EXTENDED NEYMAN SMOOTH GOODNESS-OF-FIT TESTS, APPLIED TO COMPETING HEAVY-TAILED DISTRIBUTIONS

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ABSTRACT

A simplified version of the Neyman (1937) "Smooth" goodness-of-fit test is extended to account for the presence of estimated model parameters, thereby removing overfitting bias. Using a Lagrange Multiplier approach rather than the Likelihood Ratio statistic proposed by Neyman greatly simplifies the calculations. Polynomials, splines, and the step function of Pearson's test are compared as alternative perturbations to the theoretical uniform distribution. The extended tests have negligible size distortion and more power than standard tests. The tests are applied to competing symmetric leptokurtic distributions with US stock return data. These are generally rejected, primarily because of the presence of skewness.

1. Introduction

A simplified version of the Neyman (1937) "Smooth" goodness-of-fit test (GFT) is extended to adjust for the presence of estimated model parameters. The effect is to reduce size distortion and to increase the power of the test. As is well known (e.g. Bai 2003), standard tests that do not account for the estimation of model parameters tend to under-reject the distribution assumed by the null hypothesis.

The heavy tails of most financial return series often lead to an easy rejection of the Gaussian distribution. A parametric distribution that is consistent with the data would allow the mean to be estimated more efficiently than would a nonparametric procedure. Furthermore, parametric distributions allow out-of-sample extreme tail probabilities to be estimated. These tail probabilities are important for the pricing of options and contingent claims, for Value at Risk calculations, and for non-financial applications such as the risk of floods and other extreme events.

When heavy tails are present, Mandelbrot (1963), Fama (1965), Samorodnitsky and Taqqu (1994), and McCulloch (1996) have suggested the use of stable distributions. Blattberg and Gonedes (1974), Hagerman (1978), Perry (1983), and Boothe and Glassman (1987) propose the Student *t* distributions. Nelson (1991) investigates the generalized error distribution (GED), while Praetz (1972) and Clark (1973) implement a mixture of normal distributions. This paper is aimed at developing an appropriate GFT that can work well with all the above distributions. Although this paper assumes identically and independently distributed (i.i.d.) errors, Percy (2006) addresses other models including ARCH and GARCH models of volatility clustering.

Section 2 introduces a simplified Lagrange Multiplier[1] (LM) version of the Neyman Smooth GFT. Section 3 extends this test to accommodate estimated parameters. Section 4 illustrates the extended test for the proposed heavy-tailed distributions with stock return data. Section 5 reports on the power of the test to discriminate between the candidate distributions, and Section 6 concludes. This paper draws heavily upon Percy (2006), which provides more theory and details than are available herein.

2. A simplified Neyman Smooth GFT with known parameters

We can develop an LM test for any distribution with known or unknown parameters by first constructing an LM test for the uniform distribution.

2.1. An LM test for uniformity

For m perturbation functions $\{\varphi_j(z),\ j=1,\ldots,m\}$ that are linearly independent, bounded, and integrate to zero on the unit interval and a vector of coefficients $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_m)' \in \Re^m$, define

$$\mathrm{g}(z;\boldsymbol{\alpha}) := 1+\sum_{j=1}^{m}\alpha_j\,\varphi_j(z),\ \ z\in[0,1].$$

For $\boldsymbol{\alpha}$ in a sufficiently small neighborhood N of the origin in $\Re^m$, $\mathrm{g}(\cdot;\boldsymbol{\alpha})$ is a probability density function, with cumulative distribution function denoted by $\mathrm{G}(\cdot;\boldsymbol{\alpha})$. This distribution nests the standard uniform distribution U(0,1) when $\boldsymbol{\alpha}=\mathbf{0}$.

Consider a random sample $\mathbf{u}=(u_1,\ldots,u_n)' \in \Re^n$. We wish to test H_0: u_i ~ i.i.d. U(0,1) vs. H_1: u_i ~ i.i.d. $\mathrm{G}(\cdot;\boldsymbol{\alpha})$ with $\boldsymbol{\alpha}\neq\mathbf{0}$. The log likelihood function of interest is

[1] Such tests are also called “efficient score” tests, “score” tests, or sometimes “Rao score” tests in honor of C.R. Rao, who first proposed this type of test.

$$\log \mathrm{L}(\boldsymbol{\alpha};\mathbf{u}) = \sum_{i=1}^{n} \log\left(1+\sum_{j=1}^{m}\alpha_j\,\varphi_j(u_i)\right),$$

and the m×1 score vector of first derivatives with respect to the α_j's is

$$\mathbf{s}(\boldsymbol{\alpha};\mathbf{u}) := \left(\partial \log \mathrm{L}/\partial\alpha_j\right)_{1\le j\le m} = \left(\sum_{i=1}^{n}\frac{\varphi_j(u_i)}{1+\sum_{h=1}^{m}\alpha_h\,\varphi_h(u_i)}\right)_{1\le j\le m}.$$

When evaluated at the null, this simplifies to

$$\mathbf{s}(\mathbf{0};\mathbf{u}) = \left(\sum_{i=1}^{n}\varphi_j(u_i)\right)_{1\le j\le m}. \tag{1}$$

Letting $Z \sim \mathrm{G}(\cdot\,;\boldsymbol{\alpha})$, the Fisher information matrix with respect to the perturbation parameters is

$\mathbf{I}(\boldsymbol{\alpha}) = \left(I_{jj'}(\boldsymbol{\alpha})\right)_{1\le j,j'\le m}$,

where

$$I_{jj'}(\boldsymbol{\alpha}) := \mathrm{E}\left(\frac{\partial \log \mathrm{g}(Z;\boldsymbol{\alpha})}{\partial\alpha_j}\frac{\partial \log \mathrm{g}(Z;\boldsymbol{\alpha})}{\partial\alpha_{j'}}\right) = \int_0^1 \frac{\varphi_j(z)\varphi_{j'}(z)}{1+\sum_{h=1}^{m}\alpha_h\,\varphi_h(z)}\,dz.$$

At the null hypothesis, this simplifies to

$$\mathbf{I}(\mathbf{0}) = \left(\int_0^1\varphi_j(z)\varphi_{j'}(z)\,dz\right)_{1\le j,j'\le m}. \tag{2}$$

The LM statistic is then

$$LM = \mathbf{s}'(\mathbf{0};\mathbf{u})\mathbf{I}^{-1}(\mathbf{0})\mathbf{s}(\mathbf{0};\mathbf{u})/n. \tag{3}$$

Since the null hypothesis is a single internal point in the set *N*, the asymptotic distribution of this statistic is χ^2 with *m* degrees of freedom under the null.

2.2. Functional form and basis

Although any set of linearly independent bounded functions integrating to 0 may be used in theory, there are numerical considerations in choosing a basis so that tractable results can be obtained. We investigated polynomials, orthogonal polynomials, splines, and B-splines (Judd 1999, Percy 2006).

