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Edited by

G. S. Maddala
Department of Economics
Arps Hall, Ohio State University
Columbus, OH, USA

C. R. Rao
Center for Multivariate Analysis
Department of Statistics, The Pennsylvania State University
University Park, PA, USA

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Financial Applications of Stable Distributions

J. Huston McCulloch

Life is a gamble, at terrible odds;
If it were a bet, you wouldn't take it.

Tom Stoppard, *Rosenkrantz and Guildenstern are Dead*

1. Introduction

Financial asset returns are the cumulative outcome of a vast number of pieces of information and individual decisions arriving continuously in time. According to the Central Limit Theorem, if the sum of a large number of iid random variates has a limiting distribution after appropriate shifting and scaling, the limiting distribution must be a member of the *stable* class (Lévy 1937, Zolotarev 1986: 6). It is therefore natural to assume that asset returns are at least approximately governed by a stable distribution if the accumulation is additive, or by a log-stable distribution if the accumulation is multiplicative.

The Gaussian is the most familiar and tractable stable distribution, and therefore either it or the log-normal has routinely been postulated to govern asset returns. However, returns are often much more leptokurtic than is consistent with normality. This naturally leads one to consider also the non-Gaussian stable distributions as a model of financial returns, as first proposed by Benoit Mandelbrot (1960, 1961, 1963a,b).

If asset returns are truly governed by the infinite-variance stable distributions, life is fundamentally riskier than in a Gaussian world. Sudden price movements like the 1987 stock market crash turn into real-world possibilities, and the risk immunization promised by "programmed trading" becomes mere wishful thinking, at best. These price discontinuities render the arbitrage argument of the celebrated Black-Scholes (1973) option pricing model inapplicable, so that we must look elsewhere in order to value options.

Nevertheless, we shall see that the Capital Asset Pricing Model works as well in the infinite-variance stable cases as it does in the normal case. Furthermore, the Black-Scholes formula may be extended to the non-Gaussian stable cases by means of a utility maximization argument. Two serious empirical objections that have been raised against the stable hypothesis are shown to be inconclusive.

Section 2 of this paper surveys the basic properties of univariate stable distributions, of continuous time stable processes, and of multivariate stable distributions. Section 3 reviews the literature on portfolio theory with stable distributions, and extends the CAPM to the most general MV stable case. Section 4 develops a formula for pricing European options with log-stable uncertainty and shows how it may be applied to options on commodities, stocks, bonds, and foreign exchange rates. Section 5 treats the estimation of stable parameters and surveys empirical applications for returns on various assets, including foreign exchange rates, stocks, commodities, and real estate. Empirical objections that have been raised against the stable hypothesis are considered, and alternative leptokurtic distributions that have been proposed are discussed.

2. Basic properties of stable distributions

2.1. Univariate stable distributions

Stable distributions $S(x;\alpha,\beta,c,\delta)$ are determined by four parameters. The *location parameter* $\delta \in (-\infty,\infty)$ shifts the distribution to the left or right, while the *scale parameter* $c \in (0,\infty)$ expands or contracts it about δ, so that

$$S(x;\alpha,\beta,c,\delta) = S((x-\delta)/c;\alpha,\beta,1,0)\ . \tag{1}$$

We will write the *standard stable distribution function with shape parameters* α and β as $S_{\alpha\beta}(x) = S(x;\alpha,\beta,1,0)$, and use $s(x;\alpha,\beta,c,\delta)$ and $s_{\alpha\beta}(x)$ for the corresponding densities. If X has distribution $S(x;\alpha,\beta,c,\delta)$, we write $X \sim S(\alpha,\beta,c,\delta)$.

The *characteristic exponent* $\alpha \in (0,2]$ governs the tail behavior and therefore the degree of leptokurtosis. When $\alpha = 2$, a normal distribution results, with variance $2c^2$. For $\alpha < 2$, the variance is infinite. When $\alpha > 1$, $\mathrm{E}X = \delta$, but if $\alpha \le 1$, the mean is undefined. The case $\alpha = 1, \beta = 0$ gives the Cauchy (arctangent) distribution.

Expansions due to Bergstrøm (1952) imply that as $x \uparrow \infty$,

$$\begin{aligned} S_{\alpha,\beta}(-x) &\sim (1-\beta)\frac{\Gamma(\alpha)}{\pi}\sin\frac{\pi\alpha}{2}x^{-\alpha}\ , \\ 1 - S_{\alpha,\beta}(x) &\sim (1+\beta)\frac{\Gamma(\alpha)}{\pi}\sin\frac{\pi\alpha}{2}x^{-\alpha}\ . \end{aligned} \tag{2}$$

When $\alpha < 2$, stable distributions therefore have one or more "Paretian" tails that behave asymptotically like $x^{-\alpha}$ and give the stable distributions infinite absolute population moments of order greater than or equal to α. In this case, the *skewness parameter* $\beta \in [-1,1]$ indicates the limiting ratio of the difference of the two tail probabilities to their sum. We here follow Zolotarev (1957) by defining β so that $\beta > 0$ indicates positive skewness for all α. If $\beta = 0$, the distribution is symmetric stable (SS). As $\alpha \uparrow 2$, β loses its effect and becomes unidentified.

