An Ordinalist Proof of the Quasi-Concavity of Preferences: The Case of Two Independent Goods

by

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ABSTRACT

This paper demonstrates that it is not necessary to *assume* the quasi-concavity of commodity preferences, as Hicks asserted in *Value and Capital*. Rather, at least in the simplest case of two independent goods, this property can be *deduced* from the logic of the rational application of limited means to ends of varying degrees of importance. This theorem is an application of the Austrian theory of the "marginal use." It does not require the assumption of cardinal utility, and in fact is consistent with instances of what is called "intrinsically ordinal" marginal utility.

I. Introduction

One of the standard assertions of neoclassical price theory is that the individual's preferences on goods will exhibit quasi-concavity, so that indifferences curves will be convex when viewed from the origin. Originally Edgeworth derived this assertion from diminishing cardinal marginal utility, provided the goods were independent or net rivals. In the 1930's Hicks and Allen rejected cardinal utility and replaced the old proof of quasi-concavity with a bald assumption of this property. Hicks compared his assumption with pulling a rabbit out of a hat [1946, 23] and left it at that. While it is true that preferences must be quasi-concave at any observed equilibrium it would, as Hicks himself pointed out, be desirable to have some fundamental reason for believing this property holds at unobserved points as well.

This paper shows that, at least in the simplest case of two independent goods, it is still possible to prove the quasi-concavity of commodity preferences, using the Austrian theory of the marginal use.[1] This theory involves an intrinsically ordinal notion of marginal utility.

II. Preliminary Concepts

In this paper we will use the symbol $\succ$ to denote a transitive ordering relation such that $a \succ b$ precludes $b \succ a$. The symbols $>$, $\geq$, etc. denote numerical inequalities. The symbol = may also represent set equality.

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[1] For a restatement and extension of this theory, see McCulloch [1977].

Definition: A set S is linearly ordered by a relation $\succ$ if for all distinct elements a and b of S, either $a \succ b$ or $b \succ a$.

Definition: W* denotes the set of subsets of the set W:

$$W^* = \{P \mid P \subset W\}.$$

Definition: A relation Φ on a set of subsets W* is said to have the property of unrelatedness[2] if for all $P, Q \subset W$ (and therefore $P, Q \in W^*$),

$$P \, \Phi \, Q \text{ iff } P - Q \, \Phi \, Q - P.$$

In effect, unrelatedness means that elements of W are "unrelated", so that they may be added on or taken away from both sides of a relation without disturbing the relation.

Definition: A set of subsets W* is unrelatedly ordered by an ordering relation $\succ$ if $\succ$ has the property of unrelatedness.

Theorem 1: Given W* unrelatedly ordered by $\succ$, and subsets P, Q, R and S of P such that $R \succ S$ and $P \cap R = Q \cap S = \emptyset$, then $P \succ Q$ implies $P \cup R \succ Q \cup S$.

Proof: See Krantz et. al. [1971, 211].

Corollary 1 to Theorem 1: W* is assumed to be unrelatedly ordered. Given finite partitions $\mathcal{P} = (P_1, P_2, \ldots P_n)$ of $P \subset W$ and $\mathcal{Q} = (Q_1, Q_2, \ldots Q_m)$ of $Q \subset W$ and a one-to-one mapping ϕ which maps $\mathcal{Q}$ into $\mathcal{P}$ such that for each $Q_i \in \mathcal{Q}$, $\phi(Q_i) \succ Q_i$ or $\phi(Q_i) = Q_i$, and $\phi(Q_i) \succ Q_i$ holds for at least one i, then $P \succ Q$ in either of the following cases:

i) $m = n$.

ii) $n > m$ and $P_i \succ \emptyset$ for all $P_i \in \mathcal{P}$

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[2] The term "additive" used by Kraft et. al. [1959] is rejected here because in an economic theory context, that term connotes "additive separability", a distinct concept.

<u>Corollary 2 to Theorem 1</u>: Given W* additively ordered and finite partitions $\mathcal{P} = (P_1, P_2, \ldots P_n)$ of $P \subset W$ and $\mathcal{Q} = (Q_1, Q_2, \ldots Q_n)$ of $Q \subset W$. Then $P \succ Q$ implies $P_i \succ Q_j$ for at least one i and j.

<u>Corollary 3 to Theorem 1</u>: Given W* linearly and additively ordered and subsets P, Q, R, S of W. Then

$$P \succ Q$$
$$R \prec S$$
and
$$R \subset P$$
$$S \subset Q$$
imply
$$P - R \succ Q - S.$$

<u>Definition</u>: An ordering $\succ$ on W* is <u>positive</u> if for all non-empty subsets P of W, $P \succ \emptyset$.

<u>Definition</u>: A linear, unrelated and positive ordering $\succ$ on a set of subsets W* will be said to be <u>Austrian</u>.

<u>Definition</u>: An ordering $\succ$ on W* and an ordering $\succ'$ on Z* are <u>isomorphic</u> if there exists a one-to-one and onto mapping ϕ: $W^* \to Z^*$ such that for subsets P and Q of W,

$$\phi(P) = \bigcup_{w \in P} \phi(\{w\}),$$

and

$$P \succ Q \quad \text{iff} \quad \phi(P) \succ' \phi(Q),$$

Thus, the two linear orderings I and II in Table 1 are isomorphic since II can be obtained from I through the permutation $a \to b$, $b \to c$, $c \to a$. I and III are non-isomorphic. All 12 Austrian orderings on {a, b, c}* are isomorphic to either I or III. (In Table 1, "ab" is used to represent the subset {a, b}, etc., in order to avoid unnecessary clutter.)

Table 1

I	II	III
abc	abc	abc
ab	bc	ab
ac	ab	ac
a	b	bc
bc	ac	a
b	c	b
c	a	c
∅	∅	∅

Figure 1 depicts a partial ordering isomorphic to one contained in every Austrian ordering on W* for W = {a, b, c, d, e}. Table 2 contains 16 non-isomorphic Austrian orderings on this W*.

<u>Definition</u>: A linear and unrelated ordering $\succ$ on W* for a countable set W is <u>essentially cardinal</u>[3] if for each element w_i of W there exists a real number ρ_i such that for any subsets P and Q of W, the sums below exist and

$$P \succ Q \quad \text{iff} \quad \sum_{w_i \varepsilon P} \rho_i > \sum_{w_i \varepsilon Q} \rho_i .$$

<u>Definition</u>: A linear and unrelated ordering $\succ$ on W* is <u>intrinsically ordinal</u> if it is not essentially cardinal.

<u>Conjecture</u>: (de Finetti, 1949). Every complete and unrelated ordering on W* is essentially cardinal.

De Finetti's conjecture was proven false by Kraft <u>et. al</u>. [1959]. For example, each of the orderings in Table 2 is intrinsically ordinal. Orderings 1 through 5 all contain the four relations

$$\begin{aligned} ae &\succ bc \\ bce &\succ ad \\ b &\succ ce \\ \text{and} \quad cd &\succ be. \end{aligned}$$

If any of these orderings were essentially cardinal with ρ values α, β, γ, δ and ε corresponding to a, b, c, d, and e, we would have

$$\begin{aligned} \alpha + \varepsilon &> \beta + \gamma \\ \beta + \gamma + \varepsilon &> \alpha + \delta \\ \text{and} \quad \beta &> \gamma + \varepsilon \\ \gamma + \delta &> \beta + \varepsilon, \end{aligned}$$

which imply

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[3] "Arises from a measure" in Kraft <u>et. al.</u> [1959].

