Estimation of Risk Neutral Measures using the Generalized Two-Factor Log-Stable Option Pricing Model

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Abstract

We construct a simple representative agent model to provide a theoretical framework for the log-stable option pricing model. We also implement a new parametric method for estimating the risk neutral measure (RNM) using a generalized two-factor log-stable option pricing model. Under the generalized two-factor log-stable uncertainty assumption, the RNM for the log of price is a convolution of two exponentially tilted stable distributions. Since the RNM for generalized two-factor log-stable uncertainty is expressed in terms of its Fourier Transform, we introduce a simple extension of the Fast Fourier Transform inversion procedure in order to reduce computational errors in option pricing. The generalized two-factor log-stable RNM has a very flexible parametric form for approximating other probability distributions. Thus, this model provides a sufficiently accurate tool for estimating the RNM from the observed option prices even if the log-stable assumption might not be satisfied. We estimate the RNM for the S&P 500 index options and find that the generalized two-factor log-stable model gives better performance than the Black-Scholes model, the finite moment log-stable model (Carr and Wu, 2003), and the orthogonal log-stable model (McCulloch, 2003) in fitting the observed option prices.

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1 Introduction

Derivative prices provide a valuable source of information for measuring market participants' perception of future development of underlying asset prices. Particularly, cross section data on options, all of which expire at the same time but have different strike prices, contain rich information about the underlying asset price distribution in the future. Call (put) options are only valuable to the extent that there is a chance that the underlying asset will be worth more (less) than the strike price so that the option comes to be exercised. Thus, the market prices of the options provide information about the probability that market attaches to an asset being within a range of possible prices at maturity date.

The option values are determined by the investors' risk preference and the statistical probability distribution of the underlying asset price, the so-called Frequency (probability) Measure (FM).[1] Alternatively, the value of option may be evaluated as a discounted expected value of the future payoffs of the option under the risk-adjusted distribution of the underlying asset price, the so-called Risk-Neutral (probability) Measure (RNM), as in the risk-neutral valuation of Ross (1976) and Cox and Ross (1976). The RNM allows us to price any derivative of the particular underlying asset with the same time to maturity by a present value of their expected payoffs in an arbitrage-free market. Inversely, the implied RNM may be extracted from the derivative prices by reversing the process of obtaining prices from pricing models. In a complete arbitrage-free market, a unique RNM can be recovered from a complete set of European option prices using the relationship proposed in Ross (1976) and Breeden and Litzenberger (1978).

The famous Black-Scholes (BS) option pricing model (1973) provides a market's *ex-ante* estimate of the underlying asset's price distribution at maturity under the log-normal assumption. The lognormal assumption suggests that implied volatilities[2] should be constant for all strike prices because only one volatility parameter governs the underlying stochastic process on which all options are priced. However, practitioners and researchers have noted that option prices tend to systematically violate the constant volatility assumption in the BS model. Rubinstein (1985) documented the first such systematic violations of BS model prices. The typical market volatilities implied from option prices often have an asymmetric U-shaped structure with

[1] The FM governs the empirically observable distribution of underlying asset prices. It is also called the physical probability measure, the objective probability measure, or the real world probability measure.

[2] Among the parameters entering the Black-Scholes formula only the volatility cannot be observed. However, using an observed option price an implied volatility can be computed by inverting the option pricing formula. Often the implied volatility is called the Black-Scholes implied volatility.

respect to strike prices, commonly referred to as the ”volatility smile” or ”volatility smirk.”[3] The observed deviations of implied volatilities from the Black-Scholes assumption provide some information about the shape of the RNM density function implied by option prices. The volatility smirk suggests that the log returns of the underlying asset at maturity should have a skewed and leptokurtic distribution rather than a normal distribution. Thus, it is necessary to make an alternative assumption for the asset's return distribution which is consistent with the implied volatility structure. To relax the log-normal assumption of the BS model, many option pricing models have been developed by introducing additional factors such as (i) the stochastic interest rate [Amin and Jarrow (1992)]; (ii) the jump-diffusion [Bates (1991)]; (iii) the stochastic volatility [Heston (1993)]; (iv) the stochastic volatility and stochastic interest rate [Bakshi and Chen (1997)]; (v) jump diffusion with the stochastic volatility [Bates (1996)].

However, the Generalized Central Limit Theorem (GCLT)[4] provides support for stable distributions as a financial asset's log return process since financial asset returns may be considered as the multiplicatively cumulative outcome of many stochastic events. Furthermore, stable distributions can easily accommodate heavy tails and skewness of asset returns, which are commonly observed in the financial data. It is therefore reasonable to assume that log asset returns are governed by a stable distribution. In turn, asset prices themselves follow a log-stable distribution. The normal or Gaussian distribution is the most familiar member and the only one with finite variance of the stable class, but inappropriate for modeling extreme events because the probability of a substantial change is considerably smaller than the frequency observed in financial asset returns. In addition, observed asset returns are often too skewed to be normal as mentioned above. Thus, the non-Gaussian stable distributions are preferable for modeling log returns of financial assets.

In the early 1960s the asset pricing models driven by log-stable distributions had already been proposed by Fama (1963), and Mandelbrot and Taylor (1967) as an alternative to the log-normal assumption, but the fact that expected payoffs on assets and call options are infinite under most log-stable distributions led both Paul Samuelson (as quoted by Smith 1976) and Robert Merton (1976) to conjecture that assets and derivatives could not be reasonably priced under these distributions, despite their attractive feature as

[3]Before 1987, the implied volatility of the US equity index options, as a function of strike for a certain maturity, behaves like a symmetric smiled curve. The phenomenon is called the implied volatility smile. After the market crash in 1987, the implied volatility as a function of strike price is skewed towards the left. The phenomenon is regarded as implied volatility smirk.

[4]According to the Generalized Central Limit Theorem, if the sum of a large number of identically and independently distributed (IID) random variables has a non-degenerate limiting distribution after appropriate scaling and shifting, the limiting distribution must be a member of the stable class. See Zolotarev (1986).

limiting distributions under the Generalized Central Limit Theorem. These concerns of Samuelson and Merton come from the misunderstanding about the shape of the RNM. The RNM corresponding to a log-stable FM is not a simple location shift of the FM. Instead, the RNM in general has a different shape with finite moments, and leads to reasonable asset and option prices.

The option pricing model with log-stable distributions was first proposed by McCulloch (1978, 1985, 1987, 1996) using a utility maximization setting with the orthogonal log-stable uncertainty assumption.[5] Janicki et al. (1997), Popova and Ritchken (1998), and Hurst et al. (1999) have developed option pricing models under the symmetric log-stable assumption. Carr and Wu (2003) proposes the finite moment log-stable (FMLS) option pricing model by making the very restrictive assumption that the RNM for log prices have maximally negative skewness[6], which is a parametric special case of the orthogonal log-stable assumption. McCulloch (2003) reformulated the orthogonal log-stable option pricing model in the RNM context and showed how the RNM can be derived from the underlying distribution of marginal utilities in a simple representative agent model. The orthogonal log-stable uncertainty assumption allows the RNM of log-stable returns to be the convolution of two densities: one is a maximally negatively skewed stable density, and the other is an exponentially tilted maximally positively skewed stable density. However, the orthogonal assumption is too restrictive for the RNM to fit option prices observed in the markets, which makes it inappropriate to estimate the RNM from observed option prices.

In this paper, we construct the numeraire and asset choice model in an Arrow-Debreu world to provide a theoretical framework for the log-stable option pricing model. We also introduce the generalized two-factor log-stable option pricing model by relaxing the orthogonal assumption. The orthogonal assumption can be generalized by introducing two factors which are independent maximally negatively skewed standard stable variates and affect both the log marginal utility of numeraire and the log marginal utility of asset. This assumption allows the log marginal utilities to be dependent upon each other and also provides a flexible RNM probability distribution function, which is the convolution of two exponentially tilted stable distributions. Furthermore, since the generalized two-factor stable RNM is sufficiently flexible to fit observed option prices, the generalized two-factor log-stable option pricing model provides a new parametric method

[5]It is assumed that the marginal utilities of numeraire and asset follow a log-stable distribution with maximum negative skewness, respectively, and are also independently distributed so that it is called the orthogonal log-stable uncertainty assumption.

[6]They assume the max-negative skewness in order to give the returns themselves finite moments. They also incorporate max-negative skewness directly into the stable distribution describing the RNM of the underlying asset without assumptions on the frequency measure (FM).

for estimating the RNM from a cross-section of option price data.

Since there are no known closed form expressions for general stable densities, we numerically evaluate log-stable options from the characteristic function (CF) by modifying the inverse Fourier Transform approach of Carr and Madan (1999). This paper also introduces a simple extension of the Fast Fourier Transform inversion procedure in order to reduce computational errors.

To illustrate the empirical performance of the generalized two-factor log-stable model, we estimate the RNM from observed S&P 500 index option prices. We then conduct model specification tests and compare the fitting performance of four models: the BS log-normal model, the finite moment log-stable model, the orthogonal log-stable model, and the generalized two-factor log-stable model. We evaluate the models on the basis of the goodness-of-fit and find that the generalized two-factor log-stable model outperforms the others.

The rest of the paper is organized as follows. Section 2.2 discusses the theoretical relationship between option prices and the RNM. Section 2.3 presents a simple general equilibrium model for the log-stable option pricing by constructing a numeraire and asset choice problem in the Arrow-Debreu world. Section 2.4 introduces the generalized two-factor log-stable option pricing model. In Section 2.5 we estimate the RNM from S&P 500 index option prices and compare the performance of the models. Section 2.6 concludes.

2 Option prices and the RNM

The fundamental building block in modern financial economic theory is a contingent claim. The contingent claim is an asset or security whose return depends on the state of nature at a point in the future. An option is such a contingent claim because the payoff of the option depends on the price of the underlying asset at maturity date. The price of the option therefore contains information about the market participants' probability assessment of the outcome of the underlying asset price at the future maturity date. This is the basic idea behind the RNM estimation from option prices.

A particular simple and important example of the contingent claim is the Arrow-Debreu security, which pays one unit in one specific state of nature and nothing in other state. For each state, the prices of Arrow-Debreu securities, the so-called state-prices, are proportional to the RNM probability densities for the realization of the state. In a continuum of states the state prices thus are proportional with the continuous RNM probability density function (PDF).[7] Since the state prices are determined by the combination of investors' preferences, budget constraints, information structures, and the imposition of market-clearing conditions, i.e., general equilibrium, the RNM contains additional information than the statistical probability measure (FM).

A number of estimation methods have been developed to back out the RNM from option prices based on seminal work by Ross (1976), Cox and Ross (1976), and Breeden and Litzenberger (1978). Ross (1976) showed that a portfolio[8] of European call-options can be used to construct synthetic Arrow-Debreu securities, thereby establishing a relation between option prices and the RNM. By ruling out arbitrage possibilities, Cox and Ross (1976) showed that options in general, independently of investors' risk preferences, can be priced as if investors were risk neutral.[9] Consider a general European call option whose payoff is $\max(S_T - K)$, where S_T is the value of an underlying asset on maturity date T, and K is the strike price. In a complete arbitrage-free market, the price of a European call option $C(K)$ can then be computed as the discounted value of the option's expected payoff under the RNM. Formally,

$$\begin{aligned} C(K) &= e^{-r_f T} E^Q \left[\max(S_T - K, 0)\right] \\ &= e^{-r_f T} \int_K^{\infty} (S_T - K)\, \mathrm{r}(S_T)\, dS_T, \end{aligned} \tag{1}$$

where r_f is the risk free interest rate, $\mathrm{r}(S_T)$ is the RNM PDF, and E^Q is the conditional expectation on time 0 information under the RNM. As shown by Breeden and Litzenberger (1978), the RNM density $\mathrm{r}(S_T)$ can be isolated by differentiating (1) twice, yielding[10]

$$\mathrm{r}(S_T) = e^{r_f T} \left. \frac{\partial^2 C(K)}{\partial K^2} \right|_{K=S_T}. \tag{2}$$

[7]By construction, the RNMs have a probability-like interpretation, which are nonnegative and integrating to unity. For this reason the RNM density is sometimes called the risk-neutral probability density (RND).

[8]These portfolios are butterfly spreads that are composed of two long and two short call options. The strike price of both short calls is halfway in between the strike prices of the two long call options. In the limit as the difference between the strike prices goes to zero the butterfly spread's payoff becomes a Dirac-Delta function, e.g. Merton (1999).

[9]However, it does not mean that agents are assumed to be risk neutral. We are not assuming that investors are actually risk-neutral and risky assets are actually expected to earn the risk-free rate of return.

[10]By similar reasoning, the cumulative distribution function (CDF) can be obtained by differentiating a single time. This technique is used by Neuhaus (1995).

Equation (2) states that the RNM is proportional to the second derivative of the call pricing function with respect to the strike price.