Stable distributions are defined most concisely in terms of their log characteristic functions:

$$\log \mathrm{E}e^{iXt} = i\delta t + \psi_{\alpha,\beta}(ct) \ , \tag{3}$$

where

$$\psi_{\alpha,\beta}(t) = \begin{cases} -|t|^{\alpha}[1 - i\beta \operatorname{sign}(t) \tan \pi\alpha/2] \ , & \alpha \neq 1 \ , \\ -|t|\left[1 + i\beta \frac{2}{\pi} \operatorname{sign}(t) \log |t|\right] \ , & \alpha = 1 \end{cases} \tag{4}$$

is the log c.f. for $S_{\alpha\beta}(x)$[1]. The stable distribution and density may be computed either by using Zolotarev's (1986: 74, 68) proper integral representations, or by evaluating the inverse Fourier transform of the c.f. DuMouchel (1971) tabulates the stable distributions, while Holt and Crow (1973) tabulate and graph the density.[2] See also Fama and Roll (1968) and Panton (1992). A fast numerical and reasonably accurate approximation to the SS distribution and density for $\alpha \in [0.84, 2.00]$, has been developed by McCulloch(1994b).

The formulas for $S_{\alpha\beta}(x)$ are calculable for $\alpha > 2$ or $|\beta| > 1$, but the resulting function is not a proper probability distribution since one or both tails will then lie outside [0,1], as may be seen from (2). Stable distributions are therefore constrained to have $\alpha \in (0, 2]$ and $\beta \in [-1, 1]$.

Let $X \sim S(\alpha, \beta, c, \delta)$ and a be any real constant. Then (3) implies

$$aX \sim S(\alpha, \operatorname{sign}(a)\beta, |a|c, a\delta) \ . \tag{5}$$

Let $X_1 \sim S(\alpha, \beta_1, c_1, \delta_1)$ and $X_2 \sim S(\alpha, \beta_2, c_2, \delta_2)$ be independent drawings from stable distributions with a common α. Then $X_3 = X_1 + X_2 \sim S(\alpha, \beta_3, c_3, \delta_3)$, where

$$c_3^{\alpha} = c_1^{\alpha} + c_2^{\alpha} \ , \tag{6}$$

$$\beta_3 = (\beta_1 c_1^{\alpha} + \beta_2 c_2^{\alpha})/c_3^{\alpha} \ , \tag{7}$$

$$\delta_3 = \begin{cases} \delta_1 + \delta_2, \alpha \neq 1 \\ \delta_1 + \delta_2 + \frac{2}{\pi}(\beta_3 c_3 \log c_3 - \beta_1 c_1 \log c_1 - \beta_2 c_2 \log c_2), \alpha = 1 \ . \end{cases} \tag{8}$$

When $\beta_1 = \beta_2, \beta_3$ equals their common value, so that x_3 has the same shaped distribution as x_1 and x_2. This is the "stability" property of stable distributions that leads directly to their role in the CLT, and makes them particularly useful in financial portfolio theory. If $\beta_1 \neq \beta_2, \beta_3$ lies between β_1 and β_2.

For $\alpha < 2$ and $\beta > -1$, the long upper Paretian tail makes $\mathrm{E}e^{X}$ infinite. However, when $X \sim S(\alpha, -1, c, \delta)$, Zolotarev (1986: 112) has shown that

[1] (3) follows DuMouchel (1973a) and implies (1) and (5). Samorodnitsky and Taqqu (1994), following Zolotarev (1957), use (4), but give the general log c.f. as $i\mu t + c^{\alpha}\psi_{\alpha\beta}(t)$. This is equivalent to (3) for $\alpha \neq 1$, with $\mu = \delta$. For $\alpha = 1$, however, their μ becomes $\delta - (2/\pi)\beta c \log c$. McCulloch (1986) erroneously attributes to this "μ" formulation the properties of (3). See McCulloch (in press b) for details.

[2] Holt and Crow, following the 1949 work of Kolmogorov and Gnedenko, reverse the sign on β in (4) for $\alpha \neq 1$, with the unfortunate but easily corrected result that their "β"> 0 indicates *negative* skewness and vice-versa, unless $\alpha = 1$. Cf Hall (1981).

$$\log \mathrm{E}e^X = \begin{cases} \delta - c^\alpha \sec(\frac{\pi\alpha}{2}), & \alpha \neq 1 \\ \delta + \frac{2}{\pi} c \log c, & \alpha = 1 \ . \end{cases} \tag{9}$$

This formula greatly facilitates asset pricing under log-stable uncertainty.[3]

A simulated stable r.v. may be computed directly from a pair of independent uniform pseudo-random variables without using the inverse cdf by the method of Chambers, Mallows and Stuck (1976).[4]

2.2. *Continuous time stable processes*

Because stable distributions are infinitely divisible, they are particularly attractive for continous time modeling (Samuelson 1965: 15–16; McCulloch 1978). The stable generalization of the familiar Brownian motion or Wiener process is called an *α-Stable Lévy Motion*, and is the subject of two recent monographs, by Samorodnitsky and Taqqu (1994) and Janicki and Weron (1994). Such a process is a self-similar *fractal* in the sense of Mandelbrot (1983). In Peters' (1994) terminology, a *fractal* distribution is thus a stable distribution.

A standard α-Stable Lévy Motion $\xi(t)$ is a continuous time stochastic process whose increments $\xi(t+\Delta t) - \xi(t)$ are distributed $S(\alpha, \beta, \Delta t^{1/\alpha}, 0)$ for $\alpha \neq 1$ or $S(1, \beta, \Delta t, (2/\pi)\beta\Delta t \log \Delta t)$ for $\alpha = 1$, and whose non-overlapping increments are independent. Such a process has infinitessimal increments $d\xi(t) = \xi(t+dt) - \xi(t)$, with scale $dt^{1/\alpha}$. The process itself may then be reconstructed as the integral of these increments:

$$\xi(t) = \xi(0) + \int_0^t d\xi(\tau) \ .$$

The more general process $z(t) = c_0\xi(t) + \delta t$ has scale c_0 over unit time intervals and, for $\alpha \neq 1$, drift δ per unit time.