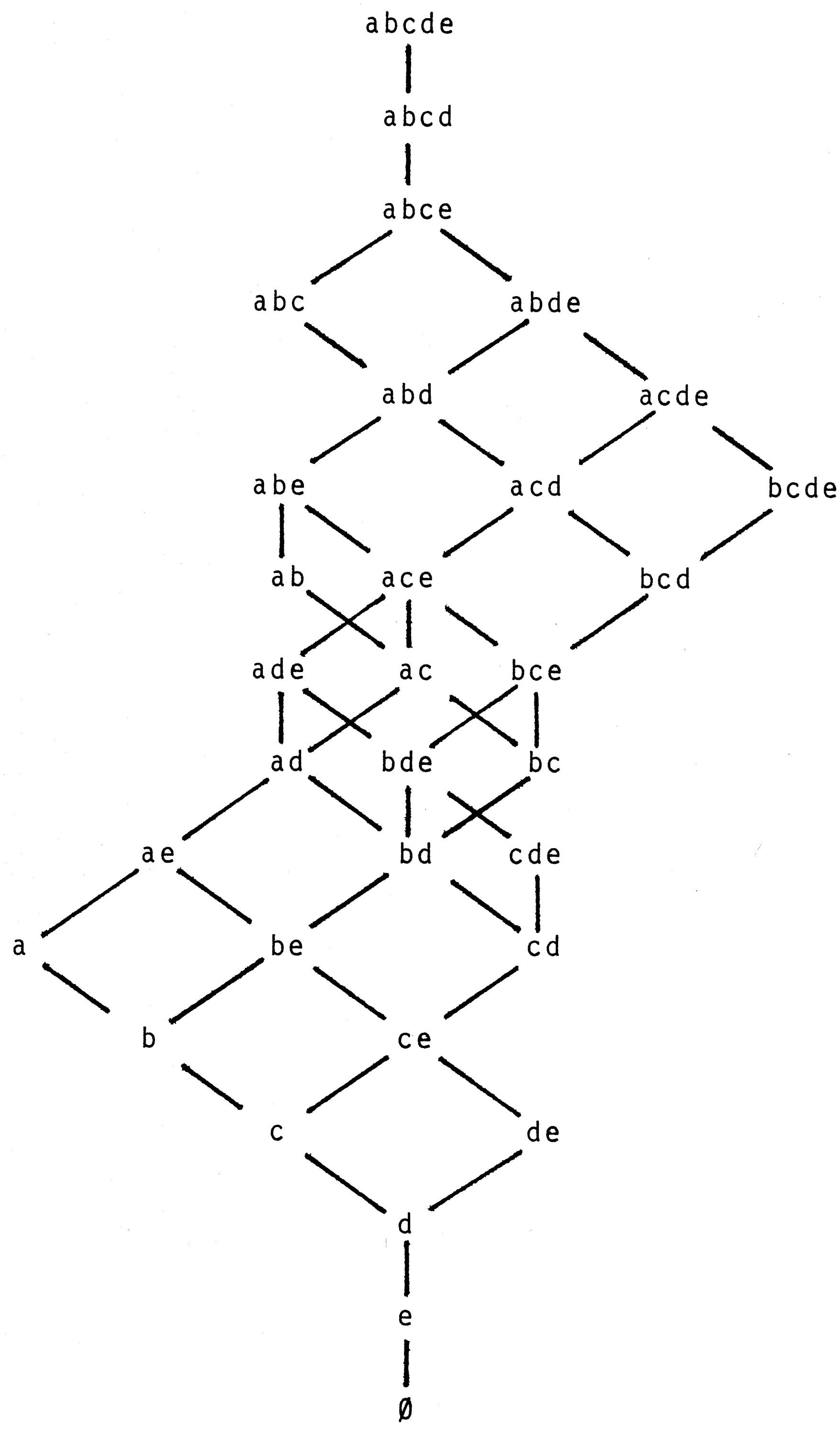

Figure 1

This partial ordering is isomorphic to one imbedded in any Austrian ordering on W when o(W) = 5.

Table 2

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
W	W	W	W	W	W	W	W	W	W	W	W	W	W	W	W
abcd	abcd	abcd	abcd	abcd	abcd	abcd	abcd	abcd	abcd	abcd	abcd	abcd	abcd	abcd	abcd
abce	abce	abce	abce	abce	abce	abce	abce	abce	abce	abce	abce	abce	abce	abce	abce
abc	abc	abde	abde	abde	abde	abc	abc	abc	abc	abde	abde	abde	abde	abde	abde
abde	abde	abc	abc	abc	abc	abde	abde	abde	abde	acde	acde	acde	abc	abc	abc
abd	abd	abd	abd	abd	acde	acde	acde	acde	acde	abc	abc	abc	acde	acde	acde
acde	acde	acde	acde	acde	abd	abd	abd	abd	abd	abd	bcde	bcde	abd	abd	bcde
acd	acd	acd	acd	acd	abe	abe	abe	abe	abe	bcde	abd	abd	bcde	bcde	abd
abe	abe	abe	abe	abe	acd	acd	acd	acd	acd	abe	abe	abe	acd	acd	acd
ab	ab	ab	ab	bcde	bcde	ace	ace	ace	bcde	acd	acd	acd	abe	abe	abe
ace	ace	ace	bcde	ab	ace	ab	ab	bcde	ace	bcd	bcd	bcd	ace	ace	ace
ac	bcde	bcde	ace	ace	ab	ac	bcde	ab	ab	ace	ace	ace	bcd	bcd	bcd
bcde	ac	ade	ade	ade	bcd	bcde	ac	ac	bcd	ade	ade	bce	ab	ab	bce
ade	ade	ac	bcd	bcd	ade	bcd	bcd	bcd	ac	ab	bce	ade	ade	bce	ab
bcd	bcd	bcd	ac	ac	ac	ade	ade	ade	ade	bce	ab	ab	bce	ade	ade
bce	bce	bce	bce	bce	ad	ad	ad	ad	ad	ac	ac	ac	bde	bde	bde
ad	ad	ad	ad	ad	bce	bce	bce	bce	bce	bde	bde	bde	ac	ac	ac
ae	ae	ae	bde	bde	bde	bc	bc	bc	bc	ad	cde	cde	ad	bc	bc
bc	bc	bde	ae	ae	bc	ae	ae	ae	bde	cde	ad	bc	bc	ad	cde
a	bde	bc	bc	bc	ae	a	bde	bde	ae	bc	bc	ad	cde	cde	ad
bde	a	a	bd	bd	cde	bde	a	cde	cde	bd	bd	bd	ae	ae	ae
bd	bd	bd	a	cde	bd	cde	cde	a	bd	ae	ae	ae	bd	bd	bd
cde	cde	cde	cde	a	a	bd	bd	bd	a	be	be	be	cd	cd	cd
cd	cd	cd	cd	cd	be	be	be	be	be	cd	cd	cd	be	be	be
be	be	be	be	be	cd	cd	cd	cd	cd	a	ce	ce	a	a	ce
b	b	b	b	b	ce	ce	ce	ce	ce	cd	a	a	ce	ce	a
ce	ce	ce	ce	ce	b	b	b	b	b	de	de	de	b	b	b
c	c	de	de	de	de	c	c	c	c	b	b	b	de	de	de
de	de	c	c	c	c	de	de	de	de	c	c	c	c	c	c
d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d
e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e
Ø	Ø	Ø	Ø	Ø	Ø	Ø	Ø	Ø	Ø	Ø	Ø	Ø	Ø	Ø	Ø

Sixteen non-isomorphic intrinsically ordinal Austrian orderings on W* for W = {a,b,c,d,e}.