For small values of *m*, the demeaned standard polynomials $\varphi_j(z) = z^j - 1/(j+1)$ are an obvious choice. For larger values of *m*, however, these standard polynomials are ill-conditioned for the matrix inversion required to compute the Fisher information matrix. Neyman-Pearson orthogonal polynomials are mathematically equivalent but computationally better conditioned in this respect. However, rounding errors still may arise in the evaluation of the score as there is a large disparity in the orders of magnitude between coefficients, even for midsize values of *m*:

> Experience with polynomials derived by truncating [Taylor] series, [especially in their use with estimating transcendental functions,] may mislead one into thinking that the use of high order polynomials does not lead to computational difficulties. However it must be appreciated that truncated [Taylor] series are not typical of polynomials in general. [Truncated Taylor series] have the special feature that the terms decrease rapidly in size for values of *x* in the range for which they are appropriate.
>
> A tendency to underestimate the difficulties involved in working with general polynomials is perhaps a consequence of one's experience in classical analysis. There it is natural to regard a polynomial as a very desirable function since it is bounded in any finite region and has derivatives of all orders. In numerical work, however, polynomials having coefficients which are more or less arbitrary are tiresome to deal with by entirely automatic procedures. (Wilkinson 1963, p. 38)

Polynomials have the further drawback that fitting one region better may require fitting other distant regions more poorly.

Cubic splines, which are piecewise cubic functions with discontinuities in the third derivative at selected knotpoints, are more pliable and better able to fit a particular interval without affecting more distant intervals as much. Simple spline basis functions can create similar numerical problems with the inversion of the information matrix but the so-called B-spline basis solves this issue, while generating the same space of functions. Thus, for much of the analysis, we used B-splines as a basis but also investigated orthogonal polynomials with a moderate number of parameters. Splines have some numerical properties that are more desirable than polynomials, while the tradeoff in the other properties is not severe.

The classic Pearson GFT arises as a special case of the LM test when each perturbation function is a step function integrating to zero, with discontinuities at *m* selected points. Piecewise linear and quadratic spline perturbation functions are also considered below.

The proposed LM GFT is closely related to the Neyman (1937) Smooth test, so named because the alternative distributions vary "smoothly" away from the null hypothesized distribution rather than with discontinuities as in the Pearson test. However, Neyman's test, which he called the Ψ^2 test (as contrasted with Pearson's χ^2 test), was based on a likelihood ratio statistic rather than an LM statistic and hence required estimating the model both under the null and the perturbed alternative. Furthermore, in order to prevent negative densities, his perturbations were exponentiated polynomials, numerically constrained to integrate to zero.

As noted already by Rayner and Best (1989), the LM simplification of the Neyman test employed here only needs to be estimated under the null, a much simpler calculation. Furthermore, since the alternative only needs to be a proper density in a neighborhood of the null hypothesis, polynomial or spline perturbations, which are easily constrained to integrate to zero, may be used directly, without exponentiation.

2.3. LM test for a completely specified distribution

The LM test for any continuous distribution with known parameters is essentially the same as that for the uniform distribution developed above.

Consider a random sample $\mathbf{y} = (y_1, \ldots, y_n)' \in \Re^n$. One would like to test: H_0: y_i ~ i.i.d. F(·), where F(·) is a completely specified continuous distribution with density f(·) = F′(·), vs. H_1: y_i ~ i.i.d. $\mathrm{G}(\mathrm{F}(\cdot)\ ;\boldsymbol{\alpha})$ with $\boldsymbol{\alpha} \neq \mathbf{0}$. Define $u_i = \mathrm{F}(y_i)$. Then, as is well known, y_i ~ F(·) if and only if u_i ~ U(0,1). Under the alternative, the density is $\mathrm{g}(\mathrm{F}(\cdot);\boldsymbol{\alpha})\mathrm{f}(\cdot)$. The log-likelihood function is then

$$\log \mathrm{L}(\boldsymbol{\alpha};\mathbf{y}) = \sum_{i-1}^{n} \log \mathrm{g}(u_i;\boldsymbol{\alpha}) + \sum_{i-1}^{n} \log \mathrm{f}(y_i).$$

Since the second summation does not depend on $\boldsymbol{\alpha}$, the derivatives necessary to calculate the LM statistic are identical to those for the test for the uniform distribution. Thus one can simply use the transformed observations u_i with the test for the uniform distribution (3). All tables and critical values that are suitable for the test of uniformity are also suitable for a general distribution.

2.4. Finite sample properties with a completely specified distribution

As noted, the LM statistic has a limiting asymptotic distribution that is chi-squared with degrees of freedom equal to *m*, the number of perturbation parameters. Simulations reported in Percy (2006) indicate that for sample size $n \geq 30$, $m \geq 5$, and test size = 0.05, the convergence to the limiting distribution is quite rapid. At the indicated values of *m*, *n*, and test size, the 95th percentile of the simulated distributions was generally within sampling error of the 95th percentile of a chi-squared random variable.