Unlike a Brownian motion, which is almost surely (a.s.) everywhere continuous, an α-Stable Lévy Motion is a.s. dense with discontinuities. Applying (2) to $S(\alpha, \beta, c_{dt}, 0)$ (cf. eqs. (18)–(19) of McCulloch 1978), the probability that $dz > x$ is

$$k_{\alpha\beta}\left(\frac{x}{c_{dt}}\right)^{-\alpha} = k_{\alpha\beta} c_o^\alpha x^{-\alpha} dt \ , \quad \text{where} \tag{10}$$

$$k_{\alpha\beta} = (1+\beta)\frac{\Gamma(\alpha)}{\pi} \sin\frac{\pi\alpha}{2} \ . \tag{11}$$

[3] The author is grateful to Vladimir Zolotarev for confirming that his Theorem 2.6.1 is, through a reparameterization, equivalent to (9). When $\alpha = 2$, (9) becomes the familiar formula $\log \mathrm{E}e^X = \mu + \sigma^2/2$.

[4] A call to IMSL subroutine GGSTA, which is based on their method, generates a simulated stable variate with argument BPRIME equal to our $\beta, c = 1$, and $\zeta = 0$, where $\zeta = \delta + \beta c \tan(\pi\alpha/2)$ for $\alpha \neq 1$ and $\zeta = \delta$ for $\alpha = 1$, rather than $\delta = 0$. See Zolotarev (1957: 454, 1987:11) and McCulloch (1986: 1121–26, in press b) concerning this shift. See also Panton (1989) for computational details concerning the CMS paper.

Eq. (10) in turn implies that values of dz greater than any threshhold $x_0 > 0$ occur at rate

$$\lambda = k_{\alpha\beta}(c_0/x_0)^\alpha \ , \tag{12}$$

and that conditional on their occurrence, they have a Pareto distribution:

$$P(dz < x | dz > x_0) = 1 - (x_0/x)^\alpha, \ \ x > x_0 \ . \tag{13}$$

Likewise, *negative* discontinuities $dz < -x_0$ also have a conditional Pareto distribution, and occur at a rate determined by (12), but with $k_{\alpha\beta}$ replaced by $k_{\alpha,-\beta}$. In the case $\alpha = 2$, $k_{\alpha\beta} = 0$, so that discontinuities a.s. never occur. With $\alpha < 2$, the frequency of discontinuities greater than x_0 in absolute value approaches infinity as $x_0 \downarrow 0$. If $\beta = \pm 1$, discontinuities a.s. occur only in the direction of the single Paretian tail.

Because the scale of $\Delta\xi$ falls to 0 as $\Delta t \downarrow 0$, an α-Stable Lévy Motion is everywhere a.s. continuous, despite the fact that it is not a.s. everywhere continuous. That is to say, every individual point t is a.s. a point of continuity, even though on any finite interval, there will a.s. be an infinite number of points for which this is not true. Even though they are a.s. dense, the points of discontinuity a.s. constitute only a set of measure zero, so that with probability one any point chosen at random will in fact be a point of continuity. Such a point of continuity will a.s. be a limit point of discontinuity points, but whose jumps approach zero as the point in question is approached.

The scale of $\Delta\xi/\Delta t$ is $(\Delta t)^{(1/\alpha)-1}$, so that if $\alpha > 1, \xi(t)$ is everywhere a.s. not differentiable, just as in the case of a Brownian motion. If $\alpha < 1, \xi(t)$ is everywhere a.s. differentiable, though of course there will be an infinite number of points (the discontinuities) for which this will not be true.

The discontinuities in an α-Stable Lévy Motion imply that the bottom may occasionally fall out of the market faster than trades can be executed, as occurred, most spectacularly, in October of 1987. When such events have a positive probability of occurrence, the portfolio risk insulation promised by "programmed trading" becomes wishful thinking, at best. Furthermore, the arbitrage argument of the Black-Scholes model (1973) cannot be used to price options, and options are not the redundant assets they would be if the underlying price were continuous.

2.3. Multivariate stable distributions

Multivariate stable distributions are in general much richer than MV normal distributions. This is because "iid" and "spherical" are not equivalent for $\alpha < 2$, and because MV stable distributions are not in general completely characterized by a simple covariation matrix as are MV normal distributions. If x_1 and x_2 are iid stable with $\alpha < 2$, their joint distribution will *not* have circular density contours. Near the center of the distribution the contours are nearly circular, but as we move away from the center, the contours have bulges in the directions of the axes (Mandelbrot 1963b: 403).

Let z be an m×1 vector of iid stable random variables, each of whose components is $S(\alpha, 1, 1, 0)$, and let $A = (a_{ij})$ be a $d \times m$ matrix of rank $d \leq m$. The $d \times 1$ vector $x = Az$ then has a d-dimensional MV stable distribution with *atoms* in the directions of each of the columns a_j of A. If any two of these columns have the same direction, say $a_2 = \lambda a_1$ for some $\lambda > 0$, they may, with no loss of generality, be merged into a single column equal to $(1 + \lambda^\alpha)^{1/\alpha} a_1$, by (5) and (6). Each atom will create a bulge in the joint density in the direction of a_j. If the columns come in pairs with opposite directions but equal norms, x will be SS.