$\alpha + 2\beta + 2\gamma + \delta + 2\varepsilon > \alpha + 2\beta + 2\gamma + \delta + 2\varepsilon$, a contradiction. Orderings 6 through 10 each contain the same four relations, with $\succ$ replaced by $\prec$. Orderings 11 through 13 each contain the four relations

$$
\begin{aligned}
&de \succ b \\
&ac \succ bde \\
\text{and}\quad &bd \succ ae \\
&be \succ cd,
\end{aligned}
$$

which would lead to the contradiction $\alpha + 2\beta + \gamma + 2\delta + 2\varepsilon > \alpha + 2\beta + \gamma + 2\delta + 2\varepsilon$ if they were essentially cardinal. Orderings 14 through 16 each contain these last four relations, with the sense reversed. Case 5 is isomorphic to the counterexample given by Kraft <u>et. al</u>.

Each of the 16 intrinsically ordinal orderings in Table 2 is isomorphic to 5! = 120 orderings obtained by permuting the elements of W. Therefore there are at least $16 \times 120 = 1920$ intrinsically ordinal Austrian orderings of the 32 elements of {a, b, c, d, e}*.

III. The Austrian Theory of Subjective Value

<u>Postulate 1</u>: Each individual has a set of <u>wants</u> W on whose satisfaction his perceived well-being depends.

<u>Postulate 2</u>: The individual's preferences define an Austrian ordering on W*.

The first postulate is well established in the Austrian literature. The second postulate is implicit in the writings of Menger, Wieser and Böhm-Bawerk from 1871 to 1914, and was employed freely by Neurath [1911], but was never made explicit.

Georgescu-Roegen [1968] does note that Menger's theory requires a rank-ordering on W*, but does not mention unrelatedness. The full postulate was finally discovered by Young [1969] and McCulloch [1977], working independently.

Definition: The use of a vector of commodities is the highest-ranking subset of W the commodities are capable of satisfying. (We tacitly assume there is such a maximal subset.)

Definition: The subjective value[4] (or utility) of a vector of commodities ($\vec{X}$) is the position of its use on the Austrian rank-ordering of W*.

Postulate 3: An individual will choose between commodity vectors according to their subjective values, as determined by their uses.

Because of Postulate 3, the individual's subjective perferences on W*, together with what he perceives (correctly or incorrectly) to be the objective technology relating commodities to the satisfaction of wants, defines a derived preference ordering $\succsim$ on commodity vectors. (Equivalence may hold for two vectors in spite of the linear ordering on W* if their uses are identical, as sometimes happens - at least in theory - with rival goods.)

Definition: Let W** be the set of all ordered pairs of disjoint subsets of W:

$$W^{**} = \{(P, Q) | P, Q \subset W, P \cap Q = \emptyset\}.$$

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[4]Subjective use-value or "value" for short in the older Austrian writings. Note that Hicks in 1946 still titles his section on utility "The Theory of Subjective Value".

An Austrian ordering on W* may be thought of as defining an additive partial ordering on W** according to the following rule: For $P' \subset P \subset W$ and $Q' \subset Q \subset W$, where $P - P' \cap Q' = Q - Q' \cap P' = \emptyset$,

$$P \succ Q \text{ iff } (P - P', Q') \succ (Q - Q', P').$$

This ordering on W** has an image of the original ordering imbedded in it, since

$$P \succ Q \text{ iff } (P, \emptyset) \succ (Q, \emptyset).$$

Definition: If $\vec{X}$ and $\Delta\vec{X}$ are non-negative commodity vectors, where Q is the use of $\vec{X}$ and P is the use of $\vec{X} + \Delta\vec{X}$, the marginal use of $\Delta\vec{X}$ given $\vec{X}$, represented by $MU(\Delta\vec{X}|\vec{X})$, is defined to be $(P - Q, Q - P)$. If $Q \subset P$ as often happens, the marginal use is $(P - Q, \emptyset)$. In this case it may be represented simply by $P - Q$.

The elements of $P - Q$ are the additional wants that are satisfied when an additional $\Delta\vec{X}$ is available. The elements of $Q - P$ are the wants (if any) which are no longer satisfied when an additional $\Delta\vec{X}$ is available. The set $Q - P$ may be non-empty when complementary technological relations are involved.

The marginal use corresponds to the German word Grenznutzen as used by Wieser, Böhm-Bawerk, and von Mises. Unfortunately, it has traditionally been translated "marginal utility," which has led to considerable confusion in the past.

Theorem 2: (The Law of the Marginal Use). For non-negative commodity vectors $\vec{X}$, $\Delta\vec{X}_1$, and $\Delta\vec{X}_2$, $\Delta\vec{X}_1$ will be preferred to $\Delta\vec{X}_2$ from a base of $\vec{X}$ iff

$$MU(\Delta\vec{X}_1|\vec{X}) \succ MU(\Delta\vec{X}_2|\vec{X}).$$

<u>Proof</u>: Let P_1 be the use of $\vec{X} + \Delta\vec{X}_1$, P_2 be the use of $\vec{X} + \Delta\vec{X}_2$, and Q be the use of X. Define

$$A = P_1 - (Q \cup P_2)$$

$$B = (P_1 \cap P_2) - Q$$

$$C = P_2 - (Q \cup P_1)$$

$$D = (P_1 \cap Q) - P_2$$

$$E = Q - (P_1 \cup P_2)$$

$$F = (Q \cap P_2) - P_1$$

$$G = P_1 \cap P_2 \cap Q$$

All these sets are pairwise disjoint. (See Figure 2.) By Postulate 3, $\Delta\vec{X}_1$ will be preferred to $\Delta\vec{X}_2$ from a base of $\vec{X}$ (i.e., $\vec{X} + \Delta\vec{X}_1$ preferred to $\vec{X} + \Delta\vec{X}_2$) iff

$$P_1 \succ P_2$$

iff

$$A \cup B \cup D \cup G \succ B \cup C \cup F \cup G$$

iff

$$A \cup D \succ C \cup F$$

iff

$$A \cup B \cup D \cup E \succ B \cup C \cup E \cup F$$

iff

$$(A \cup B, E \cup F) \succ (B \cup C, D \cup E)$$

iff

$$(P_1 - Q, Q - P_1) \succ (P_2 - Q, Q - P_2)$$

iff

$$MU(\Delta\vec{X}_1 | \vec{X}) \succ MU(\Delta\vec{X}_2 | \vec{X}),$$

which completes the proof. (If $Q \subset P_1$ and $Q \subset P_2$ as in the traditional case, the proof is considerably simpler.)