Since the RNM might embody important information about market participants' expectations concerning prices of the underlying asset in the future, methods have been developed to estimate the RNM PDF from observed option prices. As pointed out in Chang and Melick (1999), Equation (1) and (2) provide two different approaches for estimating the RNMs from observed option prices. The methods based on Equation (1) make assumptions about the form or family of the RNM for evaluating the integral in (1) and then typically use a non-linear optimization method to estimate the parameters of the RNM that minimize pricing errors, which are differences between the predicted option prices and the observed option prices. On the other hand, the methods based on Equation (2) use a variety of means to generate the call option price function $C(K)$ and then differentiate the function (either numerically or analytically) to obtain the RNM PDF. Computing the RNM by (2) requires call prices being available for continuous strike prices, but in reality just a few discretely spaced strike prices are available. These methods therefore entail directly or indirectly the construction of a continuous option pricing function from observed prices.

In this paper, we follow the first approach to estimate the RNM by making a parametric assumption that the log marginal utilities of numeraire and asset are affected by two independent standard stable variates.

3 State Prices, Pricing Kernels and the RNM

In this section, we construct the Numeraire and Asset Choice Model in the Arrow-Debreu general equilibrium economy to provide a theoretical framework for the relationship between the state prices, the pricing kernels, and the RNM.

3.1 Numeraire and Asset Choice Model

The fundamental investment-selection problem for an individual is to determine the optimal allocation of his/her wealth among the available investment opportunities. Under the expected utility maximization paradigm, each individual's consumption and investment decision is characterized as if he/she determines the probabilities of possible asset pay-offs, assigns a utility index to each possible consumption outcome, and chooses the consumption and investment policy to maximize the expected value of the index.

Consider a representative agent model in the Arrow-Debreu world, in which only a single consumption good N (numeraire) and a single underlying asset A exist. We assume that the utility function of the representative agent is random, given by:

$$U^s(N, A)$$

with random marginal utility of consumption U_N^s and random marginal utility of asset U_A^s, where s is a state variable which represents a state of random utility function.[11]

Let $S_T(s)$, which is affected by the state variable s, be the asset price at future time T and introduce Arrow-Debreu securities which pay one unit of numeraire in specified states, $S_T(s) \in [x, x + dx)$, and no payout in other states, $S_T(s) \notin [x, x + dx)$, with unconditional payment of $p(x)dx$ unit of numeraire to be made at present time 0. The price of the Arrow-Debreu security $p(x)dx$ is called the state price, which can be thought of as the insurance premium that the agent is prepared to pay in order for him/her to enjoy one unit of consumption if $S_T(s) \in [x, x + dx)$.

We assume that the representative agent receives one unit of both numeraire and asset as endowment; makes forward contracts on the asset at the forward price F; and buys/sells the Arrow-Debreu securities prior to the realization of a state s of the random utility function. The representative agent then buys/sells the asset, and consumes the numeraire after the realization of a state s.

Thus, the representative agent faces a consumption-asset choice problem as followings:[12]

$$\max_{\{N(s),A(s),B,G,Q(x)\}_{s\in\Omega,x\in[0,\infty]}} EU(N, A) \equiv \int_\Omega U(N(s), A(s))\omega(s)ds$$

$$\begin{aligned} \text{s.t.} \quad 1 &= N_0 + \int_0^\infty p(x)(B + Q(x))dx \\ N_0 + B + S_T(s)(1 - A(s)) + (S_T(s) - F)G + 1_{S_T(s)\in[x,x+dx)}Q(x) &= N(s), \quad \forall s, \end{aligned}$$

where s is the *state* of the random utility function, $s \in \Omega$; $N(s)$ is the consumption (numeraire); $A(s)$ is the amount of asset holding;[13] $S_T(s)$ is the spot price of the asset; $\omega(s)$ is the density of the state variable

[11] In the analysis of preference for flexibility (Kreps (1979), Dekel, Lipman and Rustichini's (2001)) the realization of the agent's random utility function corresponds to the realization of his subjective (emotional) state.

[12] For brevity the superscripts on the utility function $U^s(N, A)$ have been suppressed.

[13] Here, we assume no divdend for the underlying asset. Even if we introduced the dividend, the results would be same.

s, which is conditional on time 0 information; N_0 is the amount of numeraire holding before the realization of the state; B is the amount of bond holding; $1_{S_T(s)\in[x,x+dx)}$ is the payoff of the $A-D$ security (1 if $S_T(s) \in [x, x+dx)$, 0 otherwise); $p(x)dx$ is the state price of $A-D$ security; $Q(x)$ is the amount of $A-D$ security holding; F is the forward price of the asset; and G is the amount of forward contracts[14] on the asset. Asset prices, forward prices, state prices, pricing kernels, and the RNM can be easily derived from the first order conditions for the representative agent problem.

The asset price S_T at future time T may be expressed as the ratio of the marginal utilities:

$$S_T = \frac{U_A}{U_N}, \tag{3}$$

where U_N is the random future marginal utility of the numeraire in which the asset is priced, and U_A is the random future marginal utility of the asset itself.

On the other hand, the forward price F at present time 0, which is the numeraire price to be paid at time T for a contract to deliver 1 unit of the asset at future time T, may be written as the ratio of the expected marginal utilities:

$$F = \frac{\int_\Omega U_A \omega(s)ds}{\int_\Omega U_N \omega(s)ds} = \frac{EU_A}{EU_N}. \tag{4}$$

The state price $p(x)dx$, which is the unconditional payment of $p(x)dx$ units of numeraire to be made at time 0 for an Arrow-Debreu security which pays 1 if $S_T(s) \in [x, x+dx)$, otherwise 0, can be decomposed in the following way:

$$p(x)dx = m(x) \cdot f(x)dx, \tag{5}$$

where $f(x)dx$ is the probability of $S_T(s) \in [x, x+dx)$, i.e., the FM probability, while $m(x)$ is the price that we would pay to enjoy 1 unit of numeraire at time T under the condition that we know that future's state is going to be $S_T(s) \in [x, x+dx)$.[15] We call $m(x)$ the pricing kernel, which is a stochastic discount factor for the asset pricing formula.

[14]The forward contract is an contract between two parties in which the buyer agrees to pay the seller unconditional payment of F units of numeraire, in exchange for 1 unit of the asset at future time T.

[15]It is very important to note that although the agent defines $m(x)$ as if she knew S_T, it will not be equal to the price of the bond. This is because different states imply different effects on his/her utility.

Since the first order conditions imply that the state price is

$$
\begin{aligned}
p(x)dx &= \int_0^\infty p(x)dx \cdot \frac{\int_\Omega 1_{S_T(s)\in[x,x+dx)}\, U_N \frac{\omega(s)}{f(x)} ds}{\int_\Omega U_N\, \omega(s) ds}\, f(x)dx \\
&= \frac{1}{R_f}\, \frac{E(U_N|U_A/U_N = x)}{EU_N}\, f(x)dx,
\end{aligned}
$$

where R_f $(= e^{r_f T})$[16] is the gross risk-free interest rate to time T, the pricing kernel $m(x)$ is taken to be:

$$
m(x) \;=\; \frac{p(x)dx}{f(x)dx} \;=\; \frac{1}{R_f}\, \frac{E(U_N|U_A/U_N = x)}{EU_N}. \tag{6}
$$

Equation (6) shows how the preferences define the pricing kernel, translating the state changes into changes in the MRS between conditional and unconditional expected marginal utilities[17] of consumption of the representative agent.

Consider a European call option that has state dependent cash flows $\max(x - K, 0)$ when $S_T(s) \in [x, x + dx)$. The price of this call option $C(K)$ may be determined by the state prices and the possible cash flows:

$$
C(K) \;=\; \int_0^\infty \max(x - K, 0)\, p(x)\, dx. \tag{7}
$$

Substituting (5) into (7), we have

$$
\begin{aligned}
C(K) &= \int_0^\infty m(x)\, \max(x - K, 0)\, f(x)\, dx, \qquad (8) \\
&= E[m(x)\, \max(x - K, 0)]
\end{aligned}
$$

where E is the conditional expectation on time 0 information under the FM $f(x)$. Equation (8) implies that the option price should equal the expected discounted value of the option's payoff, using the investor's stochastic marginal utility to discount the payoff.

[16] In the Arrow-Debreu economy, the bond price is determined by the sum of state prices:

$$
\frac{1}{R_f} \;=\; e^{-r_f T} \;=\; \int_0^\infty p(x)dx.
$$

[17] Rigorously, the unconditional expected marginal utility, EU_N, is a conditional expectation based on information at time 0. On the other hand, the conditional expected marginal utility, $E(U_N|U_A/U_N = x)$, is a conditional expectation on time 0 information and the fact that the asset price is going to be x, $S_T(s) \in [x, x + dx)$.

Let $\mathrm{r}(x)$ be the density of the RNM at time 0 for state-contingent claims at time T. By definition of the RNM,

$$
\begin{aligned}
C(K) &= \frac{1}{R_f}\int_0^\infty \max(x-K,0)\,\mathrm{r}(x)\,dx \\
&= \frac{1}{R_f}E^Q[\max(x-K,0)]
\end{aligned} \tag{9}
$$

where E^Q is the conditional expectation on time 0 information under the RNM $r(x)$. Combining (6), (8) and (9), we finally get the RNM PDF:

$$
\begin{aligned}
\mathrm{r}(x) &= R_f\, p(x) \;=\; R_f\, m(x)\, f(x) \\
&= \frac{E(U_N|U_A/U_N = x)}{EU_N}\, f(x).
\end{aligned} \tag{10}
$$

Equation (10) states that the RNM is simply the FM, adjusted by the risk-free interest rate and the pricing kernel.

Further, combining (3), (4) and (10) yields the mean-forward price equality condition (i.e., the mean of the risk neutral distribution should equal the currently observed forward price of the underlying asset):

$$
F = \frac{EU_A}{EU_N} = \int_0^\infty x\,\mathrm{r}(x)\,dx. \tag{11}
$$

3.2 FM and RNM under Random Future Marginal Utility

To derive the RNM from the FM and pricing kernel, we start from the joint distributions of the random future marginal utilities U_N and U_A instead of assuming the random utility function $U^s(N,A)$ explicitly.

Let $g(U_N,U_A)$ be the joint PDF of U_N and U_A conditional on information at present time 0. Equation (3) implies that the FM cumulative distribution function (CDF) of S_T is

$$
\begin{aligned}
F(x) &= \Pr(S_T \le x) \;=\; \Pr(U_A \le xU_N) \\
&= \int_0^\infty \int_0^{xU_N} g(U_N,U_A)\,dU_A dU_N.
\end{aligned}
$$

Therefore, the FM probability density function (PDF) for S_T is

$$\begin{aligned} f(x) &= \int_0^\infty U_N\, g(U_N, U_A)\, dU_N \\ &= \frac{1}{x}\int_{-\infty}^{\infty} h(v_N, v_N + \log x)\, dv_N, \end{aligned}$$

where $h(v_N, v_A) = U_N U_A\, g(U_N, U_A)$ is the joint PDF of $v_N = \log U_N$ and $v_A = \log U_A$.

The FM, in terms of the PDF for $\log S_T$, is then

$$\begin{aligned} \varphi(z) &= e^z\, f(e^z) \\ &= \int_{-\infty}^{\infty} h(v_N, v_N + z)\, dv_N, \end{aligned}$$

where $z \equiv \log x$. Since

$$\begin{aligned} E(U_N | U_A/U_N = x) &= E(e^{v_N} | v_A = v_N + \log x) \\ &= \frac{\int_{-\infty}^{\infty} e^{v_N}\, h(v_N, v_N + \log x)\, dv_N}{\int_{-\infty}^{\infty} h(v_N, v_N + \log x)\, dv_N}, \end{aligned}$$

Equation (10) follows that

$$\begin{aligned} \mathrm{r}(x) &= \frac{E(U_N | U_A/U_N = x)}{EU_N}\, f(x) \\ &= \frac{1}{xEU_N}\int_{-\infty}^{\infty} e^{v_N}\, h(v_N, v_N + \log x)\, dv_N. \end{aligned}$$

The RNM PDF for the *log* of price is then

$$\begin{aligned} q(z) &= e^z\, r(e^z) \\ &= \frac{1}{EU_N}\int_{-\infty}^{\infty} e^{v_N}\, h(v_N, v_N + z)\, dv_N. \end{aligned} \tag{12}$$

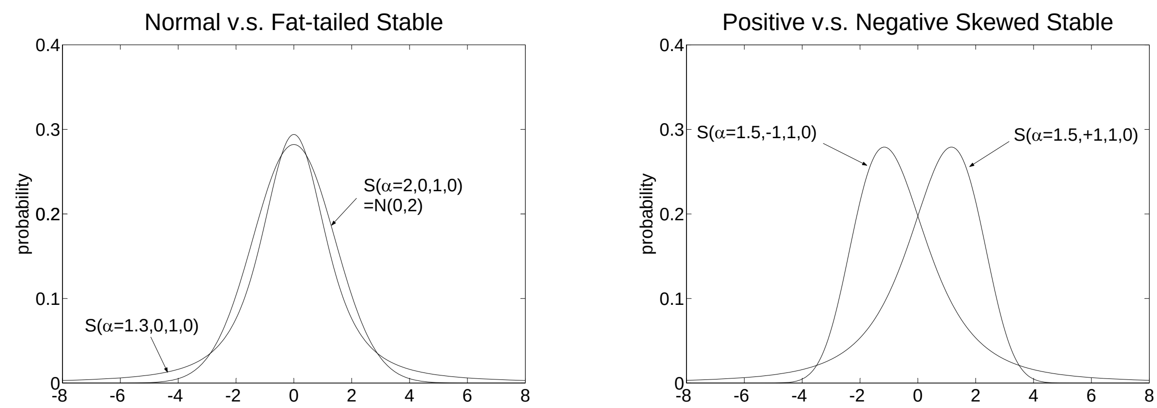


Figure 1: Illustrations of Stable Distributions with Different Parameter Values

4 Generalized Two-Factor Log-Stable Option Pricing Model

To derive a generalized two-factor log-stable RNM PDF, we rely on basic properties of stable distributions. In this section, we first present a brief review of the basic properties of stable distributions, which are essential to construct the log-stable option pricing model.