The (discrete) *spectral representation* represents a_j as $c_j s_j$, where $c_j = ||a_j||$ and $s_j = a_j/c_j$ is the point on the unit sphere $S_d \subset \boldsymbol{R}^d$ in the direction of a_j. Then x may be written

$$x = \sum_{j=1}^{m} c_j s_j z_j \ , \tag{14}$$

and for $\alpha \neq 1$ has log c.f.

$$\log \mathrm{E} e^{x' t} = \sum_{j=1}^{m} \gamma_j \psi_{\alpha 1}(s_j' t) \ , \tag{15}$$

where $\gamma_j = c_j^\alpha$.[5]

The most general MV stable distributions may be generated by contributions coming from all conceivable directions, with some or even all of the c_j in (14) infinitesimal. Abstracting from location, the log c.f. may then be written

$$\log \mathrm{E} e^{ix' t} = \int_{S_d} \psi_{\alpha 1}(s' t) \Gamma(ds) \ , \tag{16}$$

where Γ is a finite spectral measure defined on the Borel subsets of S_d.

In the case $d = 2$, (16) may be simplified to

$$\log \mathrm{E} e^{ix' t} = \int_0^{2\pi} \psi_{\alpha 1}(s_\theta' t) d\Gamma(\theta) \ , \tag{17}$$

where $s_\theta = (\cos\theta, \sin\theta)'$ is the point on the unit circle at angle θ and Γ is a nondecreasing, left-continuous function with $\Gamma(0) = 0$ and $\Gamma(2\pi) < \infty$. (Cp. Hardin, Samorodnitsky and Taqqu 1991: 585; Mittnik and Rachev 1993b: 355–56; Wu and Cambanis 1991: 86.)

Such a random vector $x = (x_1, x_2)'$ may be constructed from a maximally positively skewed ($\beta = 1$) α-stable Lévy motion $\xi(\theta)$, whose iid increments $d\xi(\theta)$ have zero drift and scale $(d\theta)^{1/\alpha}$, by

$$x = \int_0^{2\pi} s_\theta \frac{(d\Gamma(\theta))^{1/\alpha} d\xi(\theta)}{(d\theta)^{1/\alpha}} \ . \tag{18}$$

[5] Because the δ of (3) is not additive for $\alpha = 1$, $\beta \neq 0$ (see (8)), the formulas in this section require modification in this special case.

(Cp. Modarres and Nolan 1994.) This integrand has the following interpretation: If $\Gamma'(\theta)$ exists, θ contributes $s_\theta(\Gamma'(\theta))^{1/\alpha}d\xi(\theta)$ to the integral; if Γ instead jumps by $\Delta\Gamma$ at θ, θ contributes an atom $s_\theta(\Delta\Gamma)^{1/\alpha}Z_\theta$, where $Z_\theta = (d\theta)^{-1/\alpha}d\xi(\theta) \sim S(\alpha, 1, 1, 0)$ is independent of $d\xi(\theta')$ for all $\theta' \neq \theta$.

If $\boldsymbol{x}$ has such a bivariate stable distribution, and $\boldsymbol{a} = (a_1, a_2)'$ is a vector of constants,

$$\boldsymbol{a}'\boldsymbol{x} = \int_0^{2\pi} (a_1 \cos\theta + a_2 \sin\theta) \frac{(d\Gamma(\theta))^{1/\alpha} d\xi(\theta)}{(d\theta)^{1/\alpha}} \tag{19}$$

is univariate stable. By (5) and (6), $\boldsymbol{a}'\boldsymbol{x}$ will have scale determined by

$$c^\alpha(\boldsymbol{a}'\boldsymbol{x}) = \int_0^{2\pi} |a_1 \cos\theta + a_2 \sin\theta|^\alpha d\Gamma(\theta) \ . \tag{20}$$

M. Kanter (as reported by Hardin et al. 1991) showed in 1972 that if $d\Gamma$ is symmetric and $\alpha > 1$,

$$\mathrm{E}(x_2|x_1) = \kappa_{2,1} x_1 \ , \tag{21}$$

where, setting $x^{\langle a \rangle} = \mathrm{sign}(x)|x|^a$,

$$\kappa_{2,1} = \frac{1}{c^\alpha(x_1)} \int_0^{2\pi} \sin\theta (\cos\theta)^{\langle \alpha - 1 \rangle} d\Gamma(\theta) \ , \tag{22}$$

$$c^\alpha(x_1) = \int_0^{2\pi} |\cos\theta|^\alpha d\Gamma(\theta) \ . \tag{23}$$

The integral in (22) is called the *covariation of* x_2 *on* x_1. Hardin et al. (1991) demonstrate that if $d\Gamma$ is asymmetrical, $\mathrm{E}(x_2|x_1)$ is non-linear in x_1, but still is a simple function involving this $\kappa_{2,1}$. They note that (21) may be valid in the symmetric cases even for $\alpha < 1$.

If $d\Gamma$, and therefore the distribution of $\boldsymbol{x}$, is symmetric, $\psi_{\alpha 1}(\boldsymbol{s}'\boldsymbol{t})$ in (16) and (17) may be replaced by $\psi_{\alpha 0}(\boldsymbol{s}'\boldsymbol{t}) = -|\boldsymbol{s}'\boldsymbol{t}|^\alpha$, and $d\xi(\theta)$ in (18) taken to be symmetric. In this case, the integrals may be taken over any half of S_d, provided Γ is doubled.

