An Estimate of the Liquidity Premium

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The liquidity premium on U.S. government securities is quantitatively estimated and tabulated, using maturities from 1 month to 30 years. Unbiased forecasting by the market is assumed in order to get at expectations. The premium is estimated, first allowing it to take any shape and then constraining it to conform to a functional form which implies that the "normal" shape of the yield curve is monotonically increasing toward an asymptote. Tests for constancy of the premium over the post-Accord period, normality of the forecasting errors, and monotonicity of the premium with respect to maturity are performed, and the dependence of the premium on the level of interest rates is discussed.

I. Introduction

It has long been known that short-term interest rates tend to lie below long-term rates on securities with equal default risk. As early as 1935, Charles C. Abbott referred to the reversals in the usual relationship that occurred from June 1920 to January 1921 and from May 1928 to November 1929 as "the more striking in that they contradict the well-known tendency for obligations of short maturity to sell consistently at higher prices and with lower yields than obligations of equal security but longer maturity."[1]

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[1] Abbott (1935, p. 9). As it happens, the evidence Abbott cited was not statistically strong enough to demonstrate the existence of a liquidity premium (McCulloch 1973, pp. 62–63).

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A number of questions regarding this "liquidity premium" need to be answered. How large is it for various pairs of maturities? Does the premium increase monotonically with maturity? Does it vary with the level of interest rates? Is it the same for different periods in time? And is it large enough to imply that holders of longer-term bonds have a systematically higher holding period yield, after taking into account transactions costs as reflected by bid-asked spreads?

The liquidity premium is defined as the difference between a forward interest rate and the market's expectation of the corresponding future spot rate. Measurement of this premium is complicated by the fact that expectations are not directly observable. A straightforward, but in practice unreliable and limited, method is to ask market participants what they believe future interest rates will be (Kane and Malkiel 1967). Another approach is to build a model of expectation generation based on the past behavior of interest rates and, in some instances, of other variables.[2] However, these models entail strong assumptions about the set of information people take into account in forming their expectations. A third approach simply assumes that forecasting errors have mean zero, so that the subsequent realization gives us an unbiased estimator of the forecast. In one variation on this third approach, Kessel (1965) actually compares the forward rate to the subsequent spot rate. However, this variation does not make the most efficient use of the data available. Under the same basic assumption, Cagan (1969) and Roll (1970) have been able to use the term structure for two nearby points in time to obtain observations on the liquidity premium all the way out to the longest maturity observed.

Unfortunately, neither Cagan's paper nor Roll's book tells us about the size of the liquidity premium applicable to forecasts for periods longer than a few months. For instance, at the end of June 1953, the forward rate on a 10-year loan to begin 5 years in the future was 3.16 ($\pm$0.01) percent per year. Five years later, the corresponding 10-year spot rate was 2.94 ($\pm$0.02) percent per year.[3] It would be interesting to know whether such a fall was unanticipated, whether it is about the size one would expect given liquidity preference, or whether the difference was less than the mean liquidity premium, so that it actually represents an unanticipated rise. The answer has not been given in the literature for maturities of this duration.

In this paper we use a modification of the Cagan-Roll method to estimate the size of the post-Accord liquidity premium for all available maturities, together with the relevant measurement errors implied by

[2] E.g., Meiselman (1962), Diller (1969), Nelson (1972), Modigliani and Shiller (1973), Modigliani and Sutch (1966).

[3] Calculated by the author, using bid-asked mean prices of U.S. government securities and the technique described in McCulloch (1971).

this procedure, and to attack the other questions posed above. The results are compared with those obtained by other investigators.

II. The Behavior of the Postwar Liquidity Premium

In this section, we define an estimator of the liquidity premium. Its values for three pilot maturities are used to test the premium and the variance of the forecasting errors for constancy over the postwar period. The forecasting errors are tested for normality, and the dependence of the liquidity premium on the level of interest rates is discussed.

An Estimator of the Liquidity Premium

In the real world, observed bond maturities do not lie at evenly spaced discrete intervals, even though this would make our calculations easier. Nevertheless, we would expect the price of any security to be governed by a smooth discount function $\delta(t, s)$, which gives the value at time t of a dollar to be repaid at time s in the future, that is, after maturity $m = s - t$.

Corresponding to the discount function we observe at time t is a two-dimensional complex of forward interest rates $r(t, s_1, s_2)$ on hypothetical point-payment forward loans to begin at time s_1 and be repaid at time s_2, where $t \leq s_1 \leq s_2$. Most past investigators have taken m_2 (where $m_2 = s_2 - s_1$, the duration of the forward loan) as a constant with some convenient value, usually a week, month, quarter, or year, and have considered the liquidity premium corresponding only to a one-dimensional complex of forward rates as m_1 varies (where $m_1 = s_1 - t$, the period until the forward loan begins). We, on the other hand, are interested in all values of m_2, from the limit as m_2 approaches zero out to several years or even decades. In the next section, we will treat both forward rates and the liquidity premium in terms of the two variables m_1 and m_2.

However, the forward rates for different values of m_2 are far from independent. In general, forward rates with large m_2 can be obtained by averaging together appropriate forward rates with smaller m_2. It is therefore convenient for our investigation of the qualitative properties of the liquidity premium to single out a one-dimensional complex of independent forward rates, taking m_2 equal to its smallest value of interest. In our case, this is the limit as m_2 goes to zero. This "marginal" or "instantaneous" forward rate $\rho(t, s)$, where $s = t + m_1$, is related to the discount function by

$$\rho(t, s) = -100 \frac{\partial\delta(t, s)/\partial s}{\delta(t, s)}. \tag{1}$$

Because we fit a smooth curve to the discount function, we are able to evaluate this derivative. When t and s coincide, $\rho(t, t)$ becomes a "spot" rate of interest on a hypothetical loan of very short maturity, in effect a "call money" rate.

We define the liquidity premium $\pi(m)$ to be the difference between the forward rate and the expected value of the future spot rate:

$$\rho(t, s) = E_t\rho(s, s) + \pi(m), \qquad m = s - t, \tag{2}$$

where E_t denotes the expected value as of time t.[4] Of course, not all participants have the same expectations about the future. We must regard $E_t\rho(t, s)$ as some sort of market average of individual expectations.[5]

If forecasting errors are unbiased, we could compare the forward rate with the actual subsequent spot rate to get an estimator of $\pi(m)$. However, for large values of m, we would have very few such pairs of observations. We can get around this problem by observing that today's forecast of some distant future variable must be an unbiased estimator of all future forecasts of that variable. Thus, if we have observations on the term structure at points in time Δt apart, we must have:

$$E_t\rho(s, s) = E_t[E_{t+\Delta t}\rho(s, s)], \tag{3}$$

where s is some time later than $t + \Delta t$. This seems like a reasonable assertion; if we had any reason at time t to believe that our next period expectation of $\rho(s, s)$ would be any different from our current expectation, we would already have incorporated this information into our current expectation, bringing the two expectations into equality.[6]

Given observations on the term structure at times $t_1, t_2, \ldots t_j \ldots$, where $t_j - t_{j-1} = \Delta t$, we define $\Delta\pi_j(i\Delta t)$ by

$$\Delta\pi_j(i\Delta t) = \rho(t_j, t_j + i\Delta t) - \rho(t_j + \Delta t, t_j + i\Delta t). \tag{4}$$

Putting (2), (3), and (4) together, we have

$$\Delta\pi_j(i\Delta t) = \pi(i\Delta t) - \pi[(i - 1)\Delta t] + u_{i,j}, \tag{5}$$

[4] Two slightly different possible definitions of the liquidity premium are discussed in McCulloch (1973, pp. 18–19).