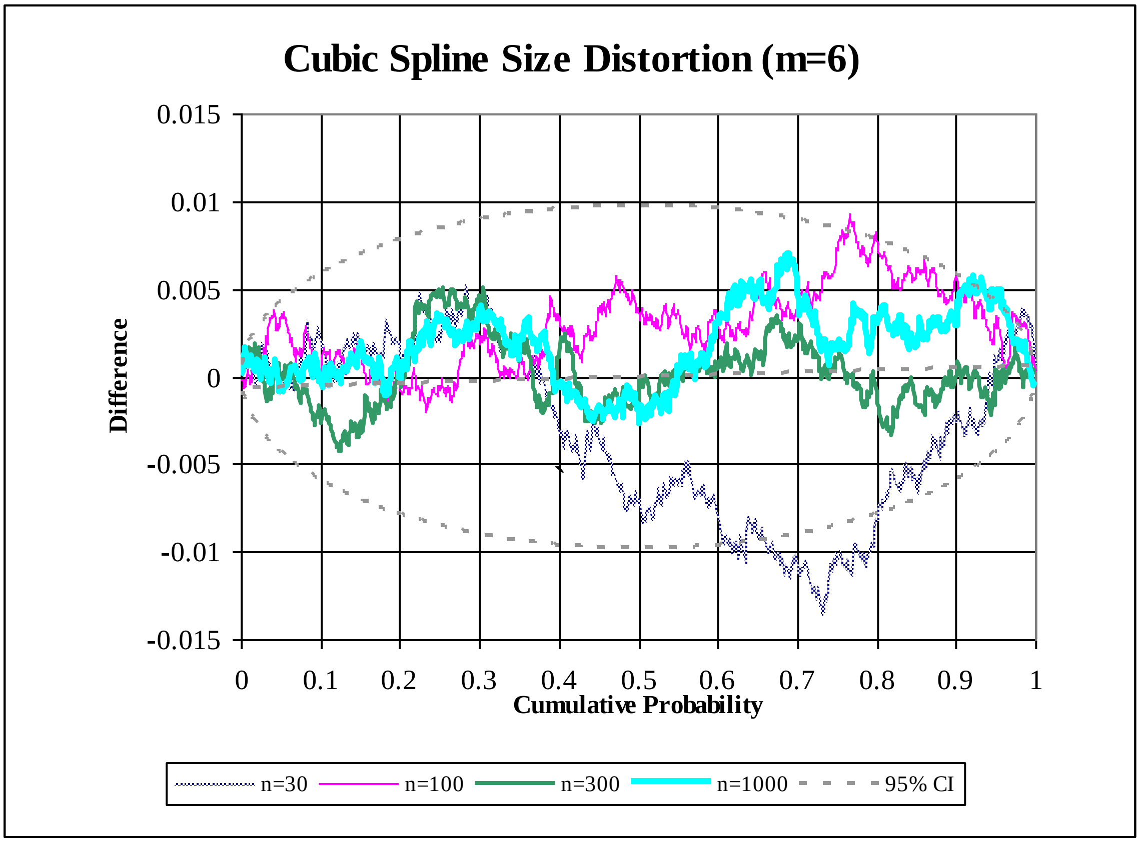

Figure 1. Size distortion of cubic spline test (m = 6).

Figure 1 shows results of the simulations for 9999 replications with a cubic spline perturbation and $m = 6$. The solid lines indicate the difference between the theoretical chi-squared distribution and the empirical distribution of the simulations. The dashed lines indicate 95% confidence intervals for the quantiles of an empirical CDF of random draws from the chi-squared distribution. Except for very small samples with $n = 30$, the correspondence is as close as could be expected at all quantiles. See Percy (2006) for illustrations of size distortion for other parameter sizes and basis functions.

3. The LM test with estimated model parameters

We now extend the LM test to the typical situation in which the values of the model parameters are not known and so must be estimated. Our null hypothesis is now H_0: y_i ~ i.i.d. $\mathrm{F}(\cdot;\boldsymbol{\theta})$, where F is a continuous distribution with density $\mathrm{f}(\cdot;\boldsymbol{\theta})$ that depends on the k×1 vector of parameters $\boldsymbol{\theta}$. This vector lies in the interior of a convex set Θ, but is otherwise unknown. The Fisher information matrix is assumed to be positive definite and finite for all interior points of this set. Let the maximum likelihood (ML) estimate of $\boldsymbol{\theta}$ under the null hypothesis be $\hat{\boldsymbol{\theta}}$ and set $\hat{\mathbf{u}} = (\hat{u}_1,\ldots,\hat{u}_n)'$ where $\hat{u}_i = \mathrm{F}(y_i;\hat{\boldsymbol{\theta}})$.

For most of the competing heavy-tailed distributions considered here, namely the symmetric stable, Student *t*, and generalized error distributions, the unknowns consist of a location, scale, and shape parameter, so that $k = 3$. For a mixture of two Gaussian distributions with a common mean, two variances, and an unknown mixing probability, $k = 4$. More generally, the location of each y_i could be determined by a possibly non-linear function $\mathrm{h}(\boldsymbol{x}_i;\boldsymbol{\beta})$, where $\boldsymbol{x}_i$ is a vector of known constants or of exogenous random variables and $\boldsymbol{\beta}$ is a vector of unknown coefficients to be estimated by ML. However, the present paper illustrates the extended test only in cases involving a common location parameter.

Our alternative hypothesis is now H_1: y_i ~ i.i.d. $\mathrm{G}(\mathrm{F}(\cdot;\boldsymbol{\theta});\boldsymbol{\alpha})$, with $\boldsymbol{\alpha} \neq \mathbf{0}$ and $\mathrm{G}(\cdot;\boldsymbol{\alpha})$ defined as above. The log likelihood is now

$$\log \mathrm{L}(\boldsymbol{\theta},\boldsymbol{\alpha};\mathbf{y}) = \sum_{i-1}^{n} \log \mathrm{g}(\mathrm{F}(y_i;\boldsymbol{\theta});\boldsymbol{\alpha}) + \sum_{i-1}^{n} \log \mathrm{f}(y_i;\boldsymbol{\theta}).$$

The full score vector

$$\mathbf{s}(\boldsymbol{\theta},\boldsymbol{\alpha};\mathbf{y}) = \begin{pmatrix} \mathbf{s}_{\boldsymbol{\theta}}(\boldsymbol{\theta},\boldsymbol{\alpha};\mathbf{y}) \\ \mathbf{s}_{\boldsymbol{\alpha}}(\boldsymbol{\theta},\boldsymbol{\alpha};\mathbf{y}) \end{pmatrix} = \begin{pmatrix} (\partial \log \mathrm{L}/\partial\theta_h)_{1\le h\le k} \\ (\partial \log \mathrm{L}/\partial\alpha_j)_{1\le j\le m} \end{pmatrix}$$

is now $(k+m)\times 1$, but $\mathbf{s}_{\boldsymbol{\theta}}\left(\hat{\boldsymbol{\theta}},\mathbf{0};\mathbf{y}\right)=\mathbf{0}$ by the ML first order conditions, while $\mathbf{s}_{\boldsymbol{\alpha}}\left(\hat{\boldsymbol{\theta}},\mathbf{0};\mathbf{y}\right)=\mathbf{s}(\mathbf{0};\hat{\mathbf{u}})$ as in (1), since $g'(\cdot;\boldsymbol{\alpha})=0$ when $\boldsymbol{\alpha}=\mathbf{0}$.

The full Fisher information matrix

$$\mathbf{I}(\boldsymbol{\theta},\boldsymbol{\alpha})=\begin{pmatrix}\mathbf{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\theta},\boldsymbol{\alpha}) & \mathbf{I}_{\boldsymbol{\theta\alpha}}(\boldsymbol{\theta},\boldsymbol{\alpha})\\ \mathbf{I}'_{\boldsymbol{\theta\alpha}}(\boldsymbol{\theta},\boldsymbol{\alpha}) & \mathbf{I}_{\boldsymbol{\alpha\alpha}}(\boldsymbol{\theta},\boldsymbol{\alpha})\end{pmatrix}$$

is now of dimension $(k+m)\times(k+m)$, with

$$\mathbf{I}_{\boldsymbol{\theta\theta}}\left(\hat{\boldsymbol{\theta}},\boldsymbol{\alpha}\right)=\left(I_{\boldsymbol{\theta\theta}}\left(\hat{\boldsymbol{\theta}},\boldsymbol{\alpha}\right)_{hh'}\right)_{1\le h,h'\le k},$$

$$\mathbf{I}_{\boldsymbol{\theta\alpha}}\left(\hat{\boldsymbol{\theta}},\boldsymbol{\alpha}\right)=\left(I_{\boldsymbol{\theta\alpha}}\left(\hat{\boldsymbol{\theta}},\boldsymbol{\alpha}\right)_{hj}\right)_{1\le h\le k,1\le j\le m}$$

$$\mathbf{I}_{\boldsymbol{\alpha\alpha}}\left(\hat{\boldsymbol{\theta}},\boldsymbol{\alpha}\right)=\left(I_{\boldsymbol{\alpha\alpha}}\left(\hat{\boldsymbol{\theta}},\boldsymbol{\alpha}\right)_{jj'}\right)_{1\le j,j'\le m}.$$