One particularly important special case of MV stable distributions is the *elliptical* class emphasized by Press (1982: 158, 172–3).[6] If $d\Gamma(\boldsymbol{s})$ in (16) simply equals a constant times $d\boldsymbol{s}$, all directions will make equal contributions to $\boldsymbol{x}$. Such a distribution will, after appropriate scaling to give the marginal distribution of each component the desired scale, have spherically symmetrical joint density $f(\boldsymbol{x}) = \phi_{\alpha d}(r)$, for some function $\phi_{\alpha d}(r)$ depending only on $r = ||\boldsymbol{x}||$, α, and the dimensionality d of $\boldsymbol{x}$. The log c.f. of such a distribution must be propor-

[6] The particular case presented here is Press's "order m" = 1. His higher order cases (with his $m > 1$) are not so useful. In (1972), Press asserted that these were the most general MV symmetric stable distributions, but in (1982: 158) concedes that this is not the case.

tional to $\psi_{\alpha 0}(||\boldsymbol{t}||) = -(\boldsymbol{t}'\boldsymbol{t})^{\alpha/2}$. Such a spherical stable distribution is also called *isotropic*.

Press prefers to select the scale factor for spherical MV stable distributions in such a way that in the standard spherical normal case, the variance of each component is unity. The univariate counterpart of this would be to replace c in (3) by $\sigma/2^{1/\alpha}$. If this is done, the *normalized scale* σ then equals $2^{1/\alpha}c$, and equals the standard deviation when $\alpha = 2$.[7] Accordingly, Press specifies what we call the *standard normalized spherical stable* log c.f. to be

$$\log \mathrm{E}e^{i\boldsymbol{x}'\boldsymbol{t}} = \psi_{\alpha 0}(||\boldsymbol{t}||)/2 = -(\boldsymbol{t}'\boldsymbol{t})^{\alpha/2}/2 \ . \tag{24}$$

In the case $d = 2$ of (17) and (18), the requisite constant value of $d\Gamma$ is, by (23),

$$d\Gamma(\theta) = \left(2\int_0^{2\pi} |\cos\omega|^\alpha d\omega\right)^{-1} d\theta \ .$$

If $\boldsymbol{z}$ has such a d-dimensional spherical stable distribution, and $\boldsymbol{x} = \boldsymbol{H}\boldsymbol{z}$ for some non-singular $d \times d$ matrix $\boldsymbol{H}$, then $\boldsymbol{x}$ will have a *d-dimensional (normalized) elliptical stable distribution* with log c.f.

$$\log \mathrm{E}\exp(i\boldsymbol{x}'\boldsymbol{t}) = -(\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t})^{\alpha/2}/2 \tag{25}$$

and joint density

$$f(x) = |\boldsymbol{\Sigma}|^{-1/2}\phi_{\alpha d}\left((\boldsymbol{x}'\boldsymbol{\Sigma}^{-1}\boldsymbol{x})^{1/2}\right) \ , \tag{26}$$

where $\boldsymbol{\Sigma} = (\sigma_{ij}) = \boldsymbol{H}\boldsymbol{H}'$. Component x_i of $\boldsymbol{x}$ will then have normalized scale $\sigma(x_i) = \sigma_{ii}^{1/2} = 2^{1/\alpha}c(x_i)$. $\boldsymbol{\Sigma}$ thus acts much like the MV normal covariance matrix, which indeed it is for $\alpha = 2$. For $\alpha > 1$, $\mathrm{E}(x_i|x_j)$ exists and equals $(\sigma_{ij}/\sigma_{jj})x_j$.[8] If $\boldsymbol{\Sigma}$ is diagonal, the components of $\boldsymbol{x}$ will be uncorrelated, in the sense $\mathrm{E}(x_i|x_j) = 0$, but *not* independent unless $\alpha = 2$.

A symmetric stable random variable C with distribution $S(\alpha, 0, c, 0)$ may be obtained as the product $BA^{2/\alpha}$, where A is distributed $S(\alpha/2, 1, c^*, 0)$ and B is distributed $S(2, 0, c, 0)$, with $c^* = (\cos(\pi\alpha/4))^{2/\alpha}$ (Samorodnitsky and Taqqu 1994: 20–21). Furthermore, if B is a spherically distributed d-vector whose components are $S(2, 0, c, 0)$, then C is also a spherically distributed d-vector, with components that are marginally $S(\alpha, 0, c, 0)$. Setting $P(||C|| < r) = P(||B||A^{2/\alpha} < r)$ then implies that our density generating function may be computed from a maximally skewed univariate stable density (see McCulloch and Panton, in press) as

[7] Ledoux and Talagrand (1991: 123) in effect make this substitution in the univariate case. We follow the traditional parameterization here, except in the MV elliptical case.