4.1 Basic Properties of Stable Distributions

Stable distributions[18] are a rich class of probability distributions that allow skewness and heavy tails and have many interesting mathematical properties. According to the Generalized Central Limit Theorem, if the sum of a large number of i.i.d. random variates has a limiting distribution after appropriate shifting and scaling, the limiting distribution must be a member of the stable class. A random variable X is stable if for X_1 and X_2 independent copies of X and any positive constants a and b,

$$a X_1 + b X_2 \overset{d}{=} c X + d \tag{13}$$

holds for some positive c and some $d \in R$, where the symbol $\overset{d}{=}$ means equality in distribution, i.e., both expressions have the same probability law. Equation (13) implies that the shape of the distribution is preserved up to scale and shift under addition.

Stable distributions $S(x; \alpha, \beta, c, \delta)$ are determined by four parameters: the characteristic exponent $\alpha \in$

[18] The stable distribution was developed by Paul Lévy, so it is also called the Lévy skew alpha-stable distribution.

$(0, 2]$, the skewness parameter $\beta \in [-1, 1]$, the scale parameter $c \in (0, \infty)$, and the location parameter $\delta \in (-\infty, \infty)$. If X has a distribution $S(x; \alpha, \beta, c, \delta)$, we write $X \sim S(x; \alpha, \beta, c, \delta)$ and use $s(x; \alpha, \beta, c, \delta)$ for the corresponding densities.

The characteristic exponent governs the tail behavior and indicates the degree of leptokurtosis. When $\alpha = 2$, its maximum permissible value, the normal distribution results, with variance $2c^2$. For $\alpha < 2$, the population variance is infinite. When $\alpha > 1$, the mean of the distribution $E(X)$ is δ. For $\alpha \leq 1$, the mean is undefined. The skewness parameter β is 0 when the distribution is symmetrical, positive when the distribution is skewed to the right, and negative when the distribution is skewed to the left. As α approaches 2, β loses its effect, and the distribution becomes symmetrical regardless of β. The location parameter δ merely shifts the distribution left or right, and the scale parameter c expands or contracts the distribution about δ in proportion to c. Figure 1 depicts stable densities with different parameter values. The left panel shows bell-shaped symmetric stable densities with $\alpha = 1.3,\ 2$. When $\alpha = 2$, the normal density results as mentioned above. As α decreases, three things occur to the density: the peak gets higher, the regions flanking the peak get lower, and the tails get heavier. The right panel shows maximally skewed stable densities with $\alpha = 1.5$ and $\beta = -1,\ 1$. The stable density is max-negatively skewed when $\beta = -1$, and max-positively skewed when $\beta = 1$.

Since there are no known closed form expressions for general stable densities[19], the most concrete way to describe all possible stable distributions is through the characteristic function (CF) or Fourier transform (FT).[20] The log CF of the general stable distribution $S(x; \alpha, \beta, c, \delta)$ is

$$\begin{aligned} \log cf_{\alpha,\beta,c,\delta}(t) &= \log E[e^{iXt}] \\ &= \begin{cases} i\delta t - |ct|^{\alpha}\left[1 - i\beta \operatorname{sgn}(t)\tan\left(\frac{\pi\alpha}{2}\right)\right], & \alpha \neq 1 \\ i\delta t + |ct|\left[1 + i\beta\frac{2}{\pi}\operatorname{sgn}(t)\log|ct|\right], & \alpha = 1, \end{cases} \end{aligned}$$

where $\alpha \in (0, 2]$ is the characteristic exponent, $\beta \in [-1, 1]$ is the skewness parameter, $c \in (0, \infty)$ is the scale

[19]There are only three cases where one can write down closed form expressions for the density: stable-normal, Cauchy and Lévy distributions.

[20]For a random variable X with density function $f(x)$, the characteristic function is defined by

$$cf(t) = E[e^{iXt}] = \int_{-\infty}^{\infty} e^{iXt} dx.$$

The function $cf(t)$ completely determines the distribution of X.

parameter, and $\delta \in (-\infty, \infty)$ is the location parameter.[21]

Two properties of stable distributions are important for deriving the generalized two-factor log-stable RNM:

Property 1: Convolution[22]

$$
\begin{aligned}
X &= \sum_{j=1}^{n} a_j X_j, \quad X_j \sim ind.\, S(\alpha, \beta_j, c_j, \delta_j) \quad \Rightarrow \\
X &\sim \mathop{\ast}_{j=1}^{n} S(\alpha, \mathrm{sgn}(a_j)\beta_j, |a_j| c_j, a_j \delta_j) \\
&\sim S(\alpha, \beta, c, \delta),
\end{aligned}
$$

where

$$
\begin{aligned}
\beta &= \frac{\sum_{j=1}^{n} |a_j|^{\alpha} c_j^{\alpha} \mathrm{sgn}(a_j)\beta_j}{c^{\alpha}}, \\
c &= \left(\sum_{j=1}^{n} |a_j|^{\alpha} c_j^{\alpha} \right)^{1/\alpha}, \text{ and} \\
\delta &= \sum_{j=1}^{n} a_j \delta_j.
\end{aligned}
$$

[21]The sgn function is defined as

$$
\mathrm{sgn}(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}
$$

[22]The convolution of f and g is written as $f \ast g$. If X and Y are two independent random variables with probability distributions f and g, respectively, then the probability distribution of the sum $z = X + Y$ is given by the convolution:

$$
(f \ast g)(z) = \int f(\tau) g(z - \tau) d\tau.
$$

Property 2: Two-sided Laplace transform

$$\begin{aligned} X &\sim S(\alpha, \beta, c, \delta), \quad \lambda \text{ complex with } Re(\lambda) \geq 0 \quad \Rightarrow \\ Ee^{-\lambda X} &= \begin{cases} -\infty, & \alpha < 2,\ \beta < 1 \\ \exp\left(-\lambda\delta - \lambda^\alpha c^\alpha \sec\left(\frac{\pi\alpha}{2}\right)\right), & \beta = 1, \end{cases} \end{aligned}$$

or equivalently,

$$Ee^{\lambda X} = \begin{cases} \infty, & \alpha < 2,\ \beta > -1 \\ \exp\left(\lambda\delta - \lambda^\alpha c^\alpha \sec\left(\frac{\pi\alpha}{2}\right)\right), & \beta = -1. \end{cases}$$

To describe the generalized two-factor log-stable RNM, it is also necessary to introduce an exponentially tilted stable distribution.[23] An exponentially tilted positively skewed stable density with parameters α, c, δ, and $\lambda > 0$ has density

$$ts_+(x; \alpha, c, \delta, \lambda) = ke^{-\lambda x} s(x; \alpha, +1, c, \delta),$$

where k is a normalizing constant to be determined. Its CF, using Property 2(Two-sided Laplace transform) with $\alpha \neq 1$, is

$$\begin{aligned} cf_{ts+}(t) &= k \int_{-\infty}^{\infty} e^{ixt} e^{-\lambda x} s(x; \alpha, +1, c, \delta) dx \\ &= k \int_{-\infty}^{\infty} e^{-(\lambda - it)x} s(x; \alpha, +1, c, \delta) dx \\ &= k \exp\left(-(\lambda - it)\delta - (\lambda - it)^\alpha c^\alpha \sec\left(\frac{\pi\alpha}{2}\right)\right). \end{aligned}$$

Since for any CF, $cf(0) \equiv 1$, we must have

$$\begin{aligned} k &= \exp\left(\lambda\delta + \lambda^\alpha c^\alpha \sec\left(\frac{\pi\alpha}{2}\right)\right) \quad \text{so that} \\ \log cf_{ts+}(t) &= i\delta t + c^\alpha \sec\left(\frac{\pi\alpha}{2}\right)(\lambda^\alpha - (\lambda - it)^\alpha). \end{aligned}$$

[23] Tilted stable distributions have already been used in the context of option pricing by Vinogaradov (2002), and Createa and Howison (2003).

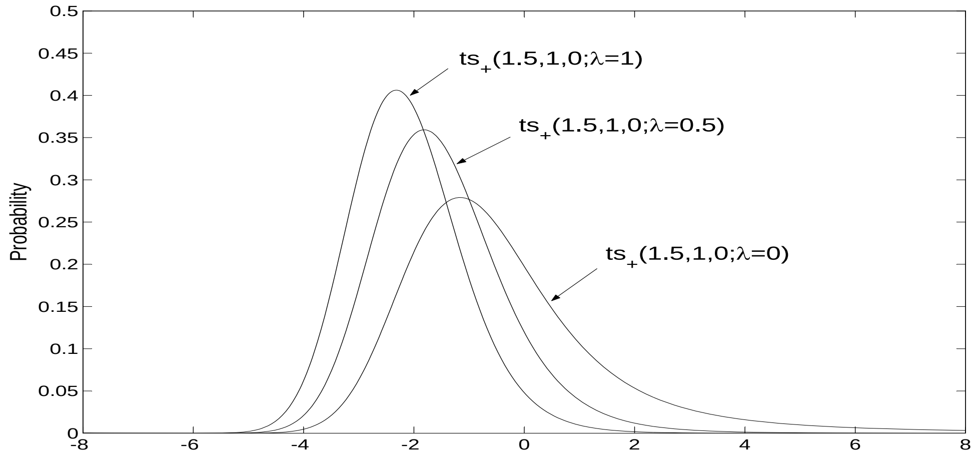


Figure 2: Exponentially Tilted Positively Skewed Stable Densities with Different Tilting Factor Values

Similarly, an exponentially tilted negatively skewed stable density with parameters α, c, δ, and $\lambda > 0$ is expressed as follows:[24]

$$
\begin{aligned}
ts_-(x; \alpha, c, \delta, \lambda) &= ke^{\lambda x} s(x; \alpha, -1, c, \delta), \\
k &= exp\left(-\lambda\delta + \lambda^\alpha c^\alpha \sec\left(\frac{\pi\alpha}{2}\right)\right), \text{ and} \\
\log cf_{ts-}(t) &= i\delta t + c^\alpha \sec\left(\frac{\pi\alpha}{2}\right)(\lambda^\alpha - (\lambda + it)^\alpha).
\end{aligned}
$$

Figure 2 illustrates exponentially tilted positively skewed stable densities with different tilting parameters $\lambda = 0,\ 0.5$, and 1. As λ increases, the upper tails get thinner. In contrast, for the exponentially tilted negatively skewed stable densities, as λ increases, the lower tails get thinner.