[5] See Williams (1938), chap. 10. Bierwag and Grove (1967) develop an ingenious model in which the market behaves as if governed by a market expectation which is an arithmetic average of individual expectations, weighted in proportion to the individuals' audacity (negative risk aversion), certainty, and wealth. Unfortunately, they deal in terms of price uncertainty, instead of consumption uncertainty. Compare Stiglitz (1970), who correctly deals with consumption uncertainty, although he assumes homogeneous expectations.

[6] This is essentially the point Samuelson makes (1965, pp. 41–49). His analysis is unnecessarily complicated by assumptions about the relation between anticipations and current forward prices. The real crux of his argument is that anticipations themselves must fluctuate randomly. The proof of our assertion is similar to the proofs he gives. Once we have identified the psychological anticipation with the mathematical expected value by our unbiased forecasting assumption, the remainder of the proof is a purely mathematical consequence of the properties of probability distributions.

where $u_{i,j}$ is a random forecasting error with mean zero. When we sum (5) over i from one to n, we obtain the following estimator of $\pi(m)$ for $m = n\Delta t$:

$$\begin{aligned} \pi_j(m) &= \sum_{i=1}^{n} \Delta\pi_j(i\Delta t), \\ &= \pi(m) - \pi(0) + \sum_{i=1}^{n} u_{i,j}, \\ &= \pi(m) + v_{m,j}. \end{aligned} \tag{6}$$

Since the $u_{i,j}$ have mean zero, the $v_{m,j}$ also have mean zero.[7]

The discount function $\delta(t, s)$ was fit for the close of each month from December 1946 to March 1966 by means of a quadratic spline.[8] The prices used were the means of bid-and-asked monthly closing offers for most fully taxable U.S. government bills, notes, and bonds.[9] Forward rates were derived from these discount curves, and from them the values $\pi_j(m)$ were calculated. Figure 1 shows monthly observations on $\pi_j(m)$ for $m = 1$ year, along with the average yield to maturity

$$\begin{aligned} \eta(t_j, m) &= \frac{1}{m} \int_{t_j}^{t_j+m} \rho(t_j, s)\, ds \\ &= -\frac{100}{m} \ln \delta(t_j, t_j + m). \end{aligned} \tag{7}$$

For ease of comparison, both are shown relative to the same origin, but with different scales.

Likelihood Ratio Tests for Homogeneity

For the purpose of analysis, we divided this interval into four approximately equal periods, as shown in table 1. These periods correspond roughly to the pre-Accord period, the first Eisenhower administration,

[7] Roll (1970, pp. 98–99) estimates the liquidity premium as the sum of first differences of forward rates, so this is basically the same method as his. However, he used forward rates spanning 1 week, while we have taken the limit as the period spanned goes to zero.

[8] See McCulloch (1971) for details of this procedure, and Rice (1969, 2:123–67) for a discussion of splines in general. Compare Williams (1938, pp. 120–24), who gives an algorithm that fits security prices exactly. His method yields a step-function forward curve (see his chart 4, p. 355), instead of one that is continuous. The method proposed by Bryan and Carleton (1972) is similar to his in this respect. Weingartner (1966) proposes a method based on successive approximations to the coupon-free yield curve. Buse (1970) raises objections that apply to a large number of other studies.

[9] These data were collected by Reuben A. Kessel from the quotation sheets of Salomon Brothers and Hutzler, Inc., of New York, and were processed under the supervision of Merton H. Miller and Myron Scholes. Reduction of these data into discount curves and forward rates was supported by a grant from the University of Chicago, Graduate School of Business.

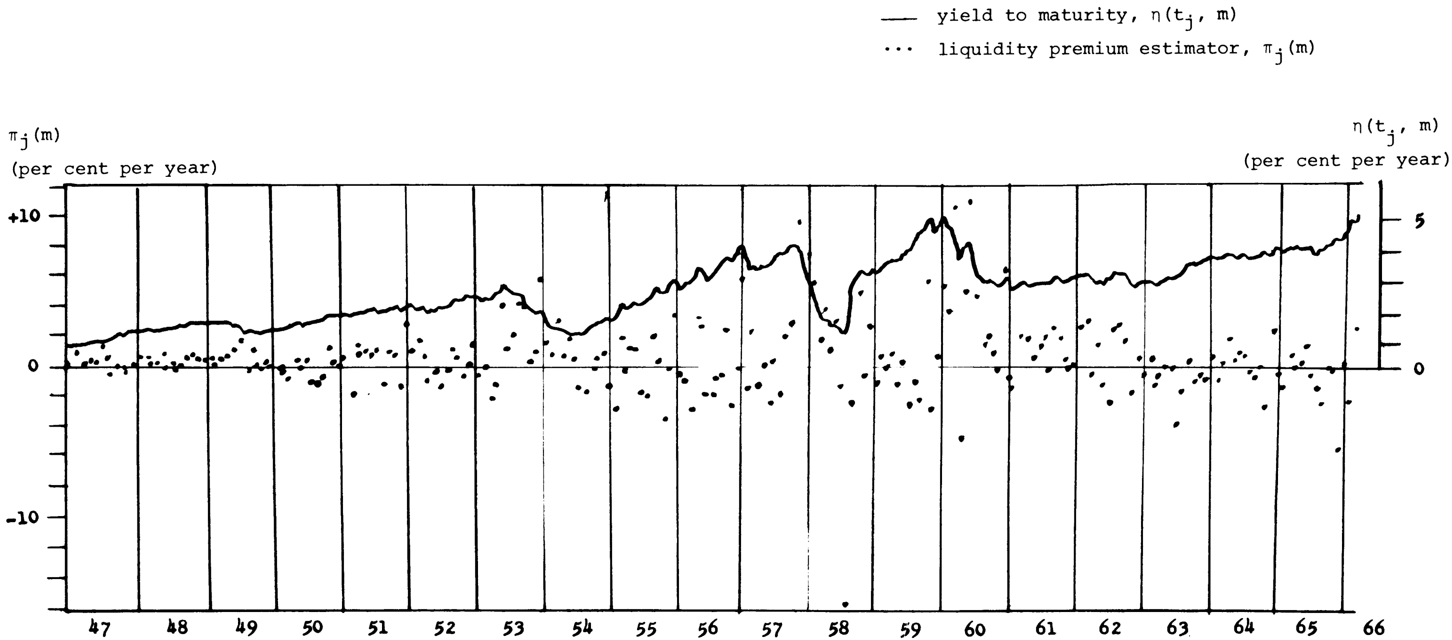

FIG. 1.—Solid line-yield to maturity, $n(t_j, m)$; dotted line-liquidity premium estimator, $\pi_j(m)$

TABLE 1

DEFINITION OF PERIODS FOR HOMOGENEITY TESTS

Period	Number of Even-numbered Observations
1. December 31, 1946–March 4, 1951	25
2. March 4, 1951–December 31, 1955	29
3. December 31, 1955–December 31, 1960...........	30
4. December 31, 1960–March 31, 1966	31

the second Eisenhower administration, and the Kennedy/Johnson administration.