Letting $Z\sim \mathrm{G}(\mathrm{F}(\cdot\,;\boldsymbol{\theta});\boldsymbol{\alpha})$, we have

$$\begin{aligned}I_{\boldsymbol{\theta\theta}}\left(\hat{\boldsymbol{\theta}},\mathbf{0}\right)_{hh'}&:=\left[\mathrm{E}\,\frac{\partial\log\left(\mathrm{g}(\mathrm{F}(Z;\boldsymbol{\theta});\boldsymbol{\alpha})\mathrm{f}(Z;\boldsymbol{\theta})\right)}{\partial\theta_h}\frac{\partial\log\left(\mathrm{g}(\mathrm{F}(Z;\boldsymbol{\theta});\boldsymbol{\alpha})\mathrm{f}(Z;\boldsymbol{\theta})\right)}{\partial\theta_{h'}}\right]_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}},\boldsymbol{\alpha}=\mathbf{0}}\\&=\left[\int_{-\infty}^{\infty}\frac{\partial\,\mathrm{f}(z;\boldsymbol{\theta})}{\partial\theta_h}\frac{\partial\,\mathrm{f}(z;\boldsymbol{\theta})}{\partial\theta_{h'}}/\,\mathrm{f}(z;\boldsymbol{\theta})dz\right]_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}},\end{aligned}$$

$$\begin{aligned}I_{\boldsymbol{\theta\alpha}}\left(\hat{\boldsymbol{\theta}},\mathbf{0}\right)_{hj}&:=\left[\mathrm{E}\,\frac{\partial\log\left(\mathrm{g}(\mathrm{F}(Z;\boldsymbol{\theta});\boldsymbol{\alpha})\mathrm{f}(Z;\boldsymbol{\theta})\right)}{\partial\theta_h}\frac{\partial\log\left(\mathrm{g}(\mathrm{F}(Z;\boldsymbol{\theta});\boldsymbol{\alpha})\mathrm{f}(Z;\boldsymbol{\theta})\right)}{\partial\alpha_j}\right]_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}},\boldsymbol{\alpha}=\mathbf{0}}\\&=\left[\int_{-\infty}^{\infty}\frac{\partial\,\mathrm{f}(z;\boldsymbol{\theta})}{\partial\theta_h}\varphi_j(\mathrm{F}(z;\boldsymbol{\theta}))dz\right]_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}},\end{aligned}$$

$$\begin{aligned}I_{\boldsymbol{\alpha\alpha}}\left(\hat{\boldsymbol{\theta}},\mathbf{0}\right)_{jj'}&:=\left[\mathrm{E}\,\frac{\partial\log\left(\mathrm{g}(\mathrm{F}(Z;\boldsymbol{\theta});\boldsymbol{\alpha})\mathrm{f}(Z;\boldsymbol{\theta})\right)}{\partial\alpha_j}\frac{\partial\log\left(\mathrm{g}(\mathrm{F}(Z;\boldsymbol{\theta});\boldsymbol{\alpha})\mathrm{f}(Z;\boldsymbol{\theta})\right)}{\partial\alpha_{j'}}\right]_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}},\boldsymbol{\alpha}=\mathbf{0}}\\&=\int_0^1\varphi_j(z)\varphi_{j'}(z)dz,\end{aligned}$$

so that $\mathbf{I}_{\boldsymbol{\alpha\alpha}}\left(\hat{\boldsymbol{\theta}},\mathbf{0}\right)=\mathbf{I}(\mathbf{0})$ as in (2).

Using the partitioned matrix inverse formula, the extended LM test statistic becomes

$$\begin{aligned} LM &= \mathbf{s}'\left(\hat{\boldsymbol{\theta}},\mathbf{0};\mathbf{y}\right)\mathbf{I}^{-1}\left(\hat{\boldsymbol{\theta}},\mathbf{0}\right)\mathbf{s}\left(\hat{\boldsymbol{\theta}},\mathbf{0};\mathbf{y}\right)/\, n \\ &= \mathbf{s}'\left(\mathbf{0},\hat{\mathbf{u}}\right)\left(\mathbf{I}(\mathbf{0}) - \mathbf{I}'_{\boldsymbol{\theta\alpha}}\left(\hat{\boldsymbol{\theta}},\mathbf{0}\right)\mathbf{I}^{-1}_{\boldsymbol{\theta\theta}}\left(\hat{\boldsymbol{\theta}},\mathbf{0}\right)\mathbf{I}_{\boldsymbol{\theta\alpha}}\left(\hat{\boldsymbol{\theta}},\mathbf{0}\right)\right)^{-1}\mathbf{s}\left(\mathbf{0},\hat{\mathbf{u}}\right)/\, n. \end{aligned}$$

By standard theory, the asymptotic distribution of this statistic under the null is again χ^2 with m degrees of freedom. The subtraction of $\mathbf{I}'_{\boldsymbol{\theta\alpha}}\mathbf{I}^{-1}_{\boldsymbol{\theta\theta}}\mathbf{I}_{\boldsymbol{\theta\alpha}}$ inside the inverse increases the extended test statistic by just enough to prevent overacceptance of the null hypothesis when $\hat{\mathbf{u}}$ is used in place of $\mathbf{u}$.

Although $\mathbf{I}(\mathbf{0})$ can readily be computed in closed form for polynomial, spline, or step perturbation functions, calculation of the remaining information components of the extended LM statistic requires numerical integrations that may depend on the distribution in question, its parameters, and the form and number of the perturbation functions. DuMouchel (1975) has tabulated $\mathbf{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\theta},\mathbf{0})$ for the stable distributions. See Percy (2006) for computational details and tables for the distributions and perturbation functions considered here.

In the extended test, finite sample critical values depend on the specific model and true parameter values. Simulations reported in Percy (2006) indicate more size distortion than with known parameters. However, size distortion can still largely be ignored, even for relatively small sample sizes.

4. The extended test on stock market returns

The extended LM test is illustrated in this section using continuously compounded percent real monthly returns, including dividends, on the CRSP value-weighted stock market index, during the 40-year period from 1/53-12/92 (480 observations), as employed in McCulloch (1997). Percy (2006) also considers the 50-year period ending in 12/2002. However, volatility clustering is more apparent in the longer series. Although estimated ARCH or GARCH parameters may be incorporated into the extended test, the present paper assumes i.i.d. errors, and hence this section uses only the shorter period.

The extended test was performed with the following symmetric distributions: (1) Gaussian, (2) symmetric stable, (3) Student *t* with a mean and scale parameter, and (4) the generalized error distribution. We also used various perturbation functions: a step function that extends the traditional Pearson test, Neyman-Legendre polynomials, and linear, quadratic, and cubic B-splines. All the splines, as well as the extended Pearson test, used equidistant knotpoints.