4.2 Generalized Two-Factor Log-Stable RNM

The generalized two-factor log-stable option pricing formula is based on distributional assumptions on the log marginal utilities of the asset $v_A (\equiv \log U_A)$ and of the numeraire $v_N (\equiv \log U_N)$. Let the two factors

[24] It is not possible to tilt a stable distribution with $\beta \in (-1, 1)$ in either direction, since then $\int e^{\lambda x} s(\alpha, \beta, c, \delta) dx$ would be infinite for any value of $\lambda \neq 0$.

u_1 and u_2 be independent maximally negatively skewed standard stable variates[25], which affect both v_A and v_N with a scale matrix C and a location vector D:

$$
\begin{aligned}
\begin{bmatrix} v_A \\ v_N \end{bmatrix} &= C \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + D \\
&= \begin{bmatrix} c_{A1} & c_{A2} \\ c_{N1} & c_{N2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \delta \\ 0 \end{bmatrix}, \quad u_j \sim ind.S(\alpha, -1, 1, 0),
\end{aligned} \tag{14}
$$

where $\forall c_{ij} \geq 0$, $i = A, N$ and $j = 1, 2$.[26] By Property 1 (Convolution) of stable distributions, both v_A and v_N are also maximally negatively stable with different scale parameters c_A and c_N, but the same exponent α:

$$
\begin{aligned}
v_A &= c_{A1}u_1 + c_{A2}u_2 + \delta \sim S(\alpha, -1, c_A, \delta) \qquad \text{and} \\
v_N &= c_{N1}u_1 + c_{N2}u_2 \sim S(\alpha, -1, c_N, 0),
\end{aligned}
$$

where $c_A = (c_{A1}^\alpha + c_{A2}^\alpha)^{1/\alpha}$ and $c_N = (c_{N1}^\alpha + c_{N2}^\alpha)^{1/\alpha}$. By Property 2 (Two-sided Laplace transform) of stable distributions, the expected marginal utilities of numeraire and asset are taken to be

$$
\begin{aligned}
EU_A &= Ee^{v_A} = e^{\delta - c_A^\alpha \sec\left(\frac{\pi\alpha}{2}\right)} \qquad \text{and} \\
EU_N &= Ee^{v_N} = e^{-c_N^\alpha \sec\left(\frac{\pi\alpha}{2}\right)}.
\end{aligned} \tag{15}
$$

Using the forward price equation (4), we have

$$
F = \frac{EU_A}{EU_N} = e^{\delta + (c_N^\alpha - c_A^\alpha)\sec\left(\frac{\pi\alpha}{2}\right)},
$$

[25]In order for the expectations in (15) to be finite, v_N and v_A must both be maximally negatively skewed, i.e., have $\beta = -1$. Therefore we have no choice but to make this assumption in order to evaluate log stable options. Nevertheless, this restriction does not prevent $\log S_T$ itself from having the general stable distribution as in (16).

[26]The scale parameters c_{ij} are not annualized. If annualized scale parameters are known, the scale parameters c_{ij} can be calculated from the annualized one:

$$
c_{ij} = \bar{c}_{ij} T^{1/\alpha},
$$

where $\bar{c}_{ij}$ are the annualized scale parameters and T is the remaining time to maturity.

which is the finite mean of the generalized two-factor log-stable RNM distribution by the mean-forward price equality condition (12).

Since v_A and v_N are both stable with a same characteristic exponent α , Property 1 implies that the FM for $\log S_T$ also follows a stable distribution with the same exponent α:[27]

$$\begin{aligned}
\log S_T &= v_A - v_N = \sum_{j=1}^{2}(c_{Aj} - c_{Nj})u_j + \delta \\
&\sim \mathop{\ast}_{j=1}^{2} S\left(\alpha, \mathrm{sgn}(c_{Nj} - c_{Aj}), |c_{Nj} - c_{Aj}|, \delta_j\right) \\
&\sim S\left(\alpha, \beta, c, \delta\right), \qquad (16)
\end{aligned}$$

where

$$\begin{aligned}
\beta &= \frac{\sum_{j=1}^{2} \mathrm{sgn}(c_{Nj} - c_{Aj})|c_{Nj} - c_{Aj}|^{\alpha}}{c^{\alpha}}, \\
c &= \left(\sum_{j=1}^{2} |c_{Nj} - c_{Aj}|^{\alpha}\right)^{1/\alpha}, \\
\delta_1 &= \delta, \text{ and } \delta_2 = 0.
\end{aligned}$$

Finally, the RNM PDF (12) and model (14) imply that the RNM PDF for $\log S_T$ is

$$\begin{aligned}
q(z) &= \frac{1}{EU_N}\int_{-\infty}^{\infty} e^{v_N} h(v_N, v_N + z) dv_N \\
&= \frac{1}{EU_N}\int_{-\infty}^{\infty} e^{-\frac{c_{N2}c_{A1} - c_{N1}c_{A2}}{c_{N1} - c_{A1}}u_2 - \frac{c_{N1}}{c_{N1} - c_{A1}}(z-\delta)} \cdot s(u_2; \alpha, -1, 1, 0) \\
&\quad s(z; \alpha, \mathrm{sgn}(c_{N1} - c_{A1}), |c_{N1} - c_{A1}|, \delta - (c_{N2} - c_{A2})u_2) du_2. \qquad (17)
\end{aligned}$$

The derivation of (17) may be found in Appendix A. The RNM PDF $q(z)$ proves to be a convolution of two exponentially tilted stable distributions:

$$q(z) = \mathop{\ast}_{j=1}^{2} ts_{\mathrm{sgn}(c_{Nj} - c_{Aj})}\left(z_j; \alpha, |c_{Nj} - c_{Aj}|, \delta_j, \left|\frac{c_{Nj}}{c_{Nj} - c_{Aj}}\right|\right) \qquad (18)$$

where $ts(\cdot)$ is the exponentially tilted stable density. Since the closed form expression of stable distributions

[27] Carr and Wu (2003) evaluate the option price under log-stable uncertainty, but only by making the very restrictive assumption that log returns have maximally negative skewness, i.e $\beta = -1$, in order to give the returns themselves finite moments. They incorporate maximum negative skewness directly into the stable distribution describing the RNM of the underlying asset.

does not exist, the RNM PDF (18) can be described by its log CF:[28]

$$
\begin{aligned}
\log cf_q(t) &= i\delta t + |c_{N1} - c_{A1}|^{\alpha} \sec\left(\frac{\pi\alpha}{2}\right)\left[\left|\frac{c_{N1}}{c_{N1} - c_{A1}}\right|^{\alpha} - \left(\left|\frac{c_{N1}}{c_{N1} - c_{A1}}\right| - \operatorname{sgn}(c_{N1} - c_{A1})it\right)^{\alpha}\right] \\
&+ |c_{N2} - c_{A2}|^{\alpha} \sec\left(\frac{\pi\alpha}{2}\right)\left[\left|\frac{c_{N2}}{c_{N2} - c_{A2}}\right|^{\alpha} - \left(\left|\frac{c_{N2}}{c_{N2} - c_{A2}}\right| - \operatorname{sgn}(c_{N2} - c_{A2})it\right)^{\alpha}\right]. \qquad (19)
\end{aligned}
$$

The proof of (18) and the derivation of (19) are given in Appendix B.

The parameters of the two exponentially tilted stable distributions are determined by the relative magnitude of elements in the scale matrix C: The skewness parameter β_j is determined by the sign of $c_{Nj} - c_{Aj}$; the scale parameter c_j depends on the absolute value of $c_{Nj} - c_{Aj}$; and the tilting factor λ_j is also affected by the relative magnitude of c_{Nj} and c_{Aj}. The location parameter δ of the RNM can be solved from the mean-forward price equality condition:

$$
F = \frac{EU_A}{EU_N} = e^{\delta + (c_N^{\alpha} - c_A^{\alpha}) \sec\left(\frac{\pi\alpha}{2}\right)} = S_0 e^{(r_f - d)T}, \qquad (20)
$$

where S_0 is the asset price at time 0, d is the dividend rate of asset, and $S_0 e^{(r-d)T}$ is the implicit forward price. This condition always holds if there are no arbitrage opportunities. Solving (20) for the location parameter δ of the RNM yields

$$
\delta = \log\left(S_0 e^{(r_f - d)T}\right) - (c_N^{\alpha} - c_A^{\alpha}) \sec\left(\frac{\pi\alpha}{2}\right).
$$

Finally, the generalized two-factor log-stable RNM of $\log S_T$ may be simply expressed as a function of the five free parameters $(\alpha, c_{N1}, c_{N2}, c_{A1}, c_{A2})$:

$$
q(z) = q(z; \alpha, c_{N1}, c_{N2}, c_{A1}, c_{A2} | S_0, r_f, d, T),
$$

where S_0 is the asset price at time 0, r_f is the risk-free interest rate, d is the dividend rate of asset, and T is the remaining time to maturity.

The generalized two-factor log-stable RNM has a very flexible parametric form with five free parameters

[28] Unless $\beta = 0$, the case $\alpha = 1$ require special treatment of both the CF and the location parameter. Therefore, the RNM PDF equation may not apply in that special case. In practice, however, this does cause problems because $\alpha = 1$ is irrelevant for an asset return's distribution.

for approximating other probability distributions, so it provides a new parametric method for estimating the RNM from a cross-section of option data. As shown in (18), the generalized two-factor log-stable RNM $q(z)$ has two additional tilting parameters which control the shapes of upper and lower tail respectively. This model thus allows a considerably accurate tool for estimating the RNM from the observed option prices even if the log-stable assumption might not be satisfied.

4.3 Special cases

The Black-Scholes log-normal Model (1973), the finite moment log-stable Model [Carr and Wu (2003)] and the Orthogonal log-Stable Model [McCulloch (1978, 1985, 1987, and 2003) and Hales (1997)] may be considered as special cases of the generalized two-factor log-stable model. In this section we describe these three models under the generalized two-factor log-stable model framework.

4.3.1 Othorgonal Log-Stable Model

The orthogonal log-stable model of McCulloch (1978, 1985, 1987, and 2003) and Hales (1997) assumes that v_A and v_N are independent with

$$\begin{aligned} v_A &\sim S(\alpha, -1, c_A, \delta) \text{ and} \\ v_N &\sim ind.\, S(\alpha, -1, c_N, 0). \end{aligned}$$

The orthogonal assumption can be expressed as a diagonal scale matrix in terms of the generalized two-factor framework:

$$\begin{bmatrix} v_A \\ v_N \end{bmatrix} = \begin{bmatrix} c_A & 0 \\ 0 & c_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \delta \\ 0 \end{bmatrix}, \quad u_j \sim ind.\, S(\alpha, -1, 1, 0), \quad j = 1, 2.$$

By the convolution property of stable distributions, the FM PDF of the orthogonal log-stable model, which is a convolution of max-positively stable density and max-negatively skewed density, is also a stable distribution:

$$\begin{aligned}\varphi(z) &= s(z_1 : \alpha, -1, c_A, \delta) \ast s(z_2 : \alpha, 1, c_N, 0) \\ &= s(z : \alpha, \beta, c, \delta),\end{aligned}$$

where $\beta = \frac{c_N^\alpha - c_A^\alpha}{c^\alpha}$ and $c = (c_N^\alpha + c_A^\alpha)^{1/\alpha}$.

The generalized log-stable RNM equation (18) implies that the RNM of the orthogonal log-stable model is a convolution of the max-negatively skewed stable density and exponentially tilted max-negatively skewed stable density:

$$\begin{aligned}q(z) &= \mathop{\ast}_{j=1}^{2} ts\left(z_j; \alpha, \mathrm{sgn}(c_{Nj} - c_{Aj}), |c_{Nj} - c_{Aj}|, \delta_j, \left|\frac{c_{Nj}}{c_{Nj} - c_{Aj}}\right|\right) \\ &= ts_-(z_1; \alpha, c_A, \delta, 0) \ast ts_+(z_2; \alpha, c_N, 0, 1) \\ &= s(z_1; \alpha, -1, c_A, \delta) \ast ts_+(z_2; \alpha, c_N, 0, 1).\end{aligned}$$

With the mean-forward price equality condition:

$$F = \frac{EU_A}{EU_N} = e^{\delta + (c_N^\alpha - c_A^\alpha)\sec\left(\frac{\pi\alpha}{2}\right)} = S_0 e^{(r_f - d)T},$$

the location parameter of the RNM can be solved as:

$$\delta = \log(S_0 e^{(r_f - d)T}) - \beta c^\alpha \sec\left(\frac{\pi\alpha}{2}\right).$$

Therefore, the orthogonal stable RNM of $\log S_T$ is directly expressed as a function of the three parameters (α, β, c) of the FM[29]:

$$q(z) = q(z; \alpha, \beta, c | S_0, r_f, d, T).$$

[29]The free parameters (β, c) are equivalent to (c_A, c_N), since $\beta = \frac{c_N^\alpha - c_A^\alpha}{c^\alpha}$ and $c = (c_N^\alpha + c_A^\alpha)^{1/\alpha}$.