Although we would expect the pure forecasting errors to be serially uncorrelated, the errors we observe may not be because of additional errors introduced in measuring $\pi_j(m)$. The term structure cannot be measured exactly, and measurement errors for forward rates of different maturities are strongly correlated. Adjacent estimators $\pi_{j-1}(m)$ and $\pi_j(m)$ are both dependent on the term structure for time t_j, so their measurement errors will not be independent. However, the measurement errors for alternate estimators are completely independent, so that if we discard alternate observations, we should satisfy the serial independence assumption necessary to estimate the mean. Using all the data would give us smaller, but erroneous, confidence intervals. In order to avoid the month of the Accord, we will use the even-numbered observations, that is, starting with the month January 31, 1947 to February 29, 1947.[10]

The premium seems to be small with a small variance during period 1, definitely positive with an intermediate variance during periods 2 and 4, and perhaps larger with the largest variance during period 3. This suggests the following hypotheses: H_1—a common mean and variance for all four periods (one mean, one variance); H_2—one mean and variance for period 1, and a second mean and variance for periods 2–4 (two means, two variances); H_3—one mean for all four periods, but separate variances for each period (one mean, four variances); H_4—one mean for period 1, a second mean for periods 2–4, and separate variances for each period (two means, four variances); H_5—a different mean and variance for each period (four means, four variances).

These hypotheses were compared to one another using the likelihood ratio test. The asymptotically χ^2 statistics are shown in table 2, along with the relevant critical values of the appropriate χ^2 distribution.

We see that we may easily reject H_1 in favor of H_2, and H_2 in favor of H_4. We may reject H_3 in favor of H_4 for $m = 3$ months but get insignificant values for this test for 1 year and 15 years. We may not,

[10] It will be shown below, in table 3, that while all the observations taken together exhibit autocorrelation, alternate ones do not.

TABLE 2

LIKELIHOOD RATIO χ^2 STATISTICS

HYPOTHESIS		MATURITY			DEGREES OF FREEDOM	CRITICAL LEVELS OF χ^2				
						Degrees of Freedom	Confidence Level			
Null	Alternative	3-month	1-year	15-year			.995	.99	.90	.80
H_1	H_2	48.3	53.6	34.0	2	1	7.9	6.6	2.7	1.6
H_2	H_4	51.9	44.2	19.8	2	2	10.6	9.2	4.6	3.2
H_3	H_4	14.7	1.2	0.0	1					
H_4	H_5	3.8	1.2	4.4	2					

NOTE.—Based on 115 even observations.

TABLE 3

MAXIMUM-LIKELIHOOD ESTIMATES OF MEANS AND STANDARD DEVIATIONS UNDER H_4

			Maturity		
Statistic	Period	Observations	3 months	1 year	15 years
$\hat{\mu}$	1	Evens	0.052	0.06	−0.8
			(0.046)	(0.12)	(1.2)
$\hat{\mu}$	2–4	Evens	0.357	0.33	−0.7
			(0.057)	(0.22)	(1.7)
$\hat{\mu}$	2–4	Odds	0.394	0.50	0.1
			(0.054)	(0.22)	(1.5)
$\hat{\sigma}$	1	Evens	0.23	0.61	5.88
$\hat{\sigma}$	2	Evens	0.58	1.67	14.03
$\hat{\sigma}$	3	Evens	1.42	4.67	27.06
$\hat{\sigma}$	4	Evens	0.38	1.68	13.25
vNR	1–4	All	1.33	1.67	1.90
		(231 observations)	(0.999999+)	(0.989)	(0.58)
vNR	1–4	Evens	1.82	1.82	1.77
		(115 observations)	(0.71)	(0.70)	(0.82)

NOTE.—Means "μ" are in percent per year. Standard deviations "σ" are in (percent per year) per $(\text{month})^{1/2}$. Confidence levels shown in parentheses beneath the von Neumann ratios (vNR) are for a two-tailed test against no autocorrelation, based on the normal approximation for large samples given in Hart (1942).

however, reject H_4 in favor of H_5. We interpret these results as meaning that there has been a fairly constant liquidity premium since the Accord, although the variance of forecasting errors has not been constant. We may not treat the pre-Accord liquidity premium as equal to the post-Accord premium, but this is to be expected, since prior to the Accord the Federal Reserve System explicitly supported the prices of government securities, so that forward rates did not necessarily reflect market forces. The difference between the pre- and post-Accord means for 1 year and 15 years is not significant due to the greater accumulation of forecasting errors for these maturities, but we assume that the same is true for these maturities as for 3 months.[11]

Table 3 shows the maximum-likelihood estimates of the means and standard deviations for the different periods under H_4. The pre-Accord premium, at least for 3 months, was significantly lower than the post-Accord premium. Standard deviations increase with maturity and are higher for period 3 and lower for period 1 than they are for periods 2 or 4. For comparison, the post-Accord means for the omitted odd observations are shown. They run a little higher than the means for the even observations, but not significantly so. Also shown are von Neumann ratios based on the H_4 residuals divided by their estimated standard deviations. As predicted above, autocorrelation is highly significant when we use all the observations (at least for 3 months and 1 year), but becomes insignificant when we use only alternative observations.

[11] Wallace (1964, pp. 25–26) also finds significantly different behavior prior to the accord than after.

TABLE 4

STANDARDIZED RANGE TESTS FOR NORMALITY

	3 months	1 year	15 years
Raw data:			
Standardized range	9.48	8.85	6.41
$\hat{\alpha}_{.96}$	1.34	1.47	1.47
After adjustment for heteroskedasticity:			
Standardized range	5.38	5.74	5.53

Confidence Intervals for Normality—90 Observations*
(Two-tailed Test)

.80	4.36–5.60
.90	4.23–5.82
.98	4.01–6.27
.99	3.95–6.44

NOTE.—Based on 90 even post-Accord observations.
* Interpolated from David, Hartley, and Pearson (1954).

Tests for Normality

Table 4 shows the "standardized range" statistic for our post-Accord observations $\pi_j(m)$. It is defined as the ratio of the sample range to the estimate of the standard deviation of the errors. When the standard deviation is treated as if it were constant, we find that we may easily reject normality at the 99 percent level for 3 months and 1 year, and at the 98 percent level for 15 years. This finding might lead us to reject, as Roll does (1970, chap. 4), normality in favor of the class of symmetric stable distributions. If so, we would arrive at the estimates for the characteristic exponent α of the stable distributions shown in the second line of table 4.[12] Our estimates of α are on the same order as those obtained by Roll.[13] However, when we allow for different variances in periods 2, 3, and 4, and divide the maximum-likelihood residuals by their respective estimated standard deviations before calculating the standardized range, we find that the resulting statistics are never significant at the 90 percent level, and only once at the 80 percent level. Although we can reject homoskedastic normality, we are unable to reject heteroskedastic normality. Provided we allow for this heteroskedasticity, we seem to be justified in using tests based on the more familiar normal distribution.[14]

[12] See Fama and Roll (1968; 1971) for techniques of estimating α. The estimator $\hat{\alpha}_{.96}$, which is used here, is based on the .04 and .96 fractiles of the sample distribution. Our "standardized range" is the same as Fama and Roll's "Studentized range."