[8] Wu and Cambanis (1991) demonstrate that $\mathrm{var}(x_i|x_j)$ actually exists in cases like this.

$$\phi_{\alpha d}(r) = \frac{\alpha}{2c^*(4\pi c^2)^{d/2}} \int_0^\infty \exp\left(-\frac{r^2}{4c^2x^2}\right) x^{\alpha/2-1} s_{\alpha/2,1}(x^{\alpha/2}/c^*)dx \tag{27}$$

where $c = 2^{-1/\alpha}$ for the Press normalization. (See also Zolotarev (1981))

3. Stable portfolio theory

Tobin (1958) noted that preferences over probability distributions for wealth w can be expressed by a two-parameter indirect utility function if all distributions under consideration are indexed by these two parameters. He further demonstrated that if utility $U(w)$ is a concave function of wealth and this two-parameter class is affine, i.e. indexed by a location and scale parameter like the stable δ and c, the indirect utility function $V(\delta, c)$ generated by expected utility maximization must be quasi-concave, while the opportunity sets generated by portfolios of risky assets and a risk-free asset will be straight lines. Furthermore, if such a two-parameter affine class is closed under addition, convex portfolios of assets will be commensurate using the same quasi-concave indirect utility function. If the class is symmetrical, even non-convex portfolios, with short sales of some assets, may be thus compared. The normal distribution of course has this closure property, as do all the stable distributions (Samuelson 1967).[9]

Fama and Miller (1972: 259–74, 313–319) show that the conclusions of the traditional Capital Asset Pricing Model (CAPM) carry over to the special class of MV SS distributions in which the relative arithmetic return $R_i = (P_i(t+1) - P_i(t))/P_i(t)$ on asset i is generated by the "market model":

$$R_i = a_i + b_i M + \varepsilon_i \ , \tag{28}$$

where a_i and b_i are asset-specific constants, $M \sim S(\alpha, 0, 1, 0)$ is a market-wide factor affecting all assets, and $\varepsilon_i \sim S(\alpha, 0, c_i, 0)$ is an asset-specific disturbance independent of M and across assets.

Under (28), the returns $\boldsymbol{R} = (R_1, \ldots R_N)'$ on N assets have an $N+1$-atom MV SS distribution of form (14), generated by

$$\boldsymbol{R} = \boldsymbol{a} + (\boldsymbol{b} \ \ \boldsymbol{I}_N)\begin{pmatrix} M \\ \boldsymbol{\varepsilon} \end{pmatrix} \ , \tag{29}$$

where $\boldsymbol{a} = (a_1, \ldots a_N)'$, etc. This distribution has N symmetrical atoms aligned with each axis, along with an $N+1$st extending into the positive orthant.

FM show that when $\alpha > 1$, diversification will reduce the effect of the firm-specific risks, as in the normal case, though at a slower rate. They note that if two different portfolios of such assets are mixed in proportions x and $(1-x)$, the scale

[9] Owen and Rabinovitch (1983) show that the general class of *elliptical* distributions also shares this property. However, except for the elliptical stable distributions, these cannot arise from the accumulation of iid shocks, and have no compelling rationale.

of the mixed portfolio will be a strictly convex function of x and therefore (providing the two portfolios have different mean returns) of its mean return. On the efficient set of portfolios, where mean is an increasing function of scale, maximized mean return will therefore be a concave function of scale, as in the normal case. Given Tobin's quasi-concavity of the indirect utility function, a tangency between the efficient frontier and an indirect utility indifference curve then implies a global expected utility maximum for an individual investor.

When trading in an artificial asset paying a riskless real return R_f is introduced, all agents will choose to mix positive or negative quantities of the risk-free asset with the market portfolio, as in the normal case. Letting $\boldsymbol{\theta} = (\theta_1, \ldots \theta_N)'$ represent the shares of the N assets in the market portfolio, the market return will be given by,

$$R_m = \boldsymbol{\theta}'\boldsymbol{R} = a_m + b_m M + \varepsilon_m \ , \tag{30}$$

where $a_m = \boldsymbol{\theta}'\boldsymbol{a}, b_m = \boldsymbol{\theta}'\boldsymbol{b}$, and $\varepsilon_m = \boldsymbol{\theta}'\varepsilon$. Thus, $(R_m, R_i)'$ will have a three-atom BV SS distribution generated by

$$\begin{pmatrix} R_m \\ R_i \end{pmatrix} = \begin{pmatrix} b_m & 1 & \theta_i \\ b_i & 0 & 1 \end{pmatrix} \begin{pmatrix} M \\ \varepsilon_{\hat{i}} \\ \varepsilon_i \end{pmatrix} \ , \tag{31}$$

where $\varepsilon_{\hat{i}} = \varepsilon_m - \theta_i \varepsilon_i$. The variability of R_m will be given by

$$c^\alpha(R_m) = b_m^\alpha + c^\alpha(\varepsilon_m) \ , \tag{32}$$

where $c^\alpha(\varepsilon_m) = \sum \theta_i^\alpha c_i^\alpha$ is the contribution of the firm-specific risks to the risk of the market portfolio.

The conventional CAPM predicts that the prices of the N assets, and therefore their mean returns a_i, will be determined by the market in such a way that

$$\mathrm{E}R_i - R_f = (ER_m - R_f)\beta_{\mathrm{CAPM}} \ , \tag{33}$$

where the CAPM "β" (not to be confused with the stable "β") is ordinarily computed as

$$\beta_{\mathrm{CAPM}} = \mathrm{cov}(R_i, R_m)/\mathrm{var}(R_m) \ . \tag{34}$$

This variance and covariance are both infinite for $\alpha < 2$. However, FM point out that the market equilibrium condition in fact only requires a) that the market portfolio be an efficient portfolio and therefore minimize its *scale* given its mean return, and b) that in $(\mathrm{E}(R), c(R))$ space, the slope of the efficient set at the market portfolio equal $(\mathrm{E}R_m - R_f)/c(R_m)$. They note that these in turn imply (33), with

$$\beta_{\mathrm{CAPM}} = \frac{1}{c(R_m)} \frac{\partial c(R_m)}{\partial \theta_i} \ . \tag{35}$$

In the finite variance case, (35) yields (34), but the variance and covariance are in fact inessential.