4.3.2 Finite Moment Log-Stable Model

The finite moment log-stable model of Carr and Wu (2003) assumes that the RNM is a max-negatively skewed log-stable distribution, i.e., $\beta = -1$. When $\beta = -1$, the RNM and FM have common density:

$$q(z) = \varphi(z) = s\left(z; \alpha, -1, c, \log F + c^{\alpha} \sec\left(\frac{\pi\alpha}{2}\right)\right).$$

The finite moment assumption, $\beta = -1$, can be expressed as the following scale matrix under the generalized two-factor framework:

$$\begin{bmatrix} v_A \\ v_N \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \delta \\ 0 \end{bmatrix}, \quad u_j \sim ind.\, S(\alpha, -1, 1, 0), \quad j = 1, 2.$$

With the mean-forward price equality condition:

$$F = \frac{EU_A}{EU_N} = e^{\delta - c^{\alpha}\sec\left(\frac{\pi\alpha}{2}\right)} = S_0 e^{(r_f - d)T},$$

the location parameter of the RNM can be solved as:

$$\delta = \log(S_0 e^{(r_f - d)T}) + c^{\alpha} \sec\left(\tfrac{\pi\alpha}{2}\right).$$

Therefore, the finite moment log-stable RNM of $\log S_T$ is simply expressed as a function of the two free parameters (α, c):

$$q(z) = q(z; \alpha, c | S_0, r_f, d, T).$$

4.3.3 Black-Scholes Log-Normal Model

The Black-Scholes option pricing model (1973) assumes the log price follows a normal distribution. When $\alpha = 2$, a stable distribution results normal with mean δ and variance $\sigma^2 = 2c^2$. Therefore, the log-normal case can be considered as a special case of the generalized two-factor stable model. The log-normal assumption can be expressed as a generalized form:

$$\begin{bmatrix} v_A \\ v_N \end{bmatrix} = \begin{bmatrix} c_{A1} & 0 \\ c_{N1} & c_{N2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \delta \\ 0 \end{bmatrix}, \quad u_j \sim ind.\, S(\alpha = 2, -1, 1, 0), \quad j = 1, 2.$$

We can pin down the location parameter from the mean-forward price equality condition:

$$\begin{aligned} \delta &= \log F - (c_N^\alpha - c_A^\alpha)\sec\left(\frac{\pi\alpha}{2}\right) \\ &= \log F - \frac{1}{2}\left(\frac{c_{N1}^2 - c_{A1}^2 + c_{N2}^2}{(c_{N1} - c_{A1})^2 + c_{N2}^2}\right)\sigma^2. \end{aligned}$$

Thus, the FM of $\log S_T$ is

$$\begin{aligned} \varphi(z) &= s(z; \alpha = 2, \beta, c, \delta) \\ &= \phi(z; \delta, \sigma^2 = 2c^2) \\ &= \phi\left(z; \log F - \frac{1}{2}\left(\frac{c_{N1}^2 - c_{A1}^2 + c_{N2}^2}{(c_{N1} - c_{A1})^2 + c_{N2}^2}\right)\sigma^2, \sigma^2\right). \end{aligned}$$

The RNM of $\log S_T$ is

$$\begin{aligned} q(z) &= ts_{\mathrm{sgn}(c_{N1} - c_{A1})}\left(z_1; 2, |c_{N1} - c_{A1}|, \delta, \left|\frac{c_{N1}}{c_{N1} - c_{A1}}\right|\right) * ts_+(z_2; 2, c_{N2}, 0, 1) \\ &= \phi\left(z; \log F - \frac{\sigma^2}{2}, \sigma^2\right), \end{aligned}$$

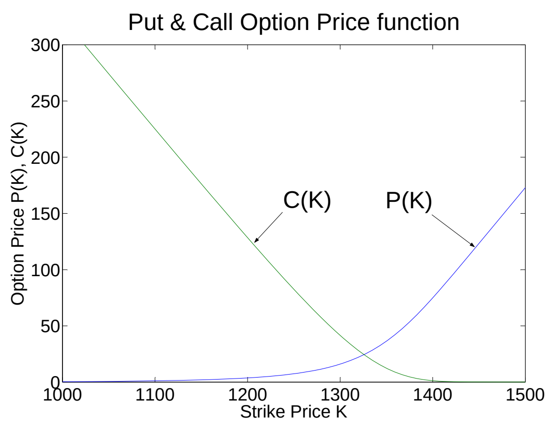
where $F = e^{(r_f - d)T} S_0$ and $\phi(\cdot)$ is the PDF of normal distributions.[30] Accordingly, the Black-Scholes log-normal RNM can be expressed as a function of only one free parameter σ:

$$q(z) = q(z; \sigma | S_0, r, d, T).$$

[30] In the log-normal case, the RNM and FM therefore both have the same Gaussian shape in terms of log price, with the same variance. They differ only in location, by the observable risk premium

$$\left(1 - \frac{c_{N1}^2 - c_{A1}^2 + c_{N2}^2}{(c_{N1} - c_{A1})^2 + c_{N2}^2}\right)\frac{\sigma^2}{2}$$

that is determined by the scale matrix, i.e., by the relative standard deviation of $\log U_N$ and $\log U_A$. This comes about because a exponentially tilted normal distribution is just another normal back again, with same variance but different mean.



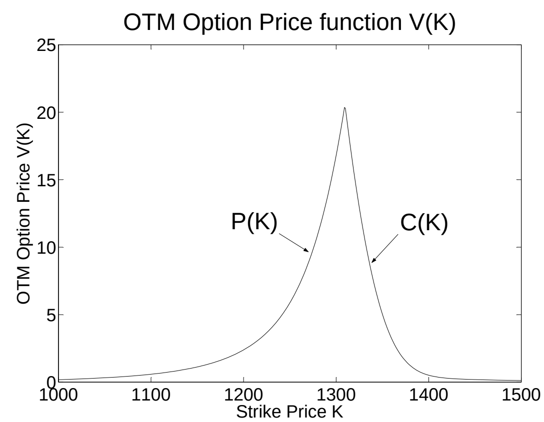


Figure 3: Option Value Functions for S&P 500 Index Options on Sep 13, 2006 (F=1,325.7)

5 RNM Estimation from Option Market Prices

In this section, we estimate the conditional RNM from a cross-section of S&P 500 index option prices using the BS lognormal model, the finite moment log-stable model, the orthogonal stable model, and the generalized two-factor stable model under the modified least square criterion. We also conduct a simple likelihood ratio test for the model selection among the competing nested models.

5.1 OTM Option value functions

Let $C(K)$ be the value, in units of numeraire to be delivered at time 0, of a European call option which gives right to the holder to purchase 1 unit of the asset in question at time T at strike price K. By the definition of the RNM, its value must be the discounted expectation of its payoff under either $\mathrm{r}(x)$ or $q(z)$:

$$\begin{aligned} C(K) &= e^{-r_f T} \int_0^{\infty} \max(x - K, 0)\mathrm{r}(x)dx \\ &= e^{-r_f T} \int_{-\infty}^{\infty} \max(e^z - K, 0)q(z)dz. \end{aligned} \tag{21}$$

Similarly, let $P(K)$ be the value of a European put option which allows the owner to sell one unit of the asset at time T at strike price K so that

$$\begin{aligned} P(K) &= e^{-r_f T} \int_0^{\infty} \max(K - x, 0)\mathrm{r}(x)dx \\ &= e^{-r_f T} \int_{-\infty}^{\infty} \max(K - e^z, 0)q(z)dz. \end{aligned} \tag{22}$$

Define the out-of-the-money (OTM) option value function by

$$V(K; \boldsymbol{\theta}) = \begin{cases} P(K; \boldsymbol{\theta}) & \text{for } K < F \\ C(K; \boldsymbol{\theta}) & \text{for } K \geq F \end{cases}, \tag{23}$$

where $\boldsymbol{\theta}$ is the vector of the RNM parameters. Using the put-call parity[31], we can rewrite (23) as:

$$V(K; \boldsymbol{\theta}) = \min\left(C(K; \boldsymbol{\theta}), P(K; \boldsymbol{\theta})\right). \tag{24}$$

The OTM option value function $V(K)$ is continuous at F, and is also monotonic and convex on either side of F under the arbitrage-free condition. Figure 3 illustrates option value functions using S&P 500 index options on Sep 13, 2006 (F=1,325.7). The left panel shows the call option value function $C(K)$ and the put option value function $P(K)$. The right panel shows the OTM option value function $V(K) = \min(C(K), P(K))$.

Since there are no known closed form expressions for general stable densities, the option value function $V(K)$ may be evaluated through the characteristic function (CF) or Fourier transform (FT). With no loss of generality, we may measure the asset in units such that $F = 1$. Modifying Carr and Madan (1999),[32] the

[31] In the absence of arbitrage opportunities, the following relationship holds for European option:

$$C(K) + e^{-r_f T} K = P(K) + e^{-dT} S_0,$$

so that $C(K) = P(K)$ at $K = F$. The put-call parity implies that $C(K)$, $P(K)$, and $V(K)$ are equivalent, so we use $V(K)$ instead of $C(K)$ and $P(K)$.

[32] Carr and Madan in fact base their (14) on a function which equals $P(K)$ when K is less than the spot price S_0 and $C(K)$ otherwise. This unnecessarily creates a small discontinuity which can only aggravate the Fourier inversion. The present function $V(K)$ avoids this problem, with the consequence that (25) is in fact somewhat simpler than their (14).

Fourier Transform of $v(z) \equiv V(e^z)$ is then

$$\phi_v(t) \;=\; e^{-rT}\left[\frac{cf_q(t-i)-1}{it-t^2}\right], \quad t \neq 0, \tag{25}$$

where cf_q is the CF of the RNM pdf $q(z)$. When $t = 0$, this formula takes the value $0/0$, but the limit may be evaluated by means of l'Hôpital's rule. In the case of the generalized two-factor generalized log-stable model, this becomes

$$\begin{aligned}\phi_v(0) &= e^{-rT}\frac{cf_q'(-i)}{i}\\ &= e^{-rT}\left[\left(-\sum_{j=1}^{2}(c_{Nj}^{\alpha}-c_{Aj}^{\alpha})\sec\left(\frac{\pi\alpha}{2}\right)\right)+\alpha\sum_{j=1}^{2}\operatorname{sgn}(c_{Nj}-c_{Aj})|c_{Nj}-c_{Aj}|^{\alpha}\sec\left(\frac{\pi\alpha}{2}\right)\cdot\right.\\ &\qquad\left.\left(\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|-\operatorname{sgn}(c_{Nj}-c_{Aj})\right)^{\alpha-1}\right].\end{aligned}$$

Unfortunately, however, the function $v(z)$ has a cusp at $z = 0$ corresponding to that in $V(K)$ at $K = F$, so that numerical inversion of (25) by means of the discrete inverse FFT results in pronounced spurious oscillations in the vicinity of the cusp. The problem is that the ultra-high frequencies required to fit the cusp and its vicinity are omitted from the discrete Fourier inversion, which only integrates over a finite range of integration instead of the entire real line. Increasing the range of integration progressively reduces these oscillations, but never entirely eliminates them. However, the fact that increasing the range of integration does give improved results allows the FFT inversion results to be "Romberged" to give satisfactory results, as follows: Start with a large number of points $N = N_1$, with a log-price step $\Delta z = c\sqrt{2\pi/N}$ (or a round number in that vicinity if desired), and a frequency-domain step $\Delta t = 2\pi/(N\Delta z)$. Then quadruple N to $N_2 = 4N_1$, and then again to $N_3 = 16N_1$, halving both step sizes each time, so as to double the range of integration each time, while obtaining values for the original z grid. Each of the original N_1 z values now has 3 approximate function values v_1, v_2, and v_3 that are converging on the true value at an approximately geometric rate as the grid fineness and range of integration are successively doubled. The true value may then be approximated to a high degree of precision at each of these points simply by extrapolating the geometric series implied by the three values to infinity:

$$v_\infty = v_3 + \frac{\rho}{1-\rho}(v_3 - v_2),$$

where $\rho = (v_3 - v_2)/(v_2 - v_1)$. The residual error may then be conservatively estimated by computing v_0 using $N_0 = N_1/4$, repeating the above procedure using v_0, v_1, and v_2, and assuming that the absolute discrepancy between the two results is an upper bound on the error. It was found that for $\alpha \geq 1.3$, $N_1 = 2^{10}$ usually gives a maximum estimated error less than .0001 relative to $F = 1$, though occasionally $N_1 = 2^{14}$ is necessary.[33] Put-call parity may then be used to recover $C(K)$ and/or $P(K)$, as desired, from $V(K) \equiv v(\log K)$. The above procedure gives the value of $v(z)$ at N_1 closely spaced values of z, and therefore $V(K)$ at N_1 closely spaced values of K. Unfortunately, however, these will ordinarily not precisely include the desired exercise prices, and because of the convexity of $V(K)$ on each side of the cusp, linear interpolation may give an interpolation error in excess of the Fourier inversion computational error. Nevertheless, cubic interpolation on $C(K)$ and/or $P(K)$ using two points on each side of each desired exercise price gives very satisfactory results.