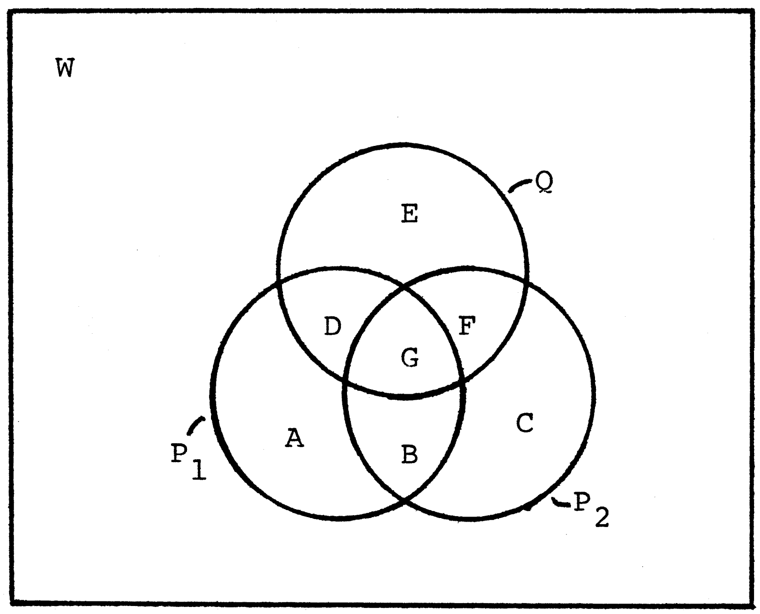

FIGURE 2

Since the marginal use determines which of several goods-increments will be preferred, its position on the ordinal wants-scale determines the "marginal utility" of the corresponding goods-increments. Note, however, that this marginal utility does not require a cardinal utility concept for its development, and indeed may be <u>intrinsically ordinal</u>.

<u>Definition</u>: A partition $(C_1, C_2, \ldots C_n)$ of the set C of commodities is an <u>independent partition</u> if there exists a partition $(W_1, W_2, \ldots W_n)$ of W such that the commodities in each C_i can only be used to satisfy wants in W_i and cannot be used, either alone or in conjunction with other commodities, to satisfy wants in $W - W_i$.

One immediate consequence of this definition is that if $(C_1, C_2, \ldots C_n)$ is an independent partition of C, the marginal use and therefore the subjective value of the goods in C_i will not depend on the quantity available of goods not in C_i.[5]

IV. Quasi-Concavity of Preferences

<u>Definition</u>: An individual's preferences on a set $\mathcal{X}$ of n-dimensional non-negative commodity vectors $\vec{X}$ are <u>quasi-concave</u> if for every point $\vec{X}_o \neq \vec{0}$ in $\mathcal{X}$ there exists an n-vector of non-negative real numbers $\vec{a}$ such that

$$\vec{a}\,\vec{X}_i > \vec{a}\,\vec{X}_o > 0$$

for all $\vec{X}_i \in \mathcal{X}$ for which $\vec{X}_i \succ \vec{X}_o$.[6]

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[5]The plausibility of an independent partition of the underlying wants-set provides a primitive basis for the separability of preferences frequently assumed in the economic literature, e.g. Burness [1976].

[6]The conventional definition of quasi-concavity in terms of the convexity of the set of points above the indifference curve or surface is useless when we consider the realistic case of discretely divisible goods, since preferences over convex combinations are then not defined. However, the concept of a separating hyperplane does continue to make sense, so we base our definition on it, and then exploit its implications.

$\mathcal{X}$ could be the set of integer vectors in the non-negative orthant, the set of real vectors in the non-negative orthant, or some more restricted set.

Theorem 3: Suppose the following conditions hold:

i) ({good 1}, {good 2}, C - {good 1, good 2}) is an independent partition of C.

ii) (A, B, W - (A $\cup$ B)) is the corresponding partition of W.

iii) Exactly n units of good 1 are necessary to satisfy any n wants in A and exactly n units of good 2 are necessary to satisfy any n wants in B.

iv) A and B are countably infinite and each contains a maximal element, say $a_1 \varepsilon A$ and $b_1 \varepsilon B$.[7]

Then preferences on $\mathcal{X}$, the set of integer-valued 2-vectors giving the quantities available of good 1 and good 2 are quasi-concave.

Before beginning the proof, let us name the elements a_i of A and b_i of B in such a way that $\{a_i\} \succ \{a_{i+1}\}$ and $\{b_i\} \succ \{b_{i+1}\}$, and define

$$A_i = \{a_1, a_2, \ldots a_i\}$$

$$B_i = \{b_1, b_2, \ldots b_i\}.$$

Lemma 1: Under the conditions of the theorem, the use of $(i, j, \vec{z})$, where i is the number of units of good 1 available, j is the number of units of good 2 available, and $\vec{z}$ is a vector corresponding to the goods other than the first two, will be $A_i \cup B_j \cup D$, where D is a subset of W - (A $\cup$ B) which does not depend on i or j.

[7] With obvious though tedious modifications to the proof, the theorem is true for A and B finite as well. We have included extra generality into the set-up of this theorem, in the hope that others will extend it to n goods with interrelationships.

Proof of Lemma 1: It is at once apparent that the use of $(i, j, \vec{Z})$ will include a subset P of A that is independent of j and $\vec{Z}$, a subset Q of B that is independent of i and $\vec{Z}$ and a subset D of $W - (A \cup B)$ that is independent of i and j. To be feasible, we must have $o(P) \leq i$ and $o(Q) \leq j$. If $P \neq A_i$, Corollary 1 of Theorem 1 implies $A_i \succ P$, whence $A_i \cup Q \cup D \succ P \cup Q \cup D$, contradicting the assumption that $P \cup Q \cup D$ is the use of $(i, j, \vec{Z})$. Therefore $P = A_i$. Similarly $Q = B_j$, which completes the proof.

Because of unrelatedness and the assumed technological independence, we may speak of A_i as the use of i units of good 1 without reference to j or $\vec{Z}$ and of B_j as the use of j units of good 2 without reference to i or $\vec{Z}$.

We may also refer to "the marginal use of p units of good 1 given q units" as

$$\mathcal{MU}_1 (p|q) = A_{p+q} - A_q.$$

Similarly,

$$\mathcal{MU}_2 (p|q) = B_{p+q} - B_q.$$

Lemma 2: Let X be the set of n-dimensional integral valued goods-vectors. If preferences on X are such that less is never preferred to more, yet these preferences are not quasi-concave, then for some point $\vec{X}_o$ in X, there exist m points $\vec{X}_1, \vec{X}_2, \ldots \vec{X}_m$ in X, $m \leq n$, each of which is preferred to $\vec{X}_o$, and non-negative real numbers $b_1, b_2, \ldots b_m$ such that either

i. $\sum_{i=1}^{m} b_i (\vec{X}_i - \vec{X}_o) = \vec{0}$, $b_i > 0$ for at least one value of i, or

$$\text{ii.}\quad \sum_{i=1}^{m} b_i\,(\vec{X}_i - \vec{X}_o) = -\,\vec{X}_o.$$

Our definition of quasi-concavity states that for each goods-vector $\vec{X}_o$ (other than the origin itself) there is a hyperplane passing through $\vec{X}_o$ with the origin on one side and the set of all points preferred to $\vec{X}_o$ on the other side (Figure 3). Lemma 2 states that otherwise $\vec{X}_o$ will either be a convex linear combination of points preferred to it (Figure 4), or else the origin will lie in a cone, with $\vec{X}_o$ at the apex, defined by points preferred to $\vec{X}_o$ (see Figure 5: from $\vec{X}_o$, the origin is the vector $-\,\vec{X}_o$).