In the market model of (28), FM show that (35) becomes[10]

$$\beta_{\mathrm{CAPM}} = \frac{b_i b_m^{\alpha-1} + \theta_i^{\alpha-1} c_i^{\alpha}}{c^{\alpha}(R_m)} \ . \tag{36}$$

As $\theta_i \downarrow 0, c(R_m) \downarrow b_m$, and hence $\beta_{\mathrm{CAPM}} \rightarrow b_i/b_m$. FM did not explore more general MV stable distributions, other than to suggest (p. 269) adding industry-specific factors to (28).

Press (1982: 379–81) demonstrates that portfolio analysis with elliptical MV stable distributions is even simpler than in the multi-atom model of FM. Let $\boldsymbol{R} - \mathrm{E}\boldsymbol{R}$ have a normalized elliptical stable distribution with log c.f. (25) and $N \times N$ covariation matrix $\boldsymbol{\Sigma}$. Then the 2×2 covariation matrix $\boldsymbol{\Sigma}^*$ of $(R_m, R_i)'$ will be

$$\boldsymbol{\Sigma}^* = \begin{pmatrix} \sigma_m^2 & \sigma_{im} \\ \sigma_{im} & \sigma_i^2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\theta}' \\ \boldsymbol{e}_i' \end{pmatrix} \boldsymbol{\Sigma} (\boldsymbol{\theta} \ \boldsymbol{e}_i) \ , \tag{37}$$

where $\boldsymbol{e}_i$ is the i^{th} unit N-vector. It can easily be shown that (35) implies

$$\beta_{\mathrm{CAPM}} = \sigma_{im}/\sigma_m^2 \ . \tag{38}$$

In the general symmetric MV stable case, not considered by either Fama and Miller or Press, $\boldsymbol{x} = (R_m - \mathrm{E}R_m, R_i - \mathrm{E}R_i)'$ will have a bivariate symmetric stable distribution of the type (17). It then may readily be shown that the Fama-Miller rule (35) implies

$$\beta_{\mathrm{CAPM}} = \kappa_{im} \ , \tag{39}$$

where $\kappa_{im} = \mathrm{E}(R_i - \mathrm{E}R_i | R_m - \mathrm{E}R_m)/(R_m - \mathrm{E}R_m)$ is as given by Kanter's formula (22) above. This generalized formulation of the stable CAPM was first noted by Gamrowski and Rachev (1994, 1995).

The possibility that $\alpha < 2$ therefore adds no new difficulties to the traditional CAPM. However, we are still left with its *original* problems. One of these is that it assumes that there is a *single* consumption good consumed at a single point in time. If there are several goods with variable relative prices, or several points in time with a non-constant real interest rate structure, there may in effect be different CAPM β's for different types of consumption risk.

A second problem with the CAPM is that if *arithmetic* returns have a stable distribution with $\alpha > 1$ and $c > 0$, there is a positive probability that any individual stock price, or even wealth and therefore consumption as a whole, will go negative. Ziemba (1974) considers restrictions on the utility function that will keep expected utility and expected marginal utility finite under these circumstances, but a non-negative distribution would be preferred, given free disposal and limited liability, not to mention the difficulty of negative consumption. A further complication is that it is more reasonable to assume that relative, rather than absolute, arithmetic returns are homoskedastic over time. Yet if relative one-

[10] This follows immediately from their (7.51), when the "efficient portfolio" considered there is the market portfolio.

period arithmetic returns have any iid distribution, then over multiple time periods they will accumulative *multiplicatively*, not additively as required to retain a stable distribution.

A normal or stable distribution for *logarithmic* asset returns, $\log(P_i(t+1)/P_i(t))$, keeps asset prices non-negative, and could easily arise from the multiplicative accumulation of returns. However, the log-normal or log-stable is no longer an *affine* two-parameter class of distributions, and so Tobin's demonstration of the quasi-concavity of the indirect utility function may no longer be invoked. Furthermore, while the closure property of stable distributions under addition implies that log-normal and log-stable distributions are closed under *multiplication*, as may take place for an individual stock over time, it does not imply that they are closed under *addition*, as takes place under portfolio formation. A portfolio of log-normal or log-stable stocks therefore does not necessarily have a distribution in the same class. As a consequence, such portfolios may not be precisely commensurate in terms of *any* two-parameter indirect utility function, whether quasi-concave or not.

Conceivably, two random variables might have a joint distribution with log-stable marginals, whose contours are somehow deformed in such a way that *linear* combinations of them are nevertheless still log-stable. However, Boris Mityagin (in McCulloch and Mityagin 1991) has shown that this cannot be the case if the *log*-stable marginal distributions have finite mean, i.e. $\alpha = 2$ or $\beta = -1$. This result makes it highly unlikely that the infinite mean cases would have the desired property, either.

In the Gaussian case, the latter set of problems has been avoided by focussing on continuous time Wiener processes, for which negative outcomes may be ruled out by a log-normal assumption, but for which instantaneous logarithmic and relative arithmetic returns differ only by a drift term governed by Itô's lemma. With $\alpha < 2$, however, the discontinuities in continuous-time stable processes make even instantaneous logarithmic and relative arithmetic returns behave fundamentally differently.