5.2 Modified Least Square Criterion

By using the OTM option value function (23), option pricing models can be expressed as a non-linear regression with the parameters of the RNM:

$$\begin{aligned} V_i &= V(K_i; \boldsymbol{\theta} | S_0, r_f, d, T) + \epsilon_i, \quad i = 1, \ldots, N \\ &= V_i(\boldsymbol{\theta}) + \epsilon_i, \end{aligned} \tag{26}$$

where $\boldsymbol{\theta}$ is the vector of the RNM parameters, V_i is an observed OTM option price, $V_i(\boldsymbol{\theta})$ is the theoretical OTM option price, and ϵ_i is the pricing error of the OTM option with strike price K_i. Consequently, we may apply non-linear regression techniques to the model (26) in order to estimate the parameters $\boldsymbol{\theta}$ of the RNM.

If we observe transaction prices for all options across strike prices at the same time, the Non-linear Least Squares (NLS) criterion would be proper for estimating the RNM parameters:

$$\begin{aligned} \min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \quad L(SSE) &= \sum_{i=1}^{N} (V_i - V_i(\boldsymbol{\theta}))^2 \\ &= \boldsymbol{\epsilon}^\top \boldsymbol{\epsilon}, \end{aligned}$$

[33]For the financially less relevant values of $\alpha < 1.3$, the infinite first derivative of the imaginary part of (25) at the origin causes additional computational problems. These problems become even worse for $\alpha < 1$, when the imaginary part becomes discontinuous at the origin.

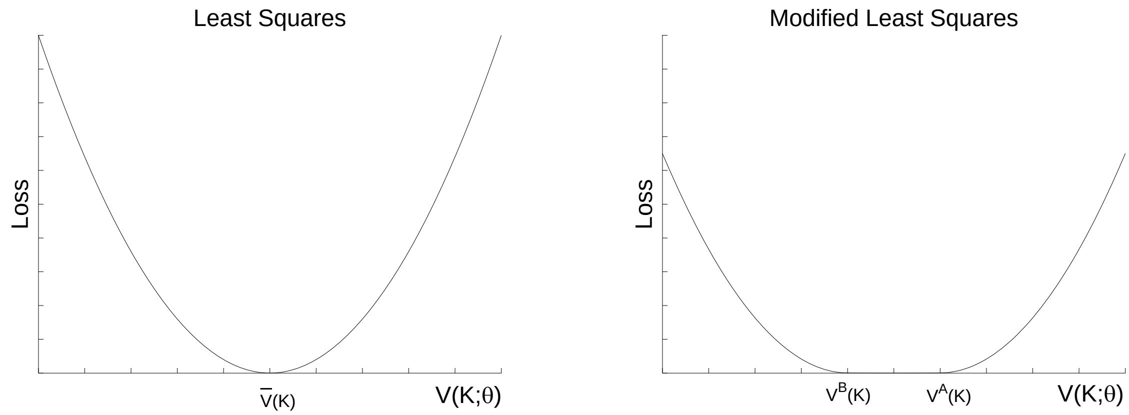


Figure 4: Loss Functions of NLS and MLS

where $\boldsymbol{\epsilon} = [\epsilon_1 \ \epsilon_2 \ \cdots \ \epsilon_N]^\top$ is the vector of pricing errors.[34]

Unfortunately the transaction prices are recorded with substantial measurement errors due to non-synchronous trading so that we instead use bid and ask quote prices. The bid-ask average prices have been used as an alternative of the transaction prices in many studies. The NLS criterion, however, does not fully exploit the additional information coming from the individual bid and ask quote prices because it only utilizes the bid-ask average OTM option prices. In our study we therefore use a modified least squares (MLS) criterion under which the loss value (Modified SSE, MSSE) increases only when the theoretical prices fall outside of the bid-ask price range. However, since multiple solutions are possible under the MLS criterion, we add an arbitrary small ordinary least square term to guarantee a unique solution:

$$\min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \quad L(MSSE) \quad = \quad \sum_{i=1}^{N} \Big[\left(V_i^B - V_i(\boldsymbol{\theta})\right)_+^2 + \left(V_i(\boldsymbol{\theta}) - V_i^A\right)_+^2 + \lambda \left(\bar{V}_i - V_i(\boldsymbol{\theta})\right)^2 \Big],$$

where V_i^B is the OTM option bid price, V_i^A is the OTM option ask price, $\bar{V}_i = (V_i^B + V_i^A)/2$ is the OTM option bid-ask average price at strike price K_i, $X_+ = \max(0, X)$, and λ is a small constant[35]. Figure 4 illustrates the loss functions based on the two criteria. The left panel shows the NLS loss function, and the right panel

[34]We cannot use the weighted least square criterion because the variance structure of pricing errors is not known. To construct variance structure of pricing errors, it is necessary to consider different source of pricing errors: non-synchronous trading, price discreteness, and the bid-ask bounce. Unfortunately, there is no simple or generally accepted manner for modeling all of these effects.

[35]In our study, we set $\lambda = 0.01$.

presents the MLS loss function.

5.3 Empirical results of the RNM estimation

5.3.1 Data

We have estimated the parametric risk-neutral density using the cross-section data on the S&P 500 index options traded at the Chicago Board of Options Exchange (CBOE). The transaction prices are recorded with substantial measurement errors due to non-synchronous trading so that we use daily closing bid and ask price quotes. We have obtained 100 sets of cross-section data on the S&P 500 index options which are transacted with 2 months to maturity in 2006. We have filtered the data using the arbitrage violation conditions since the existence of arbitrage possibilities can lead to negative risk-neutral probabilities. By checking the monotonicity and convexity of the option pricing functions, we may eliminate option prices which violate the arbitrage-free condition. After eliminating the violating data, we have 8,468 option price quotes.

Since risk-free interest rates for a time of maturity exactly matching the options' time to maturity generally can not be observed, we compute implicit risk-free interest rates from the European put-call parity as suggested by Jackwerth and Rubinstein (1996). The estimation of the RNM is conducted by using the four models separately for each cross section data set.

5.3.2 Goodness-of-fit

In this section, we compare the goodness-of-fit of the RNM estimations of the four option pricing models: the BS log-normal option pricing model (BS), the finite moment log-stable option pricing model (FS), the orthogonal log-stable option pricing model (OS), and the generalized two-factor log-stable option model (GS). If option prices are exact and continuous, and if the pricing model holds exactly for every single option, the RNM parameters can be recovered with zero pricing errors between the estimated and observed prices. However, model and market imperfections introduce pricing errors. In order to compare these pricing errors, we first estimate the RNM by applying the four option pricing models to S&P 500 index (European) options under the modified least square criterion (MLS). Then, the four models are compared by examining the pricing errors associated with each model in terms of the Modified Root Mean Squared Error (MRMSE),

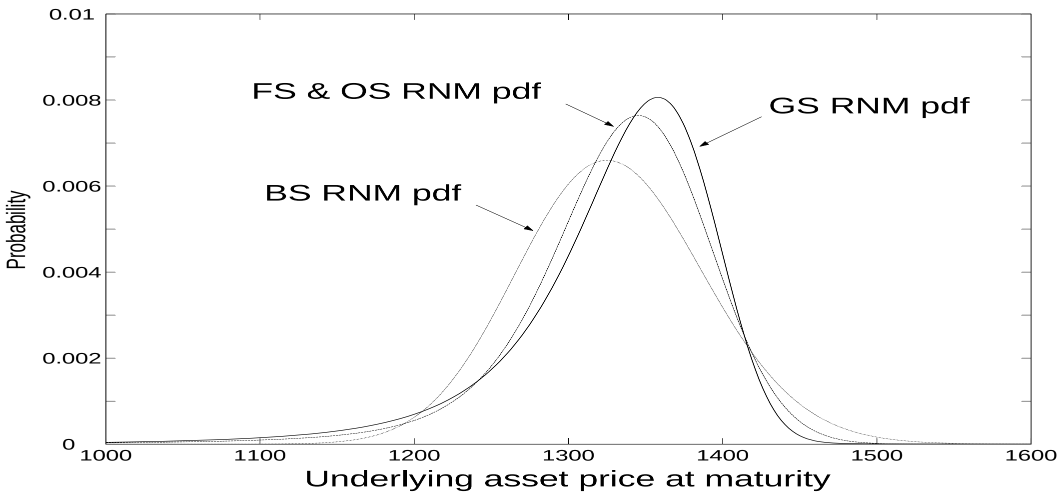


Figure 5: Estimated RNM Densities for S&P 500 Index Options on Sep. 13, 2006

BS	FS		OS[a]			GS[b]				
σ	α	c	α	c_N	c_A	α	c_{N1}	c_{N2}	c_{A1}	c_{A2}
0.116	1.743	0.080	1.743	0	0.080	1.343	0.405	0.631	0.398	0.750

a Implied FM parameters of the OS model: $\alpha = 1.743, \beta = -1$, and $c = 0.080$.
b Implied FM parameters of the GS model: $\alpha = 1.343, \beta = -0.956$, and $c = 0.121$.
Note: The entries report the sample average of the estimated parameters. The sample contains 100 sets of cross-section data on S&P 500 index options with 2 months to maturity, which are traded in 2006.

Table 1: Estimated RNM Parameters for S&P 500 Index Options

which represents the average pricing error:

$$MRMSE \quad = \quad \sqrt{\frac{1}{N-k} MSSE},$$

where $MSSE$ is the modified sum of squared errors, N is the number of strike prices, i.e., the number of observations, and k is the number of free parameters of the RNM.

The estimated RNM density functions for the four models are illustrated in Figure 5, and the estimated average RNM parameters are reported in Table 1. Also, given the RNM density from the each model, Equation (21) and (22) give predicted values for the option prices. These model predictions and the actual bid-ask

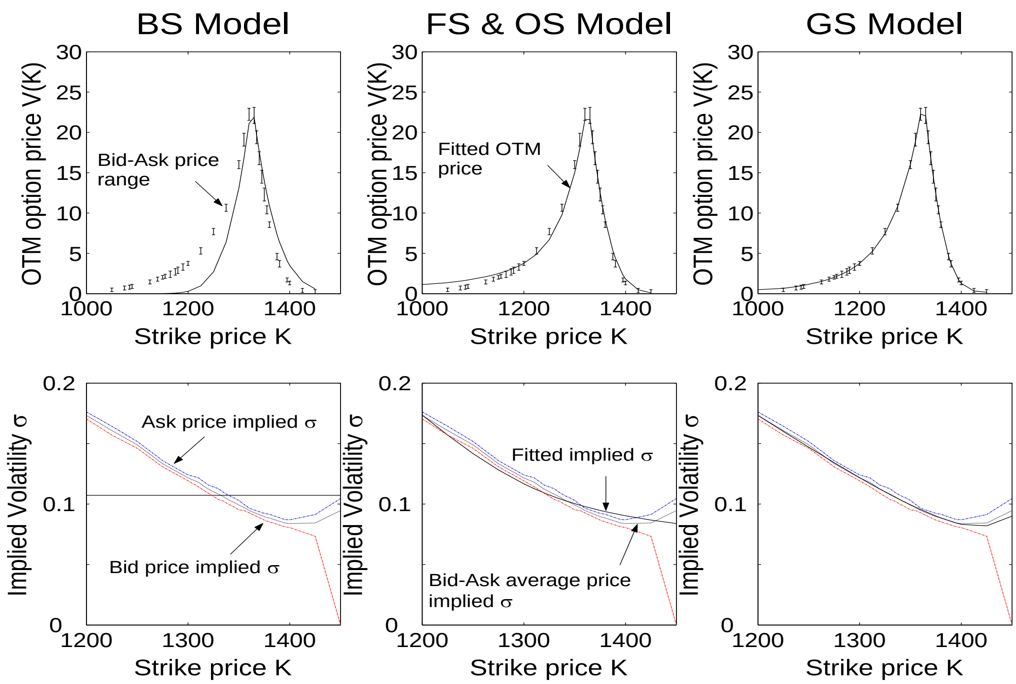


Figure 6: Fitted Option Prices and Volatility Smiles for S&P 500 Index Options on Sep. 13, 2006

prices are plotted in the upper panels in Figure 6 for an illustrative date. We calculate the volatility smiles from the predicted prices and the actual bid-ask prices, and they are plotted in the lower panels in Figure 6. Since the Black-Scholes option price is monotonically increasing in volatility, deviations of estimated volatilities from actual implied volatilities are also related with pricing errors of the option pricing model. Options with actual implied volatility above (below) the estimated volatility are underpriced (overpriced) by the option pricing model.[36]

The fitting performances of the four models are reported in Table 2 in terms of the Modified Root Mean Squared Error (MRMSE). Table 2 shows that the GS model outperforms the BS, FS and OS model with respect to the goodness-of-fit for all data sets. Note that the fitting performance of the FS and OS models are identical. This implies that there is no additional improvement from introducing an additional parameter c_N of the OS model relative to the FS model. That is to say, the S&P 500 returns have maximally negative

[36]This is based on the assumption that there is no mispricing in the observed option prices. If the model is correctly specified and the observed option prices are mispriced, options with actual implied volatility above (below) the estimated volatility are overpriced (underpriced).