[13] Roll finds values of $\hat{\alpha}$ from 1.22 to 1.72, mostly around 1.4, for 1949–64 weekly data. For 17–22 weeks to maturity, he finds some as low as 1.00 for 1959–64 (1970, p. 70). A Cauchy distribution has $\alpha = 1$, while a normal distribution has $\alpha = 2$.

[14] Press (1967) and Praetz (1972) also offer interpretations of leptokurtic residuals in terms of mixtures of normals with different variances. However, the compound distributions they suggest are homogeneous over time, in contrast with the one we propose.

Dependence of the Liquidity Premium on the Level of Interest Rates

Van Horne (1965) and Nelson (1972, p. 93) offer evidence which, they contend, supports the view that the liquidity premium decreases with the level of interest rates. On the other hand, Kessel (1965, p. 25) and Cagan (1969, p. 93) give evidence intended to show that the liquidity premium increases with the level of rates. These tests have been called into question by Telser (1967) and McCulloch (1973, pp. 36–39).

Any direct comparison between our liquidity premium estimator and the level of rates is open to two objections. First, in deriving it, we assumed the underlying premium was constant. This can bias any comparison to the extent that the unforeseen change in rates has been correlated with the level of rates. Second, the premium estimator and the level are both calculated with measurement error from the same data. This can lend some inconsistency to any regression coefficients.

Although it should be interpreted with caution, the comparison is still worth making. Figure 1 and similar charts (not shown here) for $m =$ 3 months and $m =$ 16 years show no obvious relation between our liquidity premium estimator and the level of interest rates. If there had been a pronounced relation, it would have been picked up by our maximum-likelihood test for nonconstancy of the premium since the Accord, in view of the fairly steady rise in interest rates over this period. Furthermore, direct regressions of the premium estimator on the level show no significant correlation either way for the post-Accord period.[15]

In view of the inconclusive nature of our own findings and the lack of consensus in the literature, we will assume that the post-Accord liquidity premium has been approximately independent of the level of interest rates.

III. The Post-Accord Liquidity Premium

Using the information we gathered in Section II about the behavior of the liquidity premium and the forecasting errors, we are ready to estimate the liquidity premium for a variety of maturities. We first estimate it without imposing any particular functional form on it. We find no evidence to contradict monotonicity or boundedness, so we then estimate it under the assumption that the premium monotonically approaches an asymptote. Since we are interested primarily in how free-market forces (relatively speaking) shape the term structure, and since the pre-Accord liquidity premium has been shown to be significantly different from that since the Accord, we restrict ourselves in this section to our 90 alternate post-Accord observations.

[15] See McCulloch (1973, p. 35). If period 1 is included in the regressions, the level has a positive and significant slope for $m =$ 3 months. However, it is questionable whether this is the direct effect of the lower pre-Accord rates themselves, or if the low rates are only acting as a proxy for the higher level of intervention in the securities market prior to the Accord.

TABLE 5

FREE-FORM ESTIMATES OF LIQUIDITY PREMIUM
(POST-ACCORD EVEN OBSERVATIONS)

m	n	$\hat{\pi}(m)$	$\hat{\bar{\pi}}(m)$
0	0	0.0	0.0
		(0.0)	(0.0)
1 month	90	0.19	0.09
		(0.03)	(0.02)
2 months	90	0.32	0.17
		(0.05)	(0.03)
3 months	90	0.36	0.23
		(0.06)	(0.03)
6 months	90	0.37	0.31
		(0.11)	(0.06)
9 months	90	0.32	0.32
		(0.16)	(0.08)
1 year	90	0.33	0.32
		(0.21)	(0.11)
2 years	90	0.34	0.33
		(0.40)	(0.20)
3 years	90	0.37	0.33
		(0.56)	(0.29)
5 years	90	0.47	0.39
		(0.82)	(0.44)
10 years	90	−0.68	0.15
		(1.40)	(0.75)
20 years	87	0.59	0.23
		(2.10)	(1.24)
30 years*	67	−5.24	−2.47
		(3.81)	(2.02)

NOTE.—Standard errors in parentheses.
* Periods 2 and 3 treated as if they had the same variance, due to shrinking sample size.

The Liquidity Premium and the Average Liquidity Premium

Table 5 gives our free-form estimates of the liquidity premium $\pi(m)$ and of the average liquidity premium,

$$\bar{\pi}(m) = \int_0^m \pi(x)\, dx. \tag{8}$$

The function $\pi(m)$ gives us the "typical" shape of the marginal forward interest rate curve $\rho(t, s)$, for if we expected to have the same marginal forward rates (as a function of maturity $m = s - t$) forever, we would find

$$\rho(t, s) = \rho(t, t) + \pi(s - t). \tag{9}$$

In this case, we would also expect to have the same yield curve forever and would have

$$\eta(t, m) = \eta(t, 0) + \bar{\pi}(m). \tag{10}$$

Thus, the average liquidity premium shown in table 5 gives us the "typical" shape of the yield curve. This average liquidity premium was estimated by averaging together the values

$$\bar{\pi}_j(m) = \frac{1}{m} \int_0^m \pi_j(x)\, dx, \tag{11}$$

again allowing for heteroskedasticity.[16]

The Mean Liquidity Premium

One of the most important applications of the liquidity premium is to evaluate the market's expectation of future interest rates spanning a positive interval in the future, on the basis of current forward rates. We define the mean forward rate observed at time t corresponding to a loan to begin at time $t + m_1$ and to be repaid at time $t + m_1 + m_2$ by

$$r(t, t + m_1, t + m_1 + m_2) = \frac{1}{m_2} \int_{m_1}^{m_1+m_2} \rho(t, t + m)\, dm. \tag{12}$$

In order to calculate the market's expectation of the future m_2-year yield to maturity $\eta(t + m_1, m_2)$ from this forward rate, we need to know the mean liquidity premium:

$$p(m_1, m_2) = r(t, t + m_1, t + m_1 + m_2) - E_t\eta(t + m_1, m_2). \tag{13}$$

It can be shown that

$$p(m_1, m_2) = [(m_1 + m_2)\bar{\pi}(m_1 + m_2) - m_1\bar{\pi}(m_1) - m_2\bar{\pi}(m_2)]/m_2. \tag{14}$$

Consequently,

$$p_j(m_1, m_2) = [(m_1 + m_2)\bar{\pi}_j(m_1 + m_2) - m_1\bar{\pi}_j(m_1) - m_2\bar{\pi}_j(m_2)]/m_2 \tag{15}$$

is an unbiased observation on $p(m_1, m_2)$. These observations were averaged together heteroskedastically to obtain the estimates of $p(m_1, m_2)$ shown in table 6.

It happens that when m_2 equals the observation interval Δt (1 month in our case), our estimator of the mean liquidity premium is identically equal to the difference between the 1-month holding period yield on a security of maturity $m + \Delta t$ and the certain yield on a security of Δt

[16] The integral in eq. (11) was evaluated by summing trapezoids with width $\Delta t = 1/12$ year.