<u>Proof of Lemma 2</u>: The proof follows immediately from a version of the separating hyperplane theorem given by Gale [1960, 48], Theorem 2.9. First, for any point $\vec{X}_o$ define

$$\mathcal{Y} = \{\vec{X}_i \mid \vec{X}_i \succ \vec{X}_o\}$$

and

$$\mathcal{Y}' = \{\vec{X}_i \varepsilon \mathcal{Y} \mid \vec{X}_j \leqq \vec{X}_i \text{ for no } \vec{X}_j \varepsilon \mathcal{Y},\ \vec{X}_j \neq \vec{X}_i\}$$

If $\mathcal{Y}$ is the set of points plotted in Figure 3 (exclusive of $\vec{X}_o$),' then $\mathcal{Y}'$ is the subset consisting of the encircled points. $\mathcal{Y}'$ is convenient because it is finite, with order at most equal to $n(k+1)^{n-1}$, where k is the maximal component of $\vec{X}_o$. Preferences on X will then be quasi-concave iff for each $\vec{X}_o$ there exists a vector $\vec{a}$ such that

$$\vec{a}\,\vec{X}_i > \vec{a}\,\vec{X}_o > 0$$

for all $\vec{X}_i \varepsilon \mathcal{Y}'$ corresponding to X_o.

To apply Gale's theorem, we let k be the number of elements in $\mathcal{Y}'$. Then let Gale's A matrix be $(k+1)$ X n, where the first k