	BS	FS	OS	GS
MRMSE	150.848	3.480	3.480	0.013
(1st term)	148.716	3.334	3.334	0.004
(2nd term)	2.132	0.146	0.146	0.009

Note: The 1st term of MSSE is $\sum_{i=1}^{N}\left[\left(V_i^B - V_i(\boldsymbol{\theta})\right)_+^2 + \left(V_i(\boldsymbol{\theta}) - V_i^A\right)_+^2\right]$, and the 2nd term is the ordinary least square term $\sum_{i=1}^{N}\lambda\left(\bar{V}_i - V_i(\boldsymbol{\theta})\right)^2$. The entries report the sample average of the test statistics and the corresponding P-values. The sample contains 100 sets of cross-section data on S&P 500 index options with 2 months to maturity, which are traded in 2006.

Table 2: Goodness-of-fit of the Option Pricing Models

skewness.

The BS model overprices options around the ATM price, while it underprices options with relatively large or small exercise prices. The FS and OS models perform relatively better around the ATM price, but exhibits pricing bias in both the upper and lower tails. On the other hand, the GS model shows almost perfect goodness-of-fit for all exercise prices. The first terms of the MSSE are almost zero for the GS model. This result indicates that the theoretical option prices based on the estimated RNM could fall into the bid-ask option price range across almost all strike prices.

5.3.3 Likelihood Ratio Test

Using the Kullback-Leibler Information Criterion (KLIC), Vuong (1989) proposed simple likelihood-ratio tests for the model selection among the competing models, which are non-nested or nested. We assume that the pricing errors are i.i.d. normally distributed with variance σ^2. Under such assumptions, minimizing the sum of squared pricing error is equivalent to maximizing the log likelihood function. Thus, the NLS estimates can be regarded as maximum likelihood estimates.[37]

[37]Under the MLS criterion it is not possible to compute the LR statistics so that we use NLS criterion for the model selection test.

Consider two competing models **F** and **G** whose log density functions are given by, respectively:

$$\begin{aligned}\log f(e_{\theta i};\boldsymbol{\theta}) &= -\frac{1}{2}\log\left(2\pi\sigma_f^2\right)-\frac{e_{\theta i}^2}{2\sigma_f^2} \quad \text{and}\\ \log g(e_{\gamma i};\boldsymbol{\gamma}) &= -\frac{1}{2}\log\left(2\pi\sigma_g^2\right)-\frac{e_{\gamma i}^2}{2\sigma_g^2},\end{aligned}$$

where $e_{\theta i}$ and $e_{\gamma i}$ denote the pricing error on the ith option under the model **F** and **G**, respectively, and $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$ are the parameter vectors of **F** and **G**, respectively. Since the maximum likelihood estimate for σ^2 is simply the mean squared pricing error: mse $= \mathbf{e}^\top\mathbf{e}/N$, the log likelihood functions are given by, respectively:

$$\begin{aligned}\mathcal{L}_f(\mathbf{e}_\theta;\boldsymbol{\theta}) &= \sum_{i=1}^{N}\log f(e_{\theta i};\boldsymbol{\theta}) = -\frac{N}{2}\left[1+\log(2\pi)+\log\left(\frac{\mathbf{e}_\theta{}^\top\mathbf{e}_\theta}{N}\right)\right] \quad \text{and}\\ \mathcal{L}_g(\mathbf{e}_\gamma;\boldsymbol{\gamma}) &= \sum_{i=1}^{N}\log g(e_{\gamma i};\boldsymbol{\gamma}) = -\frac{N}{2}\left[1+\log(2\pi)+\log\left(\frac{\mathbf{e}_\gamma{}^\top\mathbf{e}_\gamma}{N}\right)\right],\end{aligned}$$

where $\mathbf{e}_\theta$ and $\mathbf{e}_\gamma$ are the pricing error vectors for each model. Furthermore, the likelihood ratio between the two models (**F** and **G**) is given by:

$$\begin{aligned}LR\left(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\gamma}}\right) &= \mathcal{L}_f(\hat{\mathbf{e}}_\theta;\hat{\boldsymbol{\theta}})-\mathcal{L}_g(\hat{\mathbf{e}}_\gamma;\hat{\boldsymbol{\gamma}})\\ &= -\frac{N}{2}\left[\log\left(\frac{\hat{\mathbf{e}}_\theta^\top\hat{\mathbf{e}}_\gamma}{N}\right)-\log\left(\frac{\hat{\mathbf{e}}_\gamma^\top\hat{\mathbf{e}}_\gamma}{N}\right)\right]\\ &= -\frac{N}{2}\left[\log\left(\frac{\hat{\mathbf{e}}_\theta^\top\hat{\mathbf{e}}_\theta}{\hat{\mathbf{e}}_\gamma^\top\hat{\mathbf{e}}_\gamma}\right)\right].\end{aligned}$$

If the model **G** is nested in the model **F**, any conditional density $g(\cdot;\boldsymbol{\gamma})$ is also a conditional density $f(\cdot;\boldsymbol{\theta})$ for some $\boldsymbol{\theta}$ in $\boldsymbol{\Theta}$. Based on the KLIC, we consider the following hypotheses and definitions:[38]

$$\begin{aligned}H_0 &: \quad E\left[\log\frac{f(e_{\theta i};\boldsymbol{\theta})}{g(e_{\gamma i};\boldsymbol{\gamma})}\right]=0 : \textbf{F} \text{ and } \textbf{G} \text{ are equivalent}\\ H_A &: \quad E\left[\log\frac{f(e_{\theta i};\boldsymbol{\theta})}{g(e_{\gamma i};\boldsymbol{\gamma})}\right]>0 : \textbf{F} \text{ is better than } \textbf{G}.\end{aligned}$$

[38]Since **G** can never be better than **F**, H_A does not include the case such that **G** is better than **F**.

		Model **G**					
		BS		FS		OS	
		2LR	P-value	2LR	P-value	2LR	P-value
	FS	121.7	0.000	–	–	–	–
Model **F**	OS	121.7	0.000	0.0	1.000	–	–
	GS	266.3	0.000	144.5	0.000	144.5	0.000

Note: The test statistic is asymptotically chi-square distributed with $p-q$ degree of freedom. The entries report the sample average of the test statistics and the corresponding P-values. The sample contains 100 sets of cross-section data on S&P 500 index options with 2 months to maturity, which are traded in 2006.

Table 3: Likelihood Ratio Tests for Nested Models

If **G** is nested in **F** and **F** is correctly specified, then

$$\begin{aligned}
\text{under } H_0 \quad &: \quad 2LR(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\gamma}}) \xrightarrow{D} \chi^2_{p-q}, \\
\text{under } H_A \quad &: \quad 2LR(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\gamma}}) \xrightarrow{a.s.} +\infty,
\end{aligned}$$

where p and q are the number of parameters in models **F** and **G**, respectively.

The three stable type models (BS, FS, and OS) are nested in the GS model; the BS and FS model are nested in the OS model; and the BS model is nested in the FS model. The LR test statistics and the corresponding P-values between nested models are reported in Table 3. The test results indicate that the GS model is significantly better the BS, FS, and OS models and the OS and FS model are significantly better the BS model. However, the OS model and FS model are equivalent even though the OS model has an additional parameter relative to the FS model.

6 Concluding Remark

The generalized two-factor log-stable option pricing model is a highly integrated approach to evaluating contingent claims in the sense that it provides state prices, pricing kernels, and the risk neutral measure explicitly. The RNM can be simply derived by adjusting the FM for the state-contingent value of the numeraire. Under generalized two-factor log-stable uncertainty the RNM is expressed as a convolution of two exponentially tilted stable distributions, while the FM itself is a pure stable distribution. Furthermore, the generalized two-factor log-stable RNM has a very flexible parametric form for approximating other probability distributions. Thus, this model also provides a considerably accurate tool for estimating the RNM from the observed option prices even though the two-factor log-stable assumption might not be satisfied.

The empirical results of the RNM estimation from the S&P 500 index options shows that the generalized two-factor log-stable model gives better performance than the Black-Scholes log-normal model, the finite moment log-stable model and the orthogonal log-stable model in fitting the observed option prices. Moreover, the distributional assumption for the generalized stable model is consistent with the implied volatility structure, which violates the lognormal assumption of the Black-Scholes model.

The Black-Scholes log-normal model, the finite moment log-stable model, and the orthogonal log-stable model are nested by the generalized two-factor log-stable model. In order to verify the empirical performance of the generalized two-factor log-stable model, it is necessary to compare it with other parametric models which are not nested by it. Also, we need to examine the stability or robustness of the RNM parameter through the Monte Carlo experiment or the Bootstrap technique. Further research should be conducted on these issues to verify the empirical performance of the generalized stable option[39]

[39]See Lee (2008).

APPENDICES

A Derivation of (18)

The joint distribution of v_N and v_A can be expressed as

$$\begin{aligned} h(v_N, v_A) &= h(v_N, v_N + z) \\ &= \frac{1}{|c_{N1}c_{A2} - c_{N2}c_{A1}|} f_{U_1U_2}(u_1, u_2), \end{aligned} \tag{27}$$

where $f_{U_1U_2}(u_1, u_2)$ is the joint distribution of two factors u_1 and u_2.

Since

$$\begin{aligned} z &\equiv \log S_T = v_A - v_N \\ &= -(c_{N1} - c_{A1})u_1 - (c_{N2} - c_{A2})u_2 + \delta \end{aligned} \tag{28}$$

and

$$\begin{aligned} u_1 &= -\frac{c_{N2} - c_{A2}}{c_{N1} - c_{A1}} u_2 - \frac{1}{c_{N1} - c_{A1}} z + \frac{1}{c_{N1} - c_{A1}} \delta \\ &= -\frac{c_{N2} - c_{A2}}{c_{N1} - c_{A1}} u_2 - \frac{1}{c_{N1} - c_{A1}} (z - \delta), \end{aligned} \tag{29}$$

the log marginal utility of numerarie v_N can be written as follows:

$$\begin{aligned} v_N &= c_{N1}\left(-\frac{c_{N2} - c_{A2}}{c_{N1} - c_{A1}} u_2 - \frac{1}{c_{N1} - c_{A1}} (z - \delta)\right) + c_{N2}u_2 \\ &= \left(\frac{c_{N2}c_{N1} - c_{N2}c_{A1} - c_{N1}c_{N2} + c_{N1}c_{A2}}{c_{N1} - c_{A1}} u_2\right) - \frac{c_{N1}}{c_{N1} - c_{A1}} (z - \delta) \\ &= -\frac{c_{N2}c_{A1} - c_{N1}c_{A2}}{c_{N1} - c_{A1}} u_2 - \frac{c_{N1}}{c_{N1} - c_{A1}} (z - \delta) \end{aligned} \tag{30}$$

By substituting (27), (28) and (30) into (12), we have

$$\begin{aligned}
q(z) &= \frac{1}{EU_N}\int_{-\infty}^{\infty} e^{v_N} h(v_N, v_N+z)dv_N \\
&= \frac{1}{EU_N}\int_{-\infty}^{\infty} e^{-\frac{c_{N2}c_{A1}-c_{N1}c_{A2}}{c_{N1}-c_{A1}}u_2-\frac{c_{N1}}{c_{N1}-c_{A1}}(z-\delta)}\,\cdot \\
&\quad \frac{1}{|c_{N1}c_{A2}-c_{N2}c_{A1}|}\, f_{U_1U2}(u_1,u_2)\left|\frac{|c_{N1}c_{A2}-c_{N2}c_{A1}|}{c_{N1}-c_{A1}}\right| du_2 \\
&= \frac{1}{|c_{N1}-c_{A1}|}\frac{1}{EU_N}\int_{-\infty}^{\infty} e^{-\frac{c_{N2}c_{A1}-c_{N1}c_{A2}}{c_{N1}-c_{A1}}u_2-\frac{c_{N1}}{c_{N1}-c_{A1}}(z-\delta)}\,\cdot \\
&\quad f_{U_1U_2}\left(-\frac{z-\delta+(c_{N2}-c_{A2})u_2}{c_{N1}-c_{A1}}, u_2\right)du_2. \qquad (31)
\end{aligned}$$

By using (29), the joint distribution of u_1 and u_2 can be expressed as:

$$\begin{aligned}
& f_{U_1U_2}\left(-\frac{z-\delta+(c_{N2}-c_{A2})u_2}{c_{N1}-c_{A1}}, u_2\right) \\
=\ & s(u_2;\alpha,-1,1,0)\, s\left(-\frac{z-\delta+(c_{N2}-c_{A2})u_2}{c_{N1}-c_{A1}};\alpha,-1,1,0\right) \\
=\ & |c_{N1}-c_{A1}| s(u_2;\alpha,-1,1,0)\, s(z;\alpha,\mathrm{sgn}(c_{N1}-c_{A1}),|c_{N1}-c_{A1}|,\delta+(c_{N2}-c_{A2})u_2)
\end{aligned} \qquad (32)$$

Plugging (32) into (31), we finally have the stable RNM pdf:

$$\begin{aligned}
q(z) &= \frac{1}{EU_N}\int_{-\infty}^{\infty} e^{-\frac{c_{N2}c_{A1}-c_{N1}c_{A2}}{c_{N1}-c_{A1}}u_2-\frac{c_{N1}}{c_{N1}-c_{A1}}(z-\delta)}\cdot s(u_2;\alpha,-1,1,0) \\
&\quad s(z;\alpha,\mathrm{sgn}(c_{N1}-c_{A1}),|c_{N1}-c_{A1}|,\delta-(c_{N2}-c_{A2})u_2)du_2. \qquad (33)
\end{aligned}$$

B Proof of (19) and Derivation of (20)

The generalized two-factor log-stable RNM is a convolution of two exponentially tilted stable distributions:

$$q(z) \quad = \quad \mathop{\ast}_{j=1}^{2} ts_{\mathrm{sgn}(c_{Nj}-c_{Aj})}\left(z_j;\alpha,|c_{Nj}-c_{Aj}|,\delta_j,\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|\right)$$

Proof.