Table 1 reports maximum likelihood estimates under a Gaussian null hypothesis. Table 2 reports the extended LM test statistics and asymptotic *p* values for the hypothesis that these returns are Gaussian, with up to twelve perturbation parameters in the alternative hypotheses.

Gaussian ML Estimates				
	Estimate	Standard Error	*z*-score	Probability
Std. Deviation	4.272	0.138		
Mean	0.555	0.195	2.85	0.0044
Log L	-1378.07			

Table 1.
Gaussian maximum likelihood estimates (n = 480).

Lagrange Multiplier Test Statistics					
m	Pearson	Neyman-Legendre	Linear Spline	Quadratic Spline	Cubic Spline
1	3.88	10.51	10.51		
2	4.64	31.75	21.80	31.75	
3	13.85	31.80	28.93	31.78	31.80
4	20.46	52.50	39.40	47.45	49.81
5	21.11	53.44	44.82	50.78	52.91
6	37.84	57.99	47.56	52.38	55.51
7	38.88	69.68	49.96	55.11	61.62
8	34.65	74.90	49.83	60.08	69.18
9	32.39	90.06	51.96	67.93	79.50
10	31.62	100.98	58.56	75.55	86.55
11	34.42	103.93	61.32	77.75	88.84
12	37.27	108.61	68.34	82.20	93.03
Complement of chi square inverse of test statistic					
m	Pearson	Neyman-Legendre	Linear Spline	Quadratic Spline	Cubic Spline
1	0.0490	0.0012	0.0012		
2	0.0983	1.3e-07	1.8e-05	1.3e-07	
3	0.0031	5.8e-07	2.3e-06	5.8e-07	5.8e-07
4	0.0004	1.1e-10	5.7e-08	1.2e-09	4.0e-10
5	0.0008	2.7e-10	1.6e-08	9.6e-10	3.5e-10
6	1.2e-06	1.2e-10	1.4e-08	1.6e-09	3.7e-10
7	2.1e-06	1.7e-12	1.5e-08	1.4e-09	7.2e-11
8	3.1e-05	5.2e-13	4.4e-08	4.5e-10	7.1e-12
9	0.0002	1.6e-15	4.6e-08	3.9e-11	2.0e-13
10	0.0005	3.5e-17	6.8e-09	3.7e-12	2.6e-14
11	0.0003	3.0e-17	5.3e-09	4.0e-12	2.8e-14
12	0.0002	1.1e-17	6.6e-10	1.6e-12	1.3e-14

Table 2.
LM Test Statistics and *p* values for Gaussian null hypothesis.

The Jarque-Bera (1987) statistic for i.i.d. normality is 189.19 with this data. The *p* value for this statistic with the appropriate 2 degrees of freedom is 10^{-41}. Many of the *p* values in Table 2 are zero to several places, but none is as small as for the Jarque-Bera statistic. Jarque-Bera is designed to be sensitive to departures of skewness from 0 and of kurtosis from 3, and is unquestionably the best test to use for departures from normality that show up strongly in these moments. The extended Neyman LM statistic, on the other hand, will be sensitive to other departures from normality, depending on the number and type of basis functions, and, with sufficiently large sample size and sufficiently large values of *m*, could eventually detect distributions that were not Gaussian even if they had zero skewness and kurtosis equal to 3.

Drawing attention to the Pearson statistics momentarily, one can see that the Pearson statistics with low parameter numbers are not particularly adept at identifying the non-Gaussian nature of this data set. For $m = 1$, the significance level is 0.049 and for $m = 2$, the significance level is 0.099. Since Pearson takes no account of where an error resides *within* the bin to which it is assigned, it will often not be very sensitive to departures from the posited distribution, so we recommend against use of this statistic whenever the null is a continuous distribution, even when it is corrected, as here, for estimated parameters.

In Table 3, we estimate the symmetric stable distribution parameters by maximum likelihood using the density approximation of McCulloch (1998). As in McCulloch (1997), the estimated stable characteristic exponent α (not to be confused with the perturbation coefficients of the preceding sections) is 1.845. The algorithm fits the natural logarithm of the scale *c*, so that the asymptotic standard errors apply to log *c* rather than *c* itself. Although 1.845 is 2.63 asymptotic standard errors from the value of 2 corresponding to a Gaussian distribution, and the likelihood ratio (LR) statistic for the hypothesis of normality is 26.66, these statistics do not have their usual N(0,1) and $\chi^2(1)$ distributions, because $\alpha = 2$ is on the boundary of the permissible parameter space. Nevertheless, the simulations of McCulloch (1997) demonstrate that the 5% critical value for the LR statistic is less than 1.12 for this sample size, and that normality can be rejected with $p \ll .004$. Table 4 reports the extended LM test of the null that these real returns are symmetric stable, with the same alternatives as considered in Table 2.

Symmetric Stable ML Estimates				
	Estimate	Standard Error	*z*-score	Probability
Log scale	0.997	0.040		
Scale c	2.711			
Stable *a*	1.845	0.059		
Mean	0.673	0.182	3.70	0.0002
Log L	-1364.74			

Table 3.
Symmetric stable maximum likelihood estimates (n = 480).

Lagrange Multiplier Test Statistics					
m	Pearson	Neyman-Legendre	Linear Spline	Quadratic Spline	Cubic Spline
1	0.68	1.34	1.34		
2	1.74	1.59	1.34	1.59	
3	1.92	6.64	3.38	5.52	6.64
4	6.91	12.42	7.44	11.27	11.48
5	4.44	13.77	12.83	14.47	14.50
6	11.30	14.40	15.62	15.03	14.47
7	19.67	15.13	16.24	14.61	14.23
8	24.56	15.22	17.29	15.39	15.50
9	15.04	18.43	14.08	15.34	17.43
10	10.59	19.78	17.29	19.65	20.53
11	17.29	19.78	17.43	20.63	20.18
12	12.50	19.85	19.28	21.11	20.57
Complement of chi square inverse of test statistic					
m	Pearson	Neyman-Legendre	Linear Spline	Quadratic Spline	Cubic Spline
1	0.4111	0.2478	0.2478		
2	0.4198	0.4521	0.5128	0.4521	
3	0.5892	0.0842	0.3360	0.1377	0.0842
4	0.1407	0.0145	0.1146	0.0237	0.0217
5	0.4881	0.0171	0.0250	0.0129	0.0127
6	0.0794	0.0255	0.0160	0.0200	0.0248
7	0.0063	0.0344	0.0230	0.0414	0.0473
8	0.0018	0.0550	0.0273	0.0520	0.0501
9	0.0898	0.0305	0.1195	0.0820	0.0425
10	0.3900	0.0314	0.0682	0.0327	0.0246
11	0.0995	0.0485	0.0959	0.0375	0.0429
12	0.4066	0.0701	0.0821	0.0487	0.0570

Table 4.
LM Test Statistics and p values for symmetric stable null hypothesis.