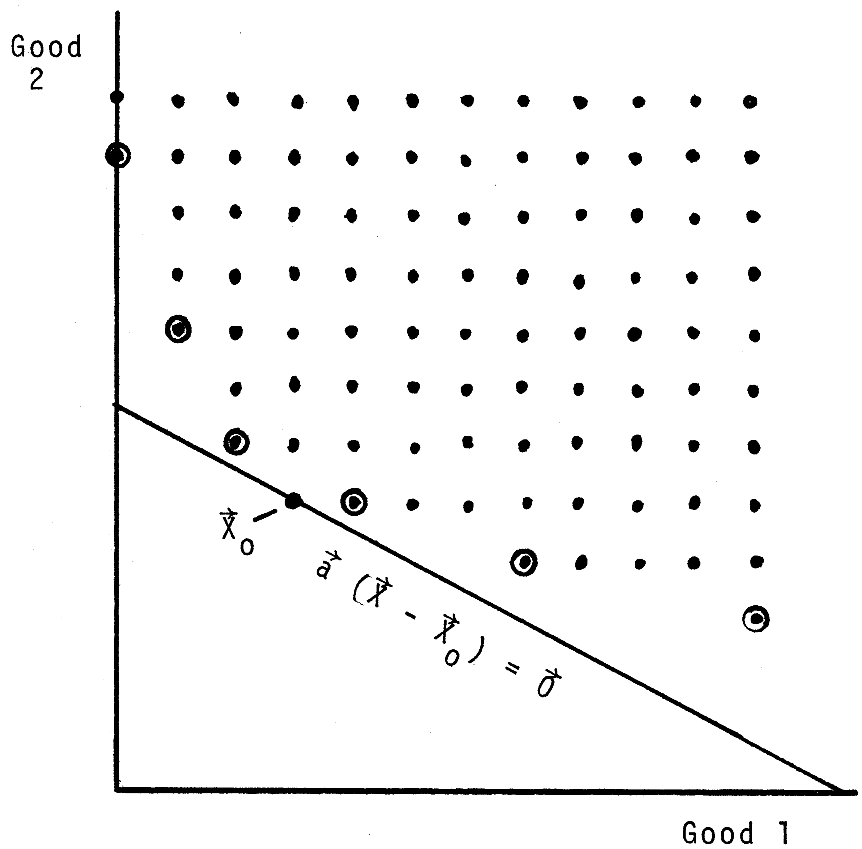


Figure 3

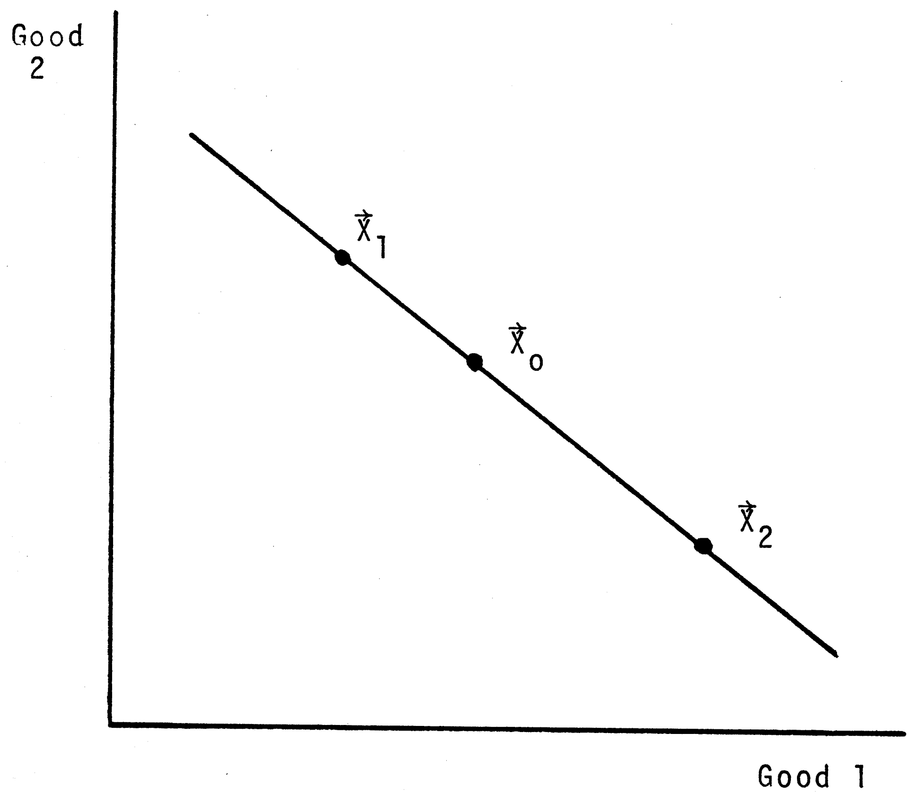
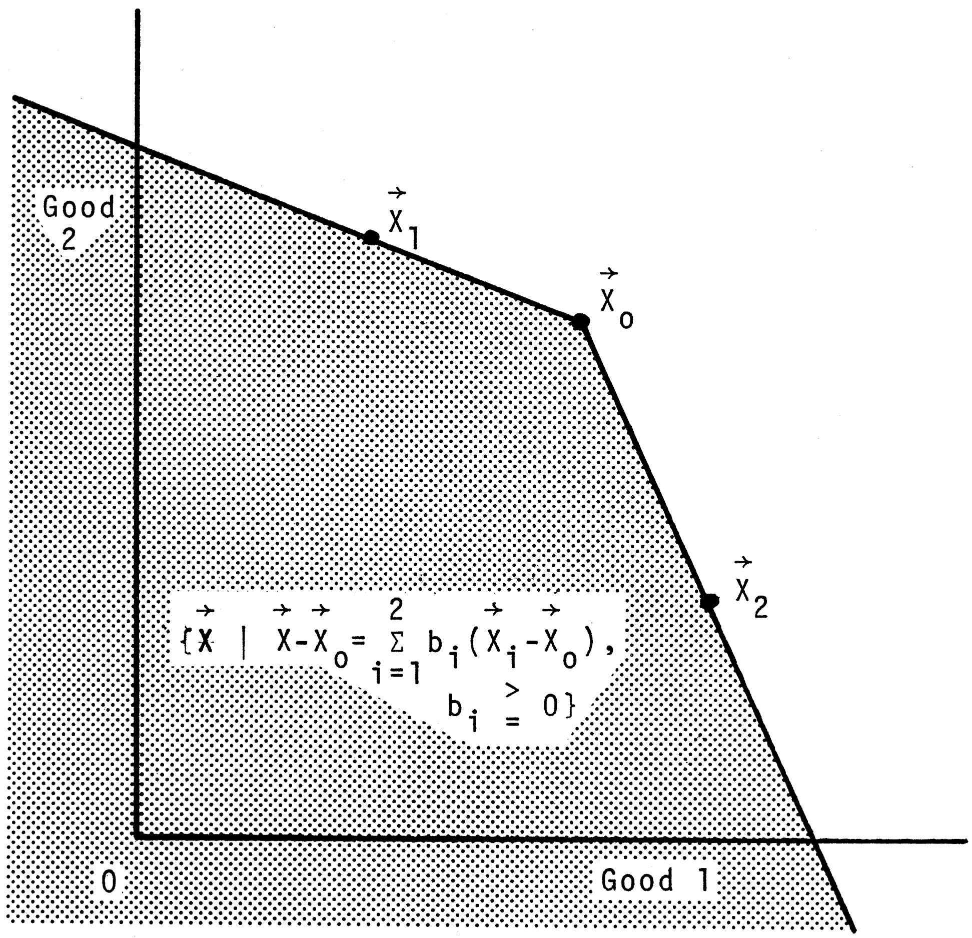


Figure 4



$\vec{X}_1$
$\vec{X}_o$
$\vec{X}_2$
Good 2
Good 1
0
$\{\vec{X} \mid \vec{X}-\vec{X}_o = \sum_{i=1}^{2} b_i(\vec{X}_i-\vec{X}_o),\ b_i \geq 0\}$

Figure 5

rows are the vectors $(\vec{X}_i - \vec{X}_o)$, and the k+1 st row is the vector $\vec{X}_o$. Let Gale's y vector be the vector of our values a_i and Gale's x vector be $(\beta_1, \beta_2, \ldots\beta_k, \beta_o)$. Then Gale's theorem implies that if preferences are <u>not</u> quasi-concave, for some $\vec{X}_o$ there will be a semi-positive vector $(\beta_1, \beta_2, \ldots\beta_k, \beta_o)$ such that

$$\sum_{i=1}^{k} \beta_i(\vec{X}_o - \vec{X}_i) + \beta_o \vec{X}_o = \vec{0}.$$

If $\beta_o = 0$, we have case i of the lemma, with $b_i = \beta_i$. If $\beta_o > 0$, we have case ii, with $b_i = \beta_i/\beta_o$. By Gale's Theorem 2.11 [1960, 50], if either system has a solution, it must have a basic solution with $m \leqq n$ non-zero b_i values. This completes the proof.

<u>Proof of Theorem 3</u>: Suppose to the contrary that preferences are <u>not</u> quasi-concave at some point X_o. Then following Lemma 2 there are two primary cases. ~~We will show that~~ In either of these cases there must exist three points

$$\vec{X}_o = (p+q,\ s+t) \quad (1)$$

$$\vec{X}_1 = (p,\ s+t+u) \quad (2)$$

and

$$\vec{X}_2 = (p+q+r,\ s) \quad (3)$$

where q, r, t and u are positive integers, p and s are non-negative integers,

$$\vec{X}_1 \succ \vec{X}_o$$

and

$$\vec{X}_2 \succ \vec{X}_o,$$

yet

$$tq \geqq ru,$$

with equality holding in case i and inequality in case ii. (See Figure 6.)

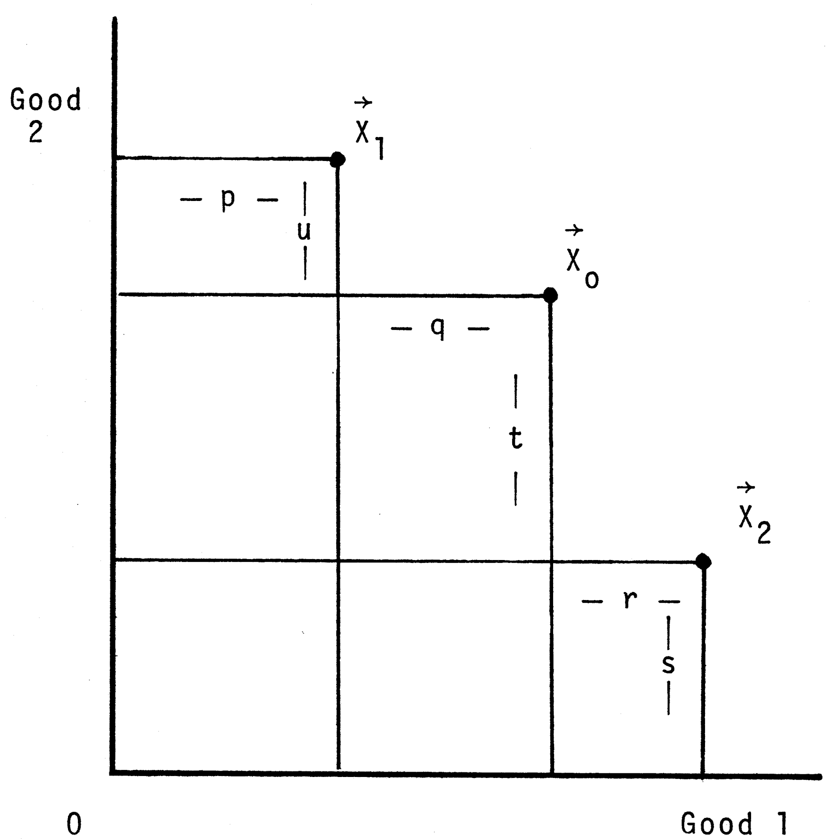


Figure 6

<u>Case i</u>: We may dismiss the possibility m=1, for then we would have $\vec{X}_1 = \vec{X}_o$, yet $\vec{X}_1 \succ \vec{X}_o$. Therefore m=2. Because under the assumptions of the theorem, less is never preferred to more, we cannot have either $\vec{X}_1 \leqq \vec{X}_o$ or $\vec{X}_2 \leqq \vec{X}_o$. We therefore must have one of them ($\vec{X}_1$) greater in the second component and less in the first component (whence $q > 0$ and $u > 0$) and the other greater in the first component and less in the second (whence $t > 0$ and $r > 0$). We have