From (33), the CF of the generalized two-factor log-stable RNM is written as:

$$\begin{aligned} cf_q(t) \quad &= \quad \int_{-\infty}^{\infty} e^{itz} q(z)dz \\ &= \quad \frac{1}{EU_N}\int_{-\infty}^{\infty} e^{-\frac{c_{N2}c_{A1}-c_{N1}c_{A2}}{c_{N1}-c_{A1}}u_2+\frac{c_{N1}}{c_{N1}-c_{A1}}\delta} s(u_2;\alpha,-1,1,0)\cdot \\ &\quad\ \int_{-\infty}^{\infty} e^{-\left(\frac{c_{N1}}{c_{N1}-c_{A1}}-it\right)z} s(z;\alpha,\mathrm{sgn}(c_{N1}-c_{A1}),|c_{N1}-c_{A1}|,\delta-(c_{N2}-c_{A2})u_2)dz\,du_2. \end{aligned} \tag{34}$$

Using the properties of sign function, we have:

$$\begin{aligned} \text{I.} \qquad & -\left(\frac{c_{N1}}{c_{N1}-c_{A1}}-it\right)z \\ & = \ -\,\mathrm{sgn}(c_{N1}-c_{A1})\left(\left|\frac{c_{N1}}{c_{N1}-c_{A1}}\right|-\mathrm{sgn}(c_{N1}-c_{A1})it\right)z \end{aligned} \tag{35}$$

$$\begin{aligned} \text{II.} \qquad & \left(-\frac{c_{N2}c_{A1}-c_{N1}c_{A2}}{c_{N1}-c_{A1}}+\mathrm{sgn}(c_{N1}-c_{A1})(c_{N2}-c_{A2})\left(\left|\frac{c_{N1}}{c_{N1}-c_{A1}}\right|-\mathrm{sgn}(c_{N1}-c_{A1})it\right)\right)u_2 \\ & = \ (c_{N2}-(c_{N2}-c_{A2}))\,u_2 \end{aligned} \tag{36}$$

$$\begin{aligned} \text{III.} \qquad & -\mathrm{sgn}(c_{N1}-c_{A1})\left(\left|\frac{c_{N1}}{c_{N1}-c_{A1}}\right|-\mathrm{sgn}(c_{N1}-c_{A1})it\right)\delta \\ & -\left(\left|\frac{c_{N1}}{c_{N1}-c_{A1}}\right|-\mathrm{sgn}(c_{N1}-c_{A1})it\right)^{\alpha}|c_{N1}-c_{A1}|^{\alpha}\sec\left(\frac{\pi\alpha}{2}\right)+\frac{c_{N1}}{c_{N1}-c_{A1}}\delta \\ & = \delta it-\left(\left|\frac{c_{N1}}{c_{N1}-c_{A1}}\right|-\mathrm{sgn}(c_{N1}-c_{A1})it\right)^{\alpha}|c_{N1}-c_{A1}|^{\alpha}\sec\left(\frac{\pi\alpha}{2}\right) \end{aligned} \tag{37}$$

By using (35), the inner integration term in (34) may be written as:

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-\mathrm{sgn}(c_{N1}-c_{A1})\left(\left|\frac{c_{N1}}{c_{N1}-c_{A1}}\right|-\mathrm{sgn}(c_{N1}-c_{A1})it\right)z}\,\cdot \\
&\quad s(z;\alpha,\mathrm{sgn}(c_{N1}-c_{A1}),|c_{N1}-c_{A1}|,\delta-(c_{N2}-c_{A2})u_2)dz \\
&= \exp\Bigg[-\mathrm{sgn}(c_{N1}-c_{A1})\left(\left|\frac{c_{N1}}{c_{N1}-c_{A1}}\right|-\mathrm{sgn}(c_{N1}-c_{A1})it\right)(\delta-(c_{N2}-c_{A2})u_2) \\
&\qquad -\left(\left|\frac{c_{N1}}{c_{N1}-c_{A1}}\right|-\mathrm{sgn}(c_{N1}-c_{A1})it\right)^{\alpha}|c_{N1}-c_{A1}|^{\alpha}\sec\left(\frac{\pi\alpha}{2}\right)\Bigg] \qquad (38)
\end{aligned}$$

By using (36), (37), and (38), the CF of the RNM may be expressed as:

$$\begin{aligned}
cf_q(t) &= \frac{1}{EU_N}e^{\delta it-\left(\left|\frac{c_{N1}}{c_{N1}-c_{A1}}\right|-\mathrm{sgn}(c_{N1}-c_{A1})it\right)^{\alpha}|c_{N1}-c_{A1}|^{\alpha}\sec\left(\frac{\pi\alpha}{2}\right)}\,\cdot \\
&\qquad \int_{-\infty}^{\infty} e^{c_{N2}-(c_{N2}-(c_{N2}-c_{A2})it)u_2}\, s(u_2;\alpha,-1,1,0)\,du_2. \qquad (39)
\end{aligned}$$

Let

$$w = (c_{N2}-c_{A2})u_2, \qquad (40)$$

so that

$$\left|\frac{du_2}{dw}\right| = \frac{1}{|c_{N2}-c_{A2}|}. \qquad (41)$$

With (40) and (41), the integral term of (39) is written as:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} e^{c_{N2}-(c_{N2}-(c_{N2}-c_{A2})it)u_2}\, s(u_2;\alpha,-1,1,0)\, du_2 \\
= &\int_{-\infty}^{\infty} e^{\left(\frac{c_{N2}}{c_{N2}-(c_{N2}-c_{A2}}-it\right)w}\, s\left(\frac{1}{c_{N2}-c_{A2}}w;\alpha,-1,1,0\right)\frac{1}{|c_{N2}-c_{A2}|}\, dw \\
= &\exp\left[-\left(\left|\frac{c_{N2}}{c_{N2}-c_{A2}}\right| - \mathrm{sgn}(c_{N2}-c_{A2})it\right)^{\alpha} |c_{N2}-c_{A2}|^{\alpha} \sec\left(\frac{\pi\alpha}{2}\right)\right].
\end{aligned} \tag{42}
$$

By Property 2 of stable distributions, the expected marginal utility of numeraire is

$$
\begin{aligned}
EU_N &= E\, e^{v_N} = \exp(\delta_2 - c_N^{\alpha} \sec\left(\frac{\pi\alpha}{2}\right)) \\
&= \exp\left(-(c_{N1}^{\alpha} + c_{N2}^{\alpha}) \sec\left(\frac{\pi\alpha}{2}\right)\right).
\end{aligned} \tag{43}
$$

By substituting (42), and (43) into (39), the CF of the RNM is taken to be

$$
\begin{aligned}
cf_q(t) = \exp\Bigg\{ & i\delta t + |c_{N1}-c_{A1}|^{\alpha} \sec\left(\frac{\pi\alpha}{2}\right)\left[\left|\frac{c_{N1}}{c_{N1}-c_{A1}}\right|^{\alpha} - \left(\left|\frac{c_{N1}}{c_{N1}-c_{A1}}\right| - \mathrm{sgn}(c_{N1}-c_{A1})it\right)^{\alpha}\right] \\
&+ |c_{N2}-c_{A2}|^{\alpha} \sec\left(\frac{\pi\alpha}{2}\right)\left[\left|\frac{c_{N2}}{c_{N2}-c_{A2}}\right|^{\alpha} - \left(\left|\frac{c_{N2}}{c_{N2}-c_{A2}}\right| - \mathrm{sgn}(c_{N2}-c_{A2})it\right)^{\alpha}\right]\Bigg\}.
\end{aligned}
$$

Finally, the log CF of the RNM is

$$
\begin{aligned}
\log cf_q(t) = \; & i\delta t + |c_{N1}-c_{A1}|^{\alpha} \sec\left(\frac{\pi\alpha}{2}\right)\left[\left|\frac{c_{N1}}{c_{N1}-c_{A1}}\right|^{\alpha} - \left(\left|\frac{c_{N1}}{c_{N1}-c_{A1}}\right| - \mathrm{sgn}(c_{N1}-c_{A1})it\right)^{\alpha}\right] \\
&+ |c_{N2}-c_{A2}|^{\alpha} \sec\left(\frac{\pi\alpha}{2}\right)\left[\left|\frac{c_{N2}}{c_{N2}-c_{A2}}\right|^{\alpha} - \left(\left|\frac{c_{N2}}{c_{N2}-c_{A2}}\right| - \mathrm{sgn}(c_{N2}-c_{A2})it\right)^{\alpha}\right].
\end{aligned} \tag{44}
$$

An exponentially tilted stable distribution

$$
\begin{aligned}
& ts_{\operatorname{sgn}(c_{Nj}-c_{Aj})}\left(z;\alpha,|c_{Nj}-c_{Aj}|,\delta_j,\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|\right) \\
&= k\, e^{-\operatorname{sgn}(c_{Nj}-c_{Aj})\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right| z} s\left(x;\alpha,\operatorname{sgn}(c_{Nj}-c_{Aj}),|c_{Nj}-c_{Aj}|,\delta_j\right)
\end{aligned}
$$

has the CF:

$$
\begin{aligned}
cf_{ts,j}(t) &= k\int_{-\infty}^{\infty} e^{-\left(\operatorname{sgn}(c_{Nj}-c_{Aj})\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|-it\right)z} s\left(z;\alpha,\operatorname{sgn}(c_{Nj}-c_{Aj}),|c_{Nj}-c_{Aj}|,\delta_j\right)dz \\
&= k\, e^{-\left(\operatorname{sgn}(c_{Nj}-c_{Aj})\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|-it\right)\delta_j-\left(\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|-\operatorname{sgn}(c_{Nj}-c_{Aj})it\right)^{\alpha}|c_{Nj}-c_{Aj}|^{\alpha}\sec\left(\frac{\pi\alpha}{2}\right)}.
\end{aligned}
$$

Since

$$
cf_{ts,j}(0) = k\,\exp\left[-\operatorname{sgn}(c_{Nj}-c_{Aj})\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|\delta_j-\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|^{\alpha}|c_{Nj}-c_{Aj}|^{\alpha}\sec\left(\frac{\pi\alpha}{2}\right)\right]\equiv 1,
$$

we must have

$$
k = \exp\left[\operatorname{sgn}(c_{Nj}-c_{Aj})\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|\delta_j+\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|^{\alpha}|c_{Nj}-c_{Aj}|^{\alpha}\sec\left(\frac{\pi\alpha}{2}\right)\right],
$$

hence

$$
cf_{ts,j}(t) = \exp\left\{i\delta_j t+|c_{Nj}-c_{Aj}|^{\alpha}\sec\left(\frac{\pi\alpha}{2}\right)\left[\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|^{\alpha}-\left(\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|-\operatorname{sgn}(c_{Nj}-c_{Aj})it\right)^{\alpha}\right]\right\}.
$$

Finally, the log CF of the tilted stable distribution is

$$
\log cf_{ts,j}(t) = i\delta_j t+|c_{Nj}-c_{Aj}|^{\alpha}\sec\left(\frac{\pi\alpha}{2}\right)\left[\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|^{\alpha}-\left(\left|\frac{c_{Nj}}{c_{Nj}-c_{Aj}}\right|-\operatorname{sgn}(c_{Nj}-c_{Aj})it\right)^{\alpha}\right]. \tag{45}
$$

Combining (44) and (45), we have

$$\log cf_q(t) \quad = \quad \sum_{j=1}^{2} \log cf_{ts,j}(t).$$

Since the log CF of the convolution of two densities is the sum of their respective log CFs, the generalized RNM $q(z)$ is such a convolution of two exponentially tilted stable densities. $\square$

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