Turning our attention to the Neyman-Legendre polynomial tests, the first three tests do not reject the symmetric stable distribution at the 5% test size. However, when we add the fourth basis function, we begin to obtain significant rejections. The first three Neyman-Legendre statistics are necessarily identical to the first statistic for the linear, quadratic, and cubic splines, respectively. For the cubic spline with more than 3 parameters and therefore at least one internal knotpoint, the results are similar to those for the Neyman-Legendre statistics. The step-function Pearson results are far more dependent on *m* than are the smooth alternatives.

Tables 5 and 6 report the same tests for the Student *t* distribution. Infinite Student *t* degrees of freedom (DOF) correspond to a proper, Gaussian distribution, so the search was parameterized in terms of reciprocal DOF rather than DOF directly.

Tables 7 and 8 report these tests for the GED. An infinite GED power parameter leads to a proper, U(-1, 1) limit, so this search was also parameterized in terms of the reciprocal of the shape parameter. In order for the Fisher information matrix to be finite, it is necessary to restrict the power to be strictly greater than 1, so as to just exclude the Laplace distribution. However, this restriction was not binding. Percy (2006) reports similar test results for a mixture of two Gaussian distributions.

Student *t* ML Estimates				
	Estimate	Standard Error	*z*-score	Probability
Log Scale	1.262	0.054		
Scale c	3.531			
Reciprocal DOF	0.155	0.043		
Deg. of Freedom	6.443			
Mean	0.716	0.182	3.95	7.9e-5
Log L	-1363.72			

Table 5.
Student *t* maximum likelihood estimates (*n* = 480)

Lagrange Multiplier Test Statistics					
m	Pearson	Neyman-Legendre	Linear Spline	Quadratic Spline	Cubic Spline
1	0.13	1.22	1.22		
2	4.46	2.65	2.87	2.65	
3	1.95	8.06	6.39	7.29	8.06
4	8.53	9.81	7.44	9.74	9.51
5	4.70	10.48	11.51	11.50	11.07
6	15.82	10.49	13.33	11.39	10.55
7	22.74	11.79	13.56	11.02	10.77
8	20.53	13.88	14.71	12.90	13.70
9	18.54	18.25	12.09	13.13	15.96
10	10.62	18.37	13.95	17.48	19.30
11	15.56	18.53	16.03	19.96	19.73
12	15.61	19.17	16.87	19.38	19.27
Complement of chi square inverse of test statistic					
m	Pearson	Neyman-Legendre	Linear Spline	Quadratic Spline	Cubic Spline
1	0.7163	0.2701	0.2701		
2	0.1077	0.2652	0.2383	0.2652	
3	0.5826	0.0449	0.0942	0.0632	0.0449
4	0.0739	0.0438	0.1143	0.0451	0.0496
5	0.4542	0.0627	0.0421	0.0423	0.0499
6	0.0148	0.1056	0.0381	0.0769	0.1032
7	0.0019	0.1077	0.0597	0.1379	0.1489
8	0.0085	0.0848	0.0651	0.1153	0.0898
9	0.0294	0.0324	0.2085	0.1569	0.0676
10	0.3882	0.0490	0.1754	0.0644	0.0367
11	0.1584	0.0701	0.1400	0.0458	0.0492
12	0.2095	0.0846	0.1544	0.0797	0.0823

Table 6.
LM Test Statistics and *p* values for Student *t* null hypothesis.

Generalized Error Distribution (GED) ML Estimates				
	Estimate	Standard Error	*z*-score	Probability
Log Scale	1.537	0.039		
Scale c	4.651			
Recip. power	0.712	6.4e-4		
GED power	1.404			
Mean	0.713	0.180	3.96	7.6e-5
Log L	-1367.76			

Table 7.
GED maximum likelihood estimates (n = 480)

Lagrange Multiplier Test Statistics					
m	Pearson	Neyman-Legendre	Linear Spline	Quadratic Spline	Cubic Spline
1	0.26	0.01	0.01		
2	4.09	0.02	0.80	0.02	
3	1.63	8.07	7.11	8.32	8.07
4	6.21	19.25	13.41	18.13	18.56
5	6.74	19.42	21.19	20.05	19.74
6	17.47	19.46	21.20	19.36	19.18
7	22.88	19.76	21.47	20.10	19.51
8	20.36	19.99	22.52	21.23	20.95
9	18.21	24.53	19.86	24.96	27.67
10	16.54	27.35	23.83	28.62	28.95
11	19.46	30.89	26.41	30.17	29.30
12	23.22	31.00	27.04	29.07	29.00
Complement of chi square inverse of test statistic					
m	Pearson	Neyman-Legendre	Linear Spline	Quadratic Spline	Cubic Spline
1	0.6134	0.9076	0.9076		
2	0.1295	0.9917	0.6701	0.9917	
3	0.6519	0.0445	0.0683	0.0398	0.0445
4	0.1839	0.0007	0.0094	0.0012	0.0010
5	0.2410	0.0016	0.0007	0.0012	0.0014
6	0.0077	0.0035	0.0017	0.0036	0.0039
7	0.0018	0.0061	0.0031	0.0054	0.0067
8	0.0091	0.0104	0.0040	0.0066	0.0073
9	0.0329	0.0035	0.0188	0.0030	0.0011
10	0.0852	0.0023	0.0081	0.0014	0.0013
11	0.0533	0.0011	0.0056	0.0015	0.0020
12	0.0260	0.0020	0.0076	0.0038	0.0039

Table 8.
LM Test Statistics and p values for GED null hypothesis.

The most easily rejected of these three distributions is the GED. These distributions have relatively thin tails compared to the others. As a general rule, the *presence* of more than the expected value of outliers (or even one extreme outlier) often allows for the rejection of null hypotheses of thin-tailed distributions, while the *absence* of outliers does not allow for as easy of a rejection of null hypotheses of heavy-tailed distributions. Of course as the sample size grows, eventually a test for a heavy-tailed distribution will decrease its *p* value if outliers do not eventually appear.

For purely computational reasons, the present paper considers only symmetric distributions. Although all three symmetric distributions considered in the present section encounter frequent rejections (with the notable exception of the Student *t* for *m* between 6 and 9), it is likely that many of these rejections would be reversed if symmetry were not imposed. The numerical approximation of Nolan (1997) permits skew-stable distributions to be fit by ML, while the Student *t* distribution can be generalized to incorporate skewness in a natural way (Kim and McCulloch 2007). It is also possible to skew the GED, albeit in an ad hoc manner, simply by expanding the horizontal axis on one side of the origin while compressing it on the other.

With limited data, densities that are similar over much of their support cannot be distinguished from one other very easily. This does not allow one to make very strong statements about the extreme tail probabilities, where the densities may differ considerably. Although this data set has only 480 monthly returns, 40 years of daily returns would yield about 10,000 observations. When daily data is used, however, the returns often become less independent and less identically distributed since there is more apparent volatility clustering, day-of-the-week effects in both mean and scale, holiday effects, end-of-year effects, and other complications. However, the extended

Neyman Smooth GFT developed here allows these additional effects to be estimated without size distortion.