$$b_1 (\vec{X}_1 - \vec{X}_o) + b_2 (\vec{X}_2 - \vec{X}_o) = \vec{0},$$

so

$$b_1(-q, u) + b_2 (r, -t) = (0,0),$$

whence

$$b_2 r = b_1 q \quad (4)$$

and

$$b_1 u = b_2 t. \quad (5)$$

Multiplying (4) through by u and (5) by q,

$$b_2 ru = b_1 qu = b_2 qt.$$

Since m necessarily equals 2, we have $b_2 \neq 0$, whence

$$qt = ru.$$

Set v equal to the greatest common divisor of q and r:

$$v = \gcd(q, r).$$

Then for integers a and b we have

$$q = av$$

$$r = bv$$

$$tav = bvu$$

$$ta = bu.$$

We have gcd (a, b) = 1, so a divides u and b divides t. Define w as

$$w = \frac{u}{a} = \frac{t}{b},$$

so that

$$u = aw,$$

and

$$t = bw.$$

Now $\vec{X}_2 \succ \vec{X}_o$ implies

$$MU_1\ (r|p+q) \succ MU_2\ (t|s).$$

Partitioning MU_1 (r|p+q) into b sets of v each and MU_2 (t|s) into b sets of w each, Corollary 2 to Theorem 1 implies

some v-element subset of MU_1 (r|p+q)

$\succ$ some w-element subset of MU_2 (t|s).

Now,

any v-element subset of MU_1 (q|p)

$\succ$ any v-element subset of MU_1 (r|p+q)

and

any w-element subset of MU_2 (t|s)

$\succ$ any w-element subset of MU_2 (u|s+t),

whence

any v-element subset of MU_1 (q|p)

$\succ$ any w-element subset of MU_2 (u|s+t). (6)

Partitioning MU_1 (q|p) into a sets of v each and MU_2 (u|s+t) into a sets of w each, Corollary 1 to Theorem 1 together with (6) implies

$$MU_1\ (q|p) \succ MU_2\ (u|s+t),$$

whence

$\vec{X}_o$ = (p+q, s+t) $\succ$ (p, s+t+u) = $\vec{X}_1$, contradicting that $\vec{X}_1 \succ \vec{X}_o$.[8]

---

[8]Case i is due to Carl Menger [1950/1871, 181-190]. The more difficult case ii is due to Smith.

<u>Case ii</u>: We again dismiss m=1, for that would imply a point $\vec{X}_1$ between $\vec{X}_o$ and $\vec{0}$ with $\vec{X}_1 \succ \vec{X}_o$, contradicting that less is never preferred to more. We therefore have m=2, so that using (1) - (3),

$$b_1 (-q, u) + b_2 (r, -t) = -\vec{X}_o \leq \vec{0}$$

for positive b_1 and b_2, where $\leq$ indicates strict inequality in at least one component.

This implies

$$b_2 r \leqq b_1 q \qquad (7)$$

and

$$b_1 u \leqq b_2 t, \qquad (8)$$

with strict inequality holding in at least one of (7) and (8). Multiplying (7) through by t and (8) by r, we have

$$b_1 ur \leqq b_2 rt \leqq b_1 qt.$$

Since $b_1 > 0$ and at least one of the inequalities is strict, this implies

$$ur < qt. \qquad (9)$$

Since less is never preferred to more, we must have $\vec{X}_1$ and $\vec{X}_2$ each greater than $\vec{X}_0$ in at least one component. Again using (1) - (3) this means

$$0 > q \text{ or } u > 0 \qquad (10)$$

and

$$r > 0 \text{ or } 0 > t \qquad (11)$$

If we had r=0, (7) would imply $q \leqq 0$, whence (10) would imply $u > 0$, and so (8) would imply $t > 0$. But this combination of r, q, u, and t is incompatible with (9), so $r \neq 0$. If we had $r > 0$, (7), (10) and (8) would imply $q > 0$, $u > 0$ and $t > 0$. If we had $r < 0$, (11), (8) and (10) would similarly imply $t < 0$, $u < 0$ and $q < 0$. Therefore either r, q, u, and t are all positive or all negative. By interchanging the roles of $\vec{X}_1$ and $\vec{X}_2$ if necessary, we may take them to be all positive, as illustrated in Figure 6.

Now for convenience, let us define

$$A' = A_{p+q+r} - A_p$$
$$\simeq \mathcal{MU}_1(q+r\,|\,p)$$

and

$$B' = B_{s+t+u} - B_s$$
$$\simeq \mathcal{MU}_2(t+u\,|\,s).$$

If gcd (q, r) $\neq$ 1, we may consider A' gcd (q, r) consecutive elements at a time, so without loss of generality, we may assume gcd (q, r) = 1. Similarly, we may assume gcd (t, u) = 1. Note that $\vec{X}_0 \prec \vec{X}_1$ implies that the q highest elements of A', considered together as a subset of A', are not as highly valued as the u lowest elements of B', so we must have

$$\text{any } q \text{ elements of } A' \prec \text{any } u \text{ elements of } B'. \tag{12}$$

Similarly,

$$\text{any } t \text{ elements of } B' \prec \text{any } r \text{ elements of } A'. \tag{13}$$

We now consider a number of subcases.

<u>Case ii.a</u>: Three sub-subcases:

1. $t = u$, $q \leqq r$
2. $q = r$, $t \leqq u$
3. $t < u$, $q < r$.

All three sub-subcases imply $tq \leqq ru$, contradicting (9).

Case ii.b: $t > u$, $q > r$. Using (12) and (13), this case implies

any q elements of A'

$\prec$ any u elements of B'

$\prec$ any t elements of B'

$\prec$ any r elements of A'

$\prec$ any q elements of A',

a contradition.

Case ii.c: $t = u$, $q > r$. Since gcd $(t, u) = 1$, we must have $t = u = 1$. We have

$MU_2\ (u|s+t)$

$\preceq$ any element of B'

$\prec$ any r elements of A'

$\prec$ any q elements of A'

$\preceq MU_1\ (q|p)$.

But this contradicts $\vec{X}_1 \succ \vec{X}_o$.

Case ii.d: $q = r$, $t > u$. Reverse the roles of goods 1 and 2 in Case ii c.

Case ii.e: Either

1. $t < u$ and $q > r$

or

2. $t > u$ and $q < r$.

Case ii e proceeds by an induction encompassing both its sub-cases. To initiate the induction, we set

$q_0 = q$

$r_0 = r$

$t_0 = t$

$u_0 = u$.