TABLE 6

FREE-FORM ESTIMATES OF MEAN LIQUIDITY PREMIUM $p(m_1, m_2)$

(Post-Accord Even Observations--Standard Errors in Parentheses)

m_1 \ m_2	0	1 mo.	2 mos.	3 mos.	6 mos.	9 mos.	1 year	2 years	3 years	5 years	10 yrs.	20 yrs.	30 yrs.
0	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)
1 month	0.19 (0.03)	0.16 (0.02)	0.12 (0.02)	0.10 (0.02)	0.04 (0.02)	0.03 (0.02)	0.02 (0.02)	0.01 (0.02)	0.01 (0.02)	0.01 (0.01)	-0.01 (0.01)	-0.00 (0.01)	-0.01 (0.01)
2 months	0.32 (0.05)	0.25 (0.04)	0.19 (0.03)	0.15 (0.04)	0.06 (0.04)	0.03 (0.04)	0.03 (0.03)	0.01 (0.03)	0.01 (0.03)	0.01 (0.03)	-0.01 (0.02)	-0.00 (0.02)	-0.03 (0.02)
3 months	0.36 (0.06)	0.29 (0.05)	0.22 (0.05)	0.16 (0.05)	0.06 (0.05)	0.04 (0.05)	0.03 (0.05)	0.02 (0.05)	0.01 (0.05)	0.01 (0.04)	-0.02 (0.03)	-0.00 (0.03)	-0.05 (0.03)
6 months	0.37 (0.11)	0.27 (0.11)	0.18 (0.11)	0.12 (0.11)	0.04 (0.11)	0.04 (0.11)	0.04 (0.10)	0.01 (0.10)	0.02 (0.09)	0.01 (0.08)	-0.05 (0.07)	-0.01 (0.06)	-0.10 (0.06)
9 months	0.32 (0.16)	0.23 (0.16)	0.15 (0.16)	0.11 (0.16)	0.05 (0.16)	0.05 (0.16)	0.05 (0.15)	0.01 (0.15)	0.02 (0.14)	0.01 (0.12)	-0.08 (0.10)	-0.01 (0.08)	-0.15 (0.10)
1 year	0.33 (0.21)	0.25 (0.21)	0.18 (0.21)	0.13 (0.21)	0.08 (0.21)	0.06 (0.20)	0.05 (0.20)	0.02 (0.20)	0.03 (0.18)	0.00 (0.16)	-0.11 (0.14)	-0.01 (0.11)	-0.20 (0.13)
2 years	0.34 (0.40)	0.24 (0.39)	0.17 (0.39)	0.12 (0.39)	0.05 (0.40)	0.04 (0.39)	0.03 (0.39)	0.03 (0.38)	0.05 (0.35)	-0.07 (0.32)	-0.24 (0.27)	0.04 (0.23)	-0.42 (0.25)
3 years	0.37 (0.56)	0.28 (0.56)	0.21 (0.56)	0.16 (0.56)	0.10 (0.55)	0.09 (0.55)	0.09 (0.55)	0.08 (0.53)	0.05 (0.51)	-0.20 (0.48)	-0.38 (0.40)	0.09 (0.34)	-0.53 (0.38)
5 years	0.47 (0.82)	0.37 (0.82)	0.29 (0.81)	0.23 (0.81)	0.12 (0.82)	0.07 (0.81)	0.02 (0.81)	-0.17 (0.80)	-0.34 (0.80)	-0.65 (0.79)	-0.66 (0.64)	-0.83 (0.62)	-1.28 (0.68)
10 years	-0.68 (1.40)	-0.77 (1.40)	-0.86 (1.40)	-0.92 (1.40)	-1.02 (1.40)	-1.07 (1.39)	-1.09 (1.39)	-1.21 (1.36)	-1.26 (1.34)	-1.33 (1.27)	-0.48 (1.16)	-1.91 (1.31)	--
20 years	0.59 (2.10)	-0.14 (2.24)	-0.21 (2.24)	-0.25 (2.25)	-0.30 (2.25)	-0.28 (2.24)	-0.24 (2.23)	0.36 (2.31)	0.60 (2.27)	-3.31 (2.49)	-3.82 (2.62)	--	--
30 years	-5.24 (3.81)	-5.38 (3.81)	-5.52 (3.81)	-5.62 (3.81)	-5.84 (3.81)	-5.95 (3.80)	-6.03 (3.79)	-6.29 (3.75)	-5.30 (3.81)	-7.67 (4.08)	--	--	--

(McCulloch 1973, pp. 43–45). Thus, our estimator is the same as Cagan's difference in holding period yields, except that we use an observation interval of 1 month, while he used 1 week. By developing it as we have, however, we are able to estimate the typical shape of the yield and forward curves exactly.[17]

Transactions Costs

If there were no transactions costs, it would follow from table 6 that for many pairs of maturities, it is worth the while of a lender who desires a short holding period to buy a longer maturity and to sell it before maturity, rather than simply to buy the shorter maturity in the first place. However, he cannot really both buy and sell at the bid-asked mean prices as implicitly assumed so far.

We cannot quantify all the components of the cost to an investor or borrower of going in and out of securities. However, the quoted bid-asked spread gives us a fair estimate of the external component of this cost, at least for large institutional investors. Subject, then, to the condition that he must buy high at the asked price and sell low at the bid price, we may investigate whether the short-term lender does better or worse to buy a longer maturity initially and sell it before maturity than simply to buy a short security.

We found that a lender who wished to lend for only 1, 2, or 3 months would have done significantly better to have bought a 2-, 4-, or 6-month security, respectively, even though it would mean having to go into the market twice instead of only once (McCulloch 1973, pp. 46–50). When we compared these differentials with the comparable values from table 6, we found that the bid-asked spread eats up only about one-third of the liquidity premium for these three pairs of maturities. However, the essentially short-term nature of the liquidity premium is evidenced by the fact that a lender who wished to lend for 4 months would not have done significantly better to have bought an 8-month security.

Miller and Orr (1967, pp. 133–51) have estimated the cost to one large nonfinancial corporation of going into and out of a given security as about \$20–\$50. Half this internal transaction cost just equals the premium (after bid-asked spread cost) on a 6-month bill held for 3 months, for investments of \$36,000–\$91,000. Therefore, a transaction would have to be in the hundreds to thousands of dollars before it definitely pays to try to exploit the liquidity premium.

[17] Culbertson (1957, p. 506) also works in terms of the difference in holding-period yields. Roll (1970, pp. 98–99) estimates the liquidity premium by a sum of differences of forward rates that is also equal to the difference in holding-period yields. In McCulloch (1973, p. 45), we show that interpolating from our table 6 for $m_1 = 6$ months and $m_2 = 1$ week gives an estimate not significantly different from Roll's.

Exponential Form Estimates of the Liquidity Premium

In tables 5 and 6 we made no assumptions about the form of the liquidity premium as a function of m. Since we put relatively little in, we got relatively little out in return. Thus, for longer maturities, where the variance of the forecasting errors is high, our estimators have very high standard errors. To return to the example we cited in Section I, the forward rate for a 10-year loan 5 years in the future from June 30, 1953 was 3.16 percent per annum. From table 6 ,we see that the premium on this rate would be -0.66 (± 0.64) percent per annum, so that a 95 percent confidence interval for the market expectation of the future 10-year spot rate extends from 2.54 percent per annum to 5.10 percent per annum (2.0 standard errors in each direction from $3.16 - [-0.66]$). The subsequently observed spot rate was 2.94 percent per annum, which lies within this interval, so we may not say for certain whether there was an unanticipated rise or fall, or no unanticipated change at all.