5. Size distortion and power with financial data parameter values

This section investigates criteria for determining the best test to use in terms of size distortion and power against the symmetric leptokurtic alternatives considered above, and also a mixture of two normals with a common mean. With the extended test, size distortion and power may depend on both the distribution and true parameter values in question. For this purpose, following Percy (2006), we use parameter estimates for percent log real monthly returns for the 50-year period January, 1953, through December, 2002. The ML parameter estimates for this longer period were as follows:

Symmetric stable: α = 1.862, log scale = 1.024, δ = 0.585

Student *t*: degrees of freedom = 6.864, log scale = 1.293, mean = 0.641

Generalized error distribution: power = 1.419, log scale = 1.568, mean = 0.670

Mixture of two Gaussians: probability of smaller standard deviation = 0.906,
log(smaller st. dev.) = 1.308, log(larger st. dev.) = 2.125, mean = 0.594

These values were used in simulations for both null hypotheses and data-generating processes. It is important to note that the conclusions to be drawn using these parameter values may not be applicable with other values.

The size distortion and power against each of the other distributions considered were investigated for each of the four leptokurtic distributions. All sizes and powers are based on simulations using 1000 samples, described in more detail in Percy (2006). All data is generated by one of the four hypothesized distributions with the indicated parameters. There are two test sizes investigated for each scenario, 0.10 and 0.05. From 1 to 20 basis parameters are tested, or 3 to 20 for the cubic splines.

Six sample sizes were considered: n = 32, 100, 316, 1000, 3162 and 10,000 (10^k, $k = {}^3/_2$, 2, ${}^5/_2$, 3, ${}^7/_2$, 4). We tested the size distortion for each extended goodness-of-fit test and did the same using conventional tests without the adjustment for estimated parameters. This produces 18 power tests per null hypothesis, based on 6 possible sample sizes with 3 possible alternative hypotheses. For each category there is a size-adjusted power for the corrected tests and non-adjusted powers for both the corrected tests and the conventional tests.

Preliminary tests indicated that the Neyman-Legendre polynomial basis and the Cubic Spline basis generally outperformed the other bases investigated (Pearson, the Quadratic Spline and the Linear Spline). Accordingly, only those two bases are compared here.

5.1. Size distortion

There is tremendous size distortion with the uncorrected conventional tests in every instance for every value of m. This distortion does not go away as the sample size increases from 32 to 10,000. It diminishes somewhat as the number of basis parameters increases, but this is still not very helpful. The distortion is, as expected, in the direction of over-acceptance.

Figure 2 illustrates the size improvement from the extended test for the symmetric stable distribution with sample size 316 and test size 0.10. It shows that the size of the extended test lies mostly within 95% confidence limits of the intended size, while that of the conventional tests lies completely outside this interval. This example is typical of the full set of figures reported in Percy (2006). For the corrected tests, there is some size distortion for smaller sample sizes, but generally much smaller than with the conventional tests. This size distortion vanishes, within sampling error, for moderate sample sizes.

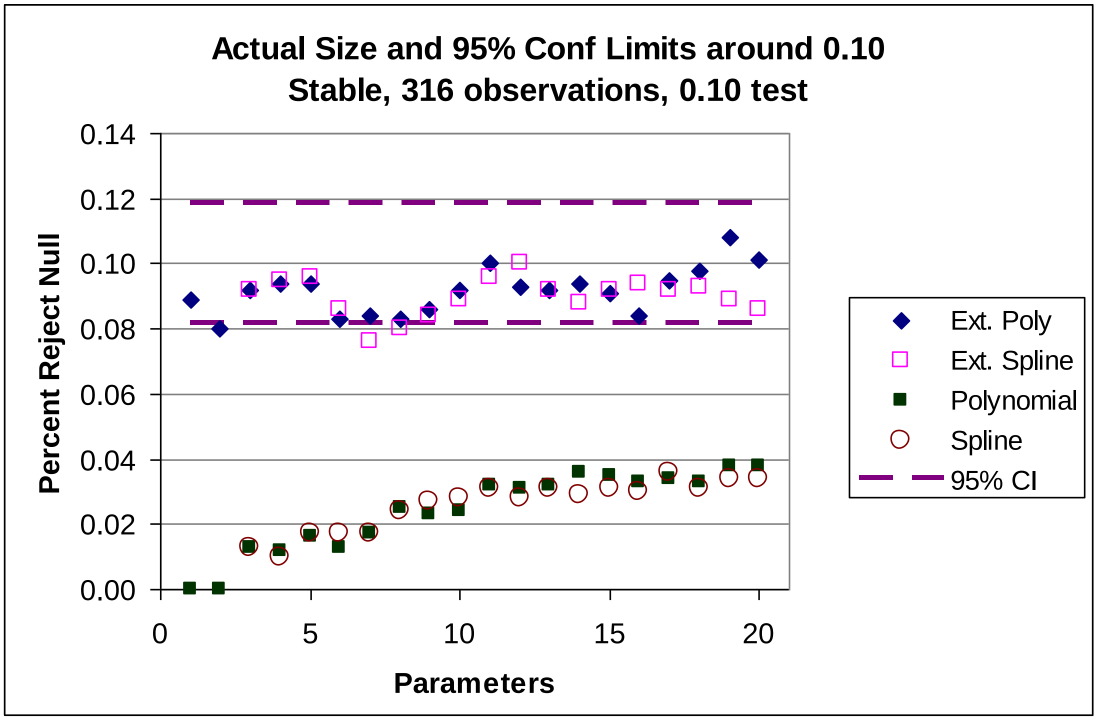

Figure 2.
Size Improvement due to Extended Test, symmetric stable null, *n* = 316, test size .10, with 95% CI bracketing nominal test size.

For the mixture distribution and as few as 100 observations, with up to $m = 6$ perturbation parameters, the extended test size distortion is undetectable. For 316 observations and the level 0.10 test, the results are just barely in the low end of the confidence interval for most values of m. For 1000 and more observations, there is no significant distortion. Distortion gradually disappears for the stable null from $n = 316$ to 1000 as well, although it is small for smaller sample sizes with small numbers of basis parameters as well. For Student nulls, distortion dies out at only 100 observations. GED nulls require around 316 observations.

5.2. Power and recommendations

The over-acceptance caused by the size distortion in the conventional tests contributes to poor power against the chosen nulls. With the conventional tests, fitting the model parameters biases against rejecting even false hypotheses. Hence, practitioners may all too often mistakenly conclude that, since their test does not reject an alternative hypothesis, they are justified in accepting the validity of the assumptions in their study. The uncorrected tests have power even less than the test size for the largest sample sizes!

The corrected tests do have more power than the conventional tests. However, it is quite difficult to tell the leptokurtic distributions under consideration from one another when the sample size is small, even with the corrected tests. When the sample size is high enough for there to be reasonable levels of power (4 to 5 times the test size), there is negligible size distortion. However, there are often significant power gains to be made by adapting the functional form and number of perturbation parameters to the specific null in question. Accordingly, recommendations are made below that are specific to the null, yet robust to the actual data-generating distribution, using parameters as fit above to stock market data.