Note that for $i = 0$,

$$r_i\ u_i < q_i\ t_i \tag{14}$$

$$\gcd\ (t_i, u_i) = 1 \tag{15}$$

$$\gcd\ (q_i, r_i) = 1 \tag{16}$$

any q_i elements of A'

$\prec$ any u_i elements of B' (17)

any t_i elements of B'

$\prec$ any r_i elements of A' (18)

and either

1. $t_i < u_i$ and $q_i > r_i$ (19)

or

2. $t_i > u_i$ and $q_i < r_i$ (20)

<u>Case ii.e.1:</u> $t_i < u_i$ and $q_i > r_i$. If $t_i = 1$, (18) implies

any element of B'

$\prec$ any r_i elements of A' (21)

Since $o(A') = q + r > q_i > r_i\ u_i$, we may apply Theorem 1 to (21) u_i times, obtaining

any u_i elements of B'

$\prec$ any $r_i\ u_i$ elements of A'

$\prec$ any q_i elements of A'

$\prec$ any u_i elements of B',

a contradiction.

If $r_i = 1$,

any t_i elements of B'

$\prec$ any element of A'. (22)

We have $B' \subset B - B_s$, and each element of $(B - B_s) - B'$ is less important than each element of B', so (22) implies

$$\text{any } t_i \text{ element of } B{-}B_s$$
$$\prec \text{any element of } A'.$$

So

$$\text{any } q_i\ t_i \text{ elements of } B{-}B_s$$
$$\prec \text{any } q_i \text{ elements of } A'$$
$$\prec \text{any } u_i \text{ elements of } B'.$$

But this is a contradiction, since $q_i\ t_i > u$ and $B' \subset B - B_s$. (The set $B - B_s$ had to be introduced here in place of B' because $q_i\ t_i$ may exceed the order of B'.) We therefore may not have either $t_i = 1$ or $r_i = 1$.

Now set

$$q_{i+1} = q_i - r_i$$
$$u_{i+1} = u_i - t_i$$
$$r_{i+1} = r_i$$
$$t_{i+1} = t_i.$$

We will now show that (14) through (20) hold for i+1 as they do for i. We have

$$r_i\ u_i < q_i\ t_i$$
$$r_i\ u_i - r_i\ t_i < q_i\ t_i - r_i\ t_i$$
$$(u_i - t_i)\ r_i < (q_i - r_i)\ t_i,$$

so

$$r_{i+1}\ u_{i+1} < q_{i+1}\ t_{i+1}. \qquad (22)$$

We also have

$$\gcd (t_{i+1}, u_{i+1}) = 1, \tag{23}$$

because any common divisor of t_{i+1} and u_{i+1} would also divide both t_i and $u_i = u_{i+1} + t_{i+1}$, and hence by (15) must be 1. Similarly,

$$\gcd (q_{i+1}, r_{i+1}) = 1. \tag{24}$$

Combining (17) and (18) using Corollary 3 to Theorem 1,

$$\begin{aligned} &\text{any } q_{i+1} = q_i - r_i \text{ elements of } A' \\ &\prec \text{any } u_{i+1} = u_i - t_i \text{ elements of } B'. \end{aligned} \tag{25}$$

Furthermore, since r_i and t_i are unchanged,

$$\begin{aligned} &\text{any } t_{i+1} \text{ elements of } B' \\ &\prec \text{any } r_{i+1} \text{ elements of } A'. \end{aligned} \tag{26}$$

We now consider whether (19) and (20) continue to hold. Unless $r_i = 1$ (which we have shown to be impossible), $\gcd (q_i, r_i) = 1$ implies $q_{i+1} \neq r_{i+1}$. Similarly, $t_{i+1} \neq u_{i+1}$. This leaves four possibilities. If $t_{i+1} > u_{i+1}$ and $q_{i+1} > r_{i+1}$ then

$$\begin{aligned} &\text{any } q_{i+1} \text{ elements of } A' \\ &\quad \prec \text{any } u_{i+1} \text{ elements of } B' \\ &\quad \prec \text{any } t_{i+1} \text{ elements of } B' \\ &\quad \prec \text{any } r_{i+1} \text{ elements of } A' \\ &\quad \prec \text{any } q_{i+1} \text{ elements of } A', \end{aligned}$$

a contradiction. Nor could we have $t_{i+1} < u_{i+1}$ and $q_{i+1} < r_{i+1}$, for that would imply

$$q_{i+1} t_{i+1} < r_{i+1} u_{i+1},$$

contradicting (22). This leaves either

1. $t_{i+1} < u_{i+1}$ and $q_{i+1} > r_{i+1}$ (27)

or

2. $t_{i+1} > u_{i+1}$ and $q_{i+1} < r_{i+1}$. (28)

Note that (22) - (28) reproduce (14) - (20) with i replaced by i+1.

Case ii.e.2: $t_i > u_i$ and $q_i < r_i$. Proceed as in Case ii.e.1, interchanging the roles of goods 1 and 2. Unless $u_i = 1$ or $q_i = 1$, (14) - (20) will hold with i replaced by i+1 for

$$q_{i+1} = q_i$$
$$u_{i+1} = u_i$$
$$r_{i+1} = r_i - q_i$$
$$t_{i+1} = t_i - u_i.$$

In either case, ii.e.1 or ii.e.2, (14) - (20) are satisfied with i incremented by 1, unless a contradiction occurs because one of the four integers equals 1, so we may return in either case to the beginning of Case ii.e. However, in either case,

$$\text{Max } (q_{i+1}, r_{i+1}, t_{i+1}, u_{i+1}) < \text{Max } (q_i, r_i, t_i, u_i).$$

Therefore we must eventually reach an i for which one of the four integers equals 1 and a contradiction occurs.

We have thus shown that assuming Theorem 3 to be false leads to a contradiction in every case. This completes its proof.

V. Conclusion

We have demonstrated that it is not necessary to assume the quasi-concavity of commodity preferences, as Hicks asserted in Value and Capital. Rather, at least in the simplest case of two independent goods, this property can be deduced from the logic of the rational application of limited means to ends of varying degrees of importance. This was done in a framework which admits of intrinsically ordinal marginal utility.

It remains to generalize this theorem to incorporate the following complications:

1. More than two independent goods.

2. Rival goods that may be used in place of one another to satisfy certain of the wants in W.

3. Complementary goods that must be used together with one another to satisfy certain wants.[9]

4. Wants that may be satisfied jointly by the application of one good, but individually if at all by the application of others.

In order to accomodate complementary goods and joint satisfaction, we have developed the concept of the marginal use at a more general level than was necessary for our Theorem 3. Similarly, Lemma 2 was made general enough to accommodate n goods.[10]

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[9] In McCulloch [1977] it is shown that the Hicks-Allen "definition" of complementary goods is not equivalent to this operational definition, and should therefore be abandoned.

[10] It is to be expected that rival goods may lead to some "flat" spots, so that goods-preferences may not be strictly quasi-concave in that case. Furthermore, as was pointed out in McCulloch [1977], we would not expect quasi-concavity (or diminishing ordinal marginal utility, for that matter), to hold, except perhaps in a qualified sense, if more than one unit of the same good were required to satisfy certain wants.

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