In order to pin the liquidity premium down with greater precision, we must make stronger assumptions. Kessel (1965) has argued that the liquidity premium should be such as to give the "typical" shape of the yield curve a monotonically increasing shape that approaches a horizontal asymptote.[18] If so, this information would eliminate much of our present uncertainty as to the behavior of $\pi(m)$ for m greater than 1 year or so.

Our free-form estimate of $\pi(m)$ rises in the first few months and then levels off. Although it falls off after 5 years, its final value (-5.24 ± 3.81 percent per year) is not significantly lower than its highest value (0.59 ± 2.10 percent per year), so its downward slope for long maturities is not necessarily significant.[19] The low values for long maturities may simply reflect the rise in interest rates over the period studied, which was probably largely unanticipated.

Since the observed liquidity premium seems not to differ significantly from the monotonically increasing-to-an-asymptote shape, we will try fitting it under that assumption. A simple two-parameter function that has these properties is

$$\pi(m) = b(1 - e^{-am}). \tag{16}$$

The parameters a and b were fit to our data by the maximum-likelihood technique as follows: free-form estimates of $\pi(m)$ for the six maturities 1 month, 3 months, 6 months, 1 year, 5 years, and 15 years were obtained

[18] Hicks (1946, p. 147) argues in favor of a monotonically increasing liquidity premium due to risk aversion in the face of interest-rate uncertainty. However, this is a non sequitur (Long 1972 and McCulloch 1973, pp. 4–10). Nevertheless, we give two alternative justifications of Kessel's monotonic shape, based on considerations involving the uncertain timing of receipts and expenditures and on the nature of a fractional reserve banking system (McCulloch 1973, pp. 10–15).

[19] A more powerful test of monotonicity is performed below.

as in table 5. Then intermaturity covariances were estimated about these means, producing a different 6×6 covariance matrix for each of the three post-Accord periods. The likelihood function was computed and maximized with respect to a and b based on the assumption that each set of six $\pi_j(m)$ constituted a multinormal drawing about the means given in (16). Given a, the maximizing value of b can be written in closed form. Different values of a were tried, until upper and lower bounds were found within .0005 of one another. The final estimator of a was placed halfway between these bounds, and the estimator of b calculated for this $\hat{a}$. The covariance matrix of the estimators of a and b was estimated by evaluating at $\hat{a}$ and $\hat{b}$ the second partial derivatives with respect to a and b of the logarithm of the likelihood function and inverting the negative of the matrix of these derivatives (Goldberger 1964, p. 131). The estimates thus obtained and their standard errors and covariance are

$$\begin{aligned} \hat{a} &= \underset{(1.068)}{6.059}(\text{yr})^{-1}, \\ \hat{b} &= \underset{(0.0738)}{0.4335} \text{ percent per year}, \\ \widehat{\text{cov}}(\hat{a}, \hat{b}) &= 0.06262. \end{aligned} \tag{17}$$

Given our formula for $\pi(m)$, we may compute $\bar{\pi}(m)$ and $p(m_1, m_2)$ by:

$$\bar{\pi}(m) = b[1 - (1 - e^{-am})/(am)], \tag{18}$$

$$p(m_1, m_2) = b(1 - e^{-am_1})(1 - e^{-am_2})/(am_2). \tag{19}$$

Standard errors for these nonlinear functions of a and b were approximated using the formula for asymptotic variances (Goldberger 1964, pp. 122–25). The results are shown in tables 7 and 8. Figures 2 and 3 show typical forward curves and yield curves incorporating the estimated values of $\pi(m)$ and $\bar{\pi}(m)$. Figure 2 uses the free-form estimates, while figure 3 uses the exponential-form estimates.

Table 6 can be used to answer a number of questions. The first column of figures tells us how much more return we would make, on average, by turning over daily in securities of maturity m than we would by turning over in "call money." (These comparisons ignore transactions costs, which of course would make it prohibitively expensive to buy a 30-year bond, hold it for a day, and then replace it by a new bond with a full 30 years to run. Our "call money" rate is the limit as maturity goes to zero of the bid-asked mean rate on Treasury bills.) The second column tells us how much more we would make by holding securities of maturity m to maturity than we would make by turning over in call money. By subtraction we also obtain the difference in yield to maturity we would obtain over a given period by holding securities of different maturities.

TABLE 7

EXPONENTIAL-FORM ESTIMATE OF LIQUIDITY PREMIUM
(POST-ACCORD EVEN OBSERVATIONS)

m	$\tilde{\pi}(m)$	$\hat{\bar{\pi}}(m)$	$\tilde{\pi}(\infty) - \tilde{\pi}(m)$	$\hat{\bar{\pi}}(\infty) - \hat{\bar{\pi}}(m)$
0	0.000	0.000	0.433	0.433
	(0.000)	(0.000)	(0.074)	(0.074)
1 month	0.172	0.093	0.262	0.340
	(0.018)	(0.010)	(0.065)	(0.069)
2 months	0.276	0.161	0.158	0.273
	(0.030)	(0.017)	(0.052)	(0.064)
3 months	0.338	0.210	0.095	0.223
	(0.040)	(0.023)	(0.040)	(0.058)
6 months	0.413	0.297	0.021	0.136
	(0.062)	(0.037)	(0.014)	(0.041)
9 months	0.429	0.339	0.005	0.094
	(0.070)	(0.046)	(0.004)	(0.030)
1 year	0.432	0.362	0.001	0.071
	(0.073)	(0.052)	(0.001)	(0.023)
2 years	0.433	0.398	0.000	0.036
	(0.074)	(0.063)	(0.000)	(0.012)
3 years	0.433	0.410	0.000	0.024
	(0.074)	(0.066)	(0.000)	(0.008)
5 years	0.433	0.419	0.000	0.014
	(0.074)	(0.069)	(0.000)	(0.005)
10 years	0.433	0.426	0.000	0.007
	(0.074)	(0.072)	(0.000)	(0.002)
20 years	0.433	0.430	0.000	0.004
	(0.074)	(0.073)	(0.000)	(0.001)
30 years	0.433	0.431	0.000	0.002
	(0.074)	(0.073)	(0.000)	(0.001)

NOTE.—Standard errors in parentheses.

For example, by holding 3-year notes to maturity, we would expect to get 0.410 − 0.210 = 0.200 percent per year more than we would by holding a sequence of 3-month bills to maturity. The standard error of this figure depends on a covariance not shown, but it must lie between |0.066 + 0.023| = 0.089 and |0.066 − 0.023| = 0.043.

The third column of figures in table 6 tells us how much more we would make by turning over daily in long maturities ($m \rightarrow \infty$) than by turning over daily in maturity m. Note that on this score there is no perceptible difference between 2-year bonds and perpetuities. The last column tells us how much more we get if we hold longs to maturity than if we hold securities of maturity m to maturity. Comparable free-form estimates do not appear in table 5 because without our exponential-form assumption (or something comparable), we have no information about what happens at infinity.

