_3 - c_1 \log c_1 - c_2 \log c_2), & \alpha = 1. \end{cases} \tag{9}$$

Furthermore, if x_1 and x_2 have the *same* stable distribution, with $c_1 = c_2$ and $\delta_1 = \delta_2$, then their average, $\bar{x} = (x_1 + x_2)/2$, will have scale $c_{\bar{x}} = c_1 2^{(1/\alpha)-1}$ and location parameter

$$\delta_{\bar{x}} = \begin{cases} & \alpha \neq 1, \\ \delta_1 + (2/\pi)\beta c_1 \log 2, & \alpha = 1. \end{cases} \tag{10}$$

When $\alpha = 1$, the average has the same scale $c_{\bar{x}} = c_1 = c_2$ as the two contributions, so that the distribution of $\bar{x}$ can differ from that of x_1 and x_2 by at most a lateral shift. Equation (10) shows that there is such a shift unless $\beta \neq 0$. This is an intrinsic property of stable distributions with $\alpha = 1$, and is independent of how location and scale are parametrized. Equation (9) shows that this shift arises as a peculiar consequence of convolution itself.

If the same x_1 and x_2 have Hall *et al.* location parameters μ_1 and μ_2, then their sum x_3 has scale c_3 as above and location parameter

$$\mu_3 = \mu_1 + u_2, \quad \text{for all } \alpha \tag{11}$$

(see [**9**, p. 10]). Parametrization (4) therefore appears to simplify, and even linearize, the behaviour under convolution when $\alpha = 1$. However, this appearance is illusory, as may be seen when we consider the Hall location parameter of the *average* of two i.i.d. stable variates x_1 and x_2:

$$\mu_{\bar{x}} = \begin{cases} \mu_1, & \alpha \neq 1, \\ \mu_1 + (2/\pi)\beta c_1 \log 2, & \alpha = 1. \end{cases} \tag{12}$$

Equation (12) looks exactly like (10), despite the deceptive simplicity of (11). Since shifting the distribution to the right or left changes μ by an equal amount, μ must reflect the same shift under averaging as does δ when $\alpha = 1$. However, it is not obvious from (11) where this shift is coming from, since the complicating terms in (9) that cause it have been hidden in μ through (5). Their effect reappears when we divide x_3 by 2 to obtain the average, because changing the scale unexpectedly introduces an extra shift into the Hall location parameter, due to the nonlinearity of (7).

In [**7**], I defined a *focus of stability* to be any quantile of a stable distribution that is invariant under averaging of i.i.d. contributions. Unless $\alpha = 1$, the scale of the average is different from that of the contributions, and therefore the focus of stability is unique. In the *convergent* cases, $\alpha > 1$, the unique focus of stability is at $\delta = \mu = Ex$, and the distribution of the average converges in toward the focus. In the *divergent* cases, $\alpha < 1$, the unique focus of stability is at $\delta = \mu$, and the distribution of the average diverges out away from the focus, while Ex is undefined. In the *Cauchy* case, $\alpha = 1$ and $\beta = 0$, *every* quantile is a focus of stability, since the distribution of the average coincides with that of the contributions. When $\alpha = 1$ and $\beta \neq 0$, however, no focus of stability exists, as the distribution of the average lies completely to the left or right of that of the contributions. In the title of this note, I have therefore identified these special stable distributions as the *afocal stable distributions*.

Stable distributions with $\alpha \neq 1$, $\delta = 0$ or $\alpha = 1$, $\beta = 0$ are sometimes classified as *strictly stable*. *Afocal stable* is therefore simply a convenient and informative synonym for the category that would otherwise have to be called *stable-yet-not-strictly-stable-after-shifting*.

Holding β, c, and either δ or μ constant, the stable c.f., and therefore the distribution itself, undergoes a discontinuity as α passes 1 unless $\beta = 0$ (see [**7**, Table VI], which shows (δ-median(x))/c as a function of α and β). However, if we define

$$\zeta = \begin{cases} \delta + \beta c \tan(\pi\alpha/2), & \alpha \neq 1, \\ \delta, & \alpha = 1, \end{cases} \tag{13}$$

then Zolotarev [**11**] has shown (for $c = 1$ and therefore trivially also for $c \neq 1$ under (3)) that the c.f., and therefore the distribution, of the new variable $z = x - \zeta$ undergoes no discontinuity as α passes unity (see [**7**, Table VII]). The discontinuity is therefore a discontinuity in the focus of stability, which we have fixed on artificially as our location parameter for $\alpha \neq 1$, rather than in the distribution itself.

In their recent monograph, Janicki and Weron [**6**, p. 23] take the position that the only acceptable value of β when $\alpha = 1$ is 0. In fact, the afocal stable distributions fit in smoothly between the convergent and divergent cases, if we just know where to look for them. With finite samples, they are statistically indistinguishable from their immediate neighbours.

Samorodnitsky and Taqqu [**9**], using (4) and following Zolotarev [**11**, p. 454], give the shift that achieves continuity in the general case as $c^\alpha\beta\tan(\pi\alpha/2)$, rather than as $c\beta\tan(\pi\alpha/2)$. It is not obvious why this shift, which is nonlinear in c, should work for $c \neq 1$, until we observe that

$$\lim_{\alpha\to 1}(c\beta\tan(\pi\alpha/2) - c^\alpha\beta\tan(\pi\alpha/2)) = (\pi/2)\beta c\log c.$$

This is just the difference between μ and δ at $\alpha = 1$, as given by (5).

Because stable distributions are infinitely divisible, they are particularly attractive for continuous time modelling. The stable generalization of the familiar Gaussian Wiener process is called an *α-stable Lévy motion* [**6**, **9**]. A standard α-stable Lévy motion $\xi(t)$ is a continuous time process whose increments $\xi(t+\Delta t) - \xi(t)$ are, in the strictly stable cases, distributed as $S(\alpha, \beta, \Delta t^{1/\alpha}, 0)$, and whose non-overlapping increments are independent. In the afocal stable case, treated in neither [**6**] nor [**9**, p. 349], the increments of a standard α-stable Lévy motion may be taken to be distributed as $S(1, \beta, \Delta t, (2/\pi)\beta\Delta t\log\Delta t)$ in terms of (3).

Formulations (3) and (4) are both completely general and equally valid, if used consistently. Unfortunately, I must confess that in [**7**], a paper in which I had hoped to clarify the topic of skew-stable distributions, I inadvertently raised the curtain on a Second Act to Hall's parametric 'Comedy of errors', by giving the c.f. as (4), with δ in place of μ, while erroneously citing (2), (6), (9) and (13) as properties of the distribution defined by it. In order to make that paper internally consistent while retaining the highly desirable properties (2) and (6), I should have added a factor of c to the t in $\log|t|$ in equation (1.1) of that paper, thereby effectively replacing (4) with (3). I employ (3) consistently in [**8**].

Stuck [**10**] makes a similar error, only the other way around. He gives the c.f. for $\alpha = 1$ in form (3), but then claims that the shift that generates continuity is $c^\alpha\beta\tan(\pi\alpha/2)$. His proof outline shows that this shift is valid for $c = 1$, and then jumps to the conclusion that it must therefore be valid for all c, while in fact this is not the case. A further complication is that Stuck switches the sign on β in (1) for $\alpha \neq 1$, thereby precluding continuity altogether, but then, in the earlier stable tradition documented by Hall, makes two separate sign errors that happen to eliminate the effect of this.

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