With a stable null: a GED distribution can start to be detected with as few as 316 observations. A pattern emerges which has a power peak at only $m = 2$ polynomial test parameters, or 3 for the cubic spline. This power peak also works well for detecting a Student distribution. The mixture of normals, with its extra parameter, is quite difficult to identify, requiring more than 3000 observations even to get modest 30% power levels. At 10,000 observations the most power comes with a large number of parameters. But at 10,000 observations, any value of m (more than 2) will have the same (100%) power levels for the GED and Student distributions. At this sample size, it is recommended to use the cubic spline, perhaps with $m = 15$ test parameters, because it has fewer numerical difficulties than the polynomial as m increases. See Percy (2006) for graphs illustrating these findings.

Under a Student *t* null hypothesis, a stable distribution can be seen almost half the time with 316 observations. Almost any value of *m* greater than 2 will work equally effectively. GED and normal mixtures are still concealed for the most part at this sample size, but a test with 2 or 3 parameters has the best chance of finding them. So, if Student *t* is the null, one may proceed with 2 or 3 test parameters, regardless of sample size.

With a GED null hypothesis, the advisable number is 4 parameters. Using a basis with only three parameters has very low power but there is a tremendous increase at 4, with small increases after that.

With a mixture of two Gaussians as the null hypothesis, there is little power to detect the other three distributions, unless one has at least 3000 observations. The pattern is a bit unusual, however, in that higher values of *m* generally yield more power. For this reason, the cubic spline is recommended here to avoid numerical inaccuracy, with about 15 parameters. At 10,000 observations, even with the possible inaccuracies, the test with 8 parameters has fairly low distortion levels, so it would be safe to use against the chosen alternatives. For further discussion, see Percy (2006).

6. Conclusion

The extended Neyman Smooth test can be used to analyze economic and financial data to probe the distribution underlying the generation of the data. Some parsimonious parametric distributions may be found that will aid inferences about levels of and relationships between economic variables. Thus, asymptotically consistent estimates of parameters are possible without either presuming normality of error terms or using solely nonparametric techniques. In that regard, these new procedures can offer new answers to old questions.

Unlike many goodness-of-fit tests, unknown model parameters can be estimated with the extended Neyman tests without prejudicing the tests. Since these tests rely on maximum likelihood techniques, they asymptotically meet the conditions of the Neyman-Pearson lemma against any simple alternative hypothesis in its parameter space. Tests with one-sided alternatives that meet these criteria qualify as Uniformly Most Powerful (UMP) tests for arbitrary significance levels.

Spline models are more tractable than polynomial models with existing double precision software, and it does not appear that this tractability is obtained at the cost of lower power in tests of interest.

Size distortion that is present in the conventional tests is lowered considerably with the extended tests, even for modest sample sizes. A related benefit is increased test power.

REFERENCES

Bai, J., 2003, Testing parametric conditional distributions of dynamic models. Review of Economics and Statistics 85, 531-549.

Blattberg, R., and N. Gonedes, 1974, A comparison of the stable and Student distributions as statistical models for stock prices. Journal of Business 47, 244-280.

Boothe, P., and D. Glassman, 1987, The statistical distribution of exchange rates. Journal of International Economics 22, 297-319.

Clark, P.K., 1973, A subordinated stochastic process model with finite variance for speculative prices. Econometrica 41, 135-155.

DuMouchel, W.H., 1975, Stable distributions in statistical inference: 2. Information from stably distributed samples. Journal of the American Statistical Association 70, 386-393.

Fama, E.F., 1965, The behavior of stock prices. Journal of Business 38, 34-105.

Hagerman, R.L., 1978, More evidence on the distribution of security returns. Journal of Finance 33, 1213-1221.

Jarque, C.M., and A.K. Bera, 1987, A test for normality of observations and regression residuals. International Statistical Review 55, 163-172.

Judd, K.L., 1999, Numerical methods in economics. MIT Press, Cambridge.

Kim, Y.I., and J.H. McCulloch, 2007, The skew-Student distribution with application to U.S. stock market returns and the equity premium. Ohio State University.

Mandelbrot, B., 1963, The variation of certain speculative prices. Journal of Business 36, 394-419.

McCulloch, J.H., 1996, Financial applications of stable distributions, in: G.S. Maddala and C.R. Rao, (Eds.), Handbook of statistics, Vol. 14: Statistical methods in finance, Elsevier Science, Amsterdam.

McCulloch, J.H., 1997, Measuring tail thickness to estimate the stable index α: A critique. Journal of Business & Economic Statistics 15, 74-81.

McCulloch, J.H., 1998, Numerical approximation of the symmetric stable distribution and density, in: R. Adler, R. Feldman, and M.S. Taqqu (Eds.), A practical guide to heavy tails, Burkhäuser, Boston.

Nelson, D.B., 1991, Conditional heteroskedasticity in asset returns: A new approach. Econometrica 59, 347-370.

Neyman, J., 1937, Smooth test for goodness of fit. Skandinavisk Aktuarietidskrift 20, 1949-1999.

Nolan, J.P., 1997, Numerical calculation of stable densities and distribution functions. Communications in Statistics – Stochastic Models 13, 759-774.

Percy, E.R., Jr., 2006, Corrected LM goodness-of-fit tests with application to stock returns, <http://www.econ.ohio-state.edu/jhm/papers/PercyDissertation.pdf>.

Perry, P.R., 1983, More evidence on the nature of the distribution of security returns. Journal of Financial and Quantitative Analysis 18, 211-221.

Praetz, P.D., 1972, The distribution of share price changes. Journal of Business 45, 49-55.

Rayner, J.C.W., and D.J. Best, 1989, Smooth tests of goodness of fit. Oxford University Press, Oxford.

Samorodnitsky, G., and M.S. Taqqu, 1994, Stable non-Gaussian random processes: Stochastic processes with infinite variance. Chapman and Hall, New York.

Wilkinson, J.H., 1963, Rounding Errors in algebraic processes. Prentice-Hall, Englewood Cliffs, NJ.

TYPOGRAPHIC NOTES

The *JE* may wish to use a double stroke font such as LaTeX Bbold or Blackboard Bold for the expectations operator E in the equation following (1), as well as on p. 9. This is unfortunately not available in MS Word Equation Editor.

Likewise, the *JE* may wish to use a cursive font such as Script MT Bold ($\mathcal{I}$) for the information matrix **I** in the equation following (1), as well as in (2), (3), and several places on p. 9. However, since the identity matrix is not used in this paper, no confusion arises from using **I**. Again, this font is not available in MS Word Equation Editor.

Every attempt has been made to use bold face for vectors and matrices. If there is any ambiguity, please contact the authors.

Vectors and matrices, when defined in terms of their elements, are consistently defined using large parentheses, as in Simon and Blume, *Mathematics for Economists*. A subscript defines the range of the subscripts, as requested by Referee 1. If the *JE* would prefer a different notation, such as square brackets as in Greene's *Econometric Analysis*, the authors will be happy to work with the production editor on this.