Our exponential-form estimates are never significantly different from our free-form estimates. Indeed, it is only rarely that they differ by more than 1 free-form standard error. However, our assumption of the exponential form greatly reduces the standard errors. The premium on

TABLE 8

EXPONENTIAL-FORM ESTIMATES OF MEAN LIQUIDITY PREMIUM $p(m_1, m_2)$

(Post-Accord Even Observations--Standard Errors in Parentheses)

m_1 \ Mm_2	0	1 mo.	2 mos.	3 mos.	6 mos.	9 mos.	1 year	2 years	3 years	5 years	10 yrs.	20 yrs.	30 yrs.
0	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)
1 month	0.17 (0.02)	0.13 (0.01)	0.11 (0.01)	0.09 (0.01)	0.05 (0.01)	0.04 (0.01)	0.03 (0.01)	0.01 (0.00)	0.01 (0.00)	0.01 (0.00)	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)
2 months	0.28 (0.03)	0.22 (0.03)	0.17 (0.03)	0.14 (0.02)	0.09 (0.02)	0.06 (0.01)	0.05 (0.01)	0.02 (0.01)	0.02 (0.00)	0.01 (0.00)	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)
3 months	0.34 (0.04)	0.27 (0.04)	0.21 (0.04)	0.17 (0.03)	0.11 (0.02)	0.07 (0.02)	0.06 (0.01)	0.03 (0.01)	0.02 (0.00)	0.01 (0.00)	0.01 (0.00)	0.00 (0.00)	0.00 (0.00)
6 months	0.41 (0.06)	0.32 (0.06)	0.26 (0.05)	0.21 (0.05)	0.13 (0.04)	0.09 (0.03)	0.07 (0.02)	0.03 (0.01)	0.02 (0.01)	0.01 (0.00)	0.01 (0.00)	0.00 (0.00)	0.00 (0.00)
9 months	0.32 (0.07)	0.34 (0.07)	0.27 (0.06)	0.22 (0.06)	0.13 (0.04)	0.09 (0.03)	0.07 (0.02)	0.04 (0.01)	0.02 (0.01)	0.01 (0.00)	0.01 (0.00)	0.00 (0.00)	0.00 (0.00)
1 year	0.43 (0.07)	0.34 (0.07)	0.27 (0.06)	0.22 (0.06)	0.14 (0.04)	0.09 (0.03)	0.07 (0.02)	0.04 (0.01)	0.02 (0.01)	0.01 (0.00)	0.01 (0.00)	0.00 (0.00)	0.00 (0.00)
2 years	0.43 (0.07)	0.34 (0.07)	0.27 (0.06)	0.22 (0.06)	0.14 (0.04)	0.09 (0.03)	0.07 (0.02)	0.04 (0.01)	0.02 (0.01)	0.01 (0.00)	0.01 (0.00)	0.00 (0.00)	0.00 (0.00)
3 years	0.43 (0.07)	0.34 (0.07)	0.27 (0.06)	0.22 (0.06)	0.14 (0.04)	0.09 (0.03)	0.07 (0.02)	0.04 (0.01)	0.02 (0.01)	0.01 (0.00)	0.01 (0.00)	0.00 (0.00)	0.00 (0.00)
5 years	0.43 (0.07)	0.34 (0.07)	0.27 (0.06)	0.22 (0.06)	0.14 (0.04)	0.09 (0.03)	0.07 (0.02)	0.04 (0.01)	0.02 (0.01)	0.01 (0.00)	0.01 (0.00)	0.00 (0.00)	0.00 (0.00)
10 years	0.43 (0.07)	0.34 (0.07)	0.27 (0.06)	0.22 (0.06)	0.14 (0.04)	0.09 (0.03)	0.07 (0.02)	0.04 (0.01)	0.02 (0.01)	0.01 (0.00)	0.01 (0.00)	0.00 (0.00)	0.00 (0.00)
20 years	0.43 (0.07)	0.34 (0.07)	0.27 (0.06)	0.22 (0.06)	0.14 (0.04)	0.09 (0.03)	0.07 (0.02)	0.04 (0.01)	0.02 (0.01)	0.01 (0.00)	0.01 (0.00)	0.00 (0.00)	0.00 (0.00)
30 years	0.43 (0.07)	0.34 (0.07)	0.27 (0.06)	0.22 (0.06)	0.14 (0.04)	0.09 (0.03)	0.07 (0.02)	0.04 (0.01)	0.02 (0.01)	0.01 (0.00)	0.01 (0.00)	0.00 (0.00)	0.00 (0.00)

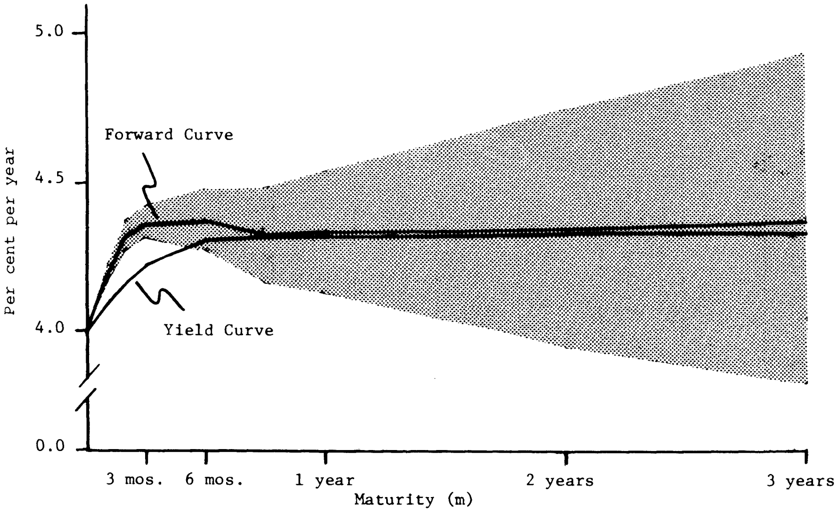

FIG. 2.—Typical shape of forward curve and yield curve under free-form assumption. Short rate assumed to be 4 percent per year. Forward curve shown with band extending ± 1 standard error.

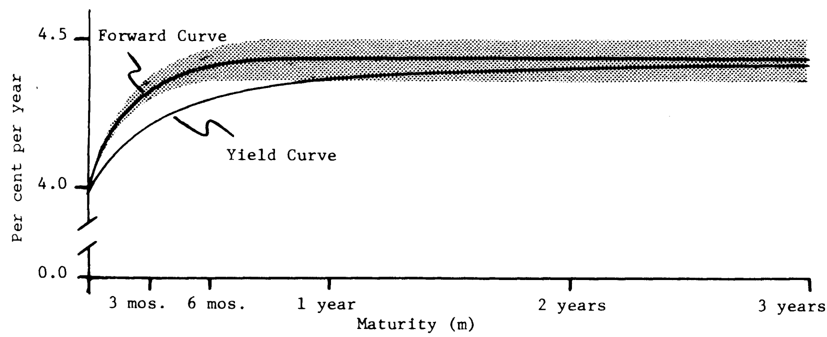

FIG. 3.—Typical shape of forward curve and yield curve under exponential form assumption. Short rate assumed to be 4 percent per year. Forward curve shown with band extending ± 1 standard error.

a 10-year rate 5 years forward is now 0.01 (± less than 0.005). Our estimate of the June 30, 1953 forecast of the June 30, 1958 10-year spot rate becomes 3.16 − 0.01 = 3.15 percent per annum. The measurement errors of the forward rate and the premium combine to give a standard error of about 0.01, so our 95 percent confidence interval now extends from 3.13 to 3.17 percent per year. The subsequent spot rate was 2.94 (±0.02) percent, so we may definitely say that the fall was unanticipated—provided, of course, we are willing to accept the assumptions of monotonicity and boundedness that led us to the exponential form.

The striking aspect of table 8 is the very low values for the mean liquidity premium when m_2 (the duration of the forward loan) is larger than a year or so. Previous investigations have given the impression that the liquidity premium is on the order of 0.5 percent per year. Our results are not inconsistent with these levels, provided m_2 is very small, as it has been in other studies. When m_2 becomes large, however, the premium in the subsequent spot rate is almost as large as that in the forward rate. The two nearly cancel out, leaving only a small residual. If we make no assumptions about the form of the premium, as in table 6, we get a very large confidence interval that gives us little information, due to the large amount of "noise" at these maturities. But then we impose monotonicity and boundedness, the additional information closes the confidence interval about a value very near zero.

Under our exponential assumption, $p(m_1, m_2)$ as defined in (19) obeys the following inequality:

$$p(m_1, m_2) < (b/a)/m_2. \tag{20}$$

This inequality gives us a convenient upper bound for the liquidity premium. The value of b/a is 0.072 percent (standard error $=$ 0.023), so we may state at the 95 percent confidence level that the mean premium is less than $0.101/m_2$ percent per year, when m_2 is measured in years.[20] If m_2 is a small fraction of a year, the liquidity premium can be substantial. But when m_2 is larger than 3 years or so, the premium is less than the precision with which forward rates can be measured.[21]

We are able to test the exponential-form estimates against the free-form estimates by means of the likelihood ratio test. With the free form we are estimating six independent means. With the exponential form, we are constraining these six values to conform to a two-parameter function, so twice the logarithm of the likelihood ratio should be compared with the χ_4^2 distribution.[22] Since its value was 4.71, which is not significant at even the 70 percent level, we may not reject the exponential form. Incidentally, this test provides a test for monotonicity. Since we may not reject this specific monotonic form, it follows a fortiori that we may not reject monotonicity in general.

[20] For a one-tailed test, the boundary of the 95 percent confidence region is approximately 1.70 standard errors above the mean.

[21] The logit form $\pi(m) = bm/(a + m)$ was tried in addition to the exponential form. Even though it increases monotonically to an asymptote, it does not imply an upper bound such as that given in eq. (20). However, the exponential form gave a slightly higher likelihood and therefore was preferred.

[22] The relevant free-form estimates for this test are ones which maximize the joint likelihood, taking into account the intermaturity covariances. These estimates are slightly different from those given in table 5, which take each maturity by itself.

Comparison with Survey Study

A survey of market expectations conducted by Kane and Malkiel[23] on April 1, 1965, affords us an opportunity to check our liquidity premium estimates against estimates that do not depend on our assumption of unbiased forecasting. Their sample consisted of 200 banks, life insurance companies, and nonfinancial corporations. Respondents varying in number from 77 to 90 ventured opinions as to the 90-day bill rate and 10-year bond rate on the various future dates listed in table 9. The standard errors of the mean market forecasts are remarkably small. For the 90-day-bill forecasts, they are actually considerably smaller than our measurement errors for the corresponding forward rates.

The survey premia are never significantly different from our free-form estimates from table 6, thanks to the high errors on the latter. And the survey premia are not significantly different from our exponential-form estimates for loans of 90 days' duration. In large measure this is due to the ambiguity of the forward rates. However, we run into trouble when we try to compare the survey premia on 10-year bonds to our table 8 estimates. For the bond 1 year in the future, the survey premium exceeds ours by four basis points or 1 standard error, and for 2 years, by 21 basis points, or 4 standard errors. This discrepancy casts some doubt on the validity of either our exponential form or the survey responses. It may be that the exponential form approaches its asymptote too quickly. In defense of our results, however, it should be noted that the survey results (taken together with the forward rates we have computed) strongly suggest that p(2 years, 10 years) is about four times p(1 year, 10 years), contradicting the widely accepted hypothesis that the premium increases with distance into the future at a decreasing rate.[24] The comparison of survey estimates of expectations with unbiased forecasting estimates like ours appears to be a fruitful field for future research.

IV. Conclusion

Without imposing any particular form on the liquidity premium, we were able to demonstrate the following: there is a liquidity premium, significantly greater than zero. This premium has been large enough since the Accord to imply that for some maturities, borrowers or lenders who desire one borrowing or lending period would do better in a different maturity, in spite of the extra costs incurred. The premium has been larger since the Accord than it was during the 4 years before the Accord.

[23] (1967). I am grateful to Malkiel for a letter explaining some of the figures given in that paper.

[24] The problem cannot be that the questionnaires were not filled out promptly on April 1, but only after a delay of variable length, for the forward rates in question were actually higher on April 30 and May 31 than they had been at the close of March.

TABLE 9

COMPARISON OF PREMIA FROM TABLES 6 AND 8 WITH THOSE DERIVED FROM KANE-MALKIEL SURVEY

Future Date	Security	Forward Rate*	Survey Expected Rate†	Survey Premium	Exponential-Form Premium	Free-Form Premium
July 1, 1965	90-day bill	4.07	3.99	.06	.17	.16
		(.13)	(.01)	(.13)	(.03)	(.05)
October 1, 1965	90-day bill	4.04	4.01	.03	.21	.12
		(.18)	(.01)	(.18)	(.05)	(.11)
January 1, 1966	90-day bill	3.84	3.96	−.12	.22	.11
		(.18)	(.02)	(.18)	(.06)	(.16)
April 1, 1966	90-day bill	3.81	3.86	−.05	.22	.13
		(.19)	(.02)	(.19)	(.06)	(.21)
April 1, 1967	90-day bill	3.93	3.66	.27	.22	.12
		(.10)	(.04)	(.11)	(.06)	(.39)
April 1, 1966	10-year bond	4.17	4.12	.05	.01	—.11
		(.03)	(.02)	(.04)	(.00)	(.14)
April 1, 1967	10-year bond	4.20	3.98	.22	.01	−.24
		(.04)	(.03)	(.05)	(.00)	(.27)

NOTE.—Standard errors in parentheses.
* Rates computed from U.S. Government security prices as of March 31, 1965, by method of McCulloch (1971). Logarithmic rates converted to banker's discount basis for comparison to bills.
† Mean of "most likely" rate for all respondents. Standard error equals standard deviation given by Kane and Malkiel (1967), divided by the square root of the number of respondents.

Finally, the variance of forecasting errors has not been constant over the period since the Accord.

We were unable to demonstrate any variation in the liquidity premium itself since the Accord, either as a function of time or as a function of the level of interest rates. We were also unable to detect nonnormality in the distribution of the forecasting errors (provided we allow for heteroskedasticity), or nonmonotonicity in the behavior of the premium as a function of maturity.

However, if we do not impose a particular form, our estimates of the mean term premium on long-term forward rates are very inaccurate. When we postulate the monotonic and bounded exponential form of (16) we obtain the relatively precise estimates contained in tables 7 and 8. In particular, we may then say that the forward rate corresponding to a future loan of duration m years is greater than the expected future spot rate, but by no more than $0.101/m$ percent per year.

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