It therefore appears that the stable CAPM, like the Gaussian CAPM, provides at best only an approximation to the equilibrium pricing of risky assets. There is, after all, nothing in theory that guarantees that asset pricing will actually have the simplicity and precision that was originally sought in the two-parameter asset pricing model.

4. Log-stable option pricing[11]

An *option* is a derivative financial security that gives its owner the right, but not the obligation, to buy or sell a specified quantity of an underlying asset at a contractual price called the *striking price* or *exercise price*, within a specified period of time. An option to buy is a *call* option, while an option to sell is a *put*

[11] This section draws heavily on, and supplants, McCulloch (1985b).

option. If the option may only be exercised on its maturity date it is said to be *European*, while if it may be exercised at any time prior to its final maturity it is said to be *American*. In practice, most options are "American," but "European" options are easier to evaluate, and under some circumstances the two will have equal value.

Black and Scholes (BS; 1973) find a precise formula for the value of a European option on a stock whose price on maturity has a log-normal distribution, by means of an arbitrage argument involving the a.s. everywhere continuous path of the stock price during the life of the option. Merton (1976) noted early on that deep-in-the money, deep-out-of-the money, and shorter maturity options tend to sell for more than their BS predicted value. Furthermore, if the BS formula were based on the true distribution, implicit volatilities calculated from it using synchronous prices for otherwise identical options with different striking prices would be constant across striking prices. In practice, the resulting implicit volatility curve instead often bends up at the ends, to form what is often referred to as the *volatility smile* (Bates 1996). This suggests that the market, at least, believes that large price movements have a higher probability, relative to small price movements, than is consistent with the log-normal assumption of the BS formula.

The logic of the BS model cannot be adapted to the log-stable case, because of the discontinuities in the time path of an α-stable Lévy process.[12] Furthermore if the log stock price is stable with $\alpha < 2$ and $\beta > -1$, the expected payoff on a call is infinite. This left Paul Samuelson (as quoted by Smith 1976: 19) "inclined to believe in [Robert] Merton's conjecture that a strict Lévy-Pareto [stable] distribution on $\log(S^*/S)$ would lead, with $1 < \alpha < 2$, to a 5-minute warrant or call being worth 100 percent of the common." Merton further conjectured (1976: 127n) that an infinite expected future price for a stock would require the risk free discount rate to be infinite, in order for the current price to be finite.

We show below that these fears are unfounded, even in the extreme case $\alpha < 1$. Furthermore, the value of European options under generalized log-stable uncertainty may be evaluated using fundamental expected utility maximization principles, rather than the BS arbitrage argument or even risk-neutrality.

4.1. Spot and forward asset prices

Let there be two assets, A_1 and A_2, that give a representative household utility $U(A_1, A_2)$, with marginal utilities U_1 and U_2. Let

$$S_T = U_2/U_1 \tag{40}$$

[12] Rachev and Samorodnitsky (1993) attempt to price a log-symmetric stable option, using a hedging argument with respect to the *directions* of the jumps in an underlying α-stable Lévy motion, but not with respect to their *magnitudes*. Furthermore, their hedge ratio is computed as a function of the *still unobserved* magnitude of the jumps. These drawbacks render their formula less than satisfactory, even apart from its difficulty of calculation. Jones (1984) calculates option values for a compound jump/diffusion process in which the jumps, and therefore the process, have infinite variance, but this is neither a stable nor a log-stable distribution.

be the random spot price of A_2 in terms of A_1 at future time T. If $\log U_1$ and $\log U_2$ are both stable with a common characteristic exponent, then $\log S_T$ will also be stable, with the same exponent. It will be apparent from context whether "S" represents the spot price of a security, as generally used in the option pricing literature, or a stable c.d.f.

Let F be the forward price in the market at present time 0 on a contract to deliver 1 unit of A_2 at time T, with unconditional payment of F units of A_1 to be made at time T. The expected utility from a position of size ε in this contract is $\mathrm{E}U(A_1 - \varepsilon F, A_2 + \varepsilon)$. Maximizing over ε and imposing the equilibrium condition $\varepsilon = 0$ yields

$$F = \mathrm{E}U_2/\mathrm{E}U_1 \ . \tag{41}$$

The expectations in (41) are both conditional on present (time 0) information.

In order for the $\mathrm{E}U_i$ to be finite when the $\log U_i$ are stable with $\alpha < 2$, the latter must both be maximally negatively skewed, i.e. have $\beta = -1$, per (9). We presently see no alternative but to make this assumption in order to evaluate log-stable options. However, this restriction does not prevent $\log S_T$ from being intermediately skew-stable, or even SS, since $\log S_T$ may receive an upper Paretian tail from U_2, as well as a lower Paretian tail from U_1, and have intermediate skewness governed by (7).

Let $u_1 \sim S(\alpha, +1, c_1, \delta_1)$ and $u_2 \sim S(\alpha, +1, c_2, \delta_2)$ be independent asset-specific maximally *positively* skewed stable variates contributing *negatively* to $\log U_1$ and $\log U_2$, respectively. In order to add some generality, let $u_3 \sim S(\alpha, +1, c_3, \delta_3)$ be a common component, contributing negatively and equally to both $\log U_1$ and $\log U_2$, and which is independent of u_1 and u_2, so that

$$\log U_1 = -u_1 - u_3 \ , \tag{42}$$

$$\log U_2 = -u_2 - u_3 \ . \tag{43}$$

Let $(\alpha, \beta, c, \delta)$ be the parameters of

$$\log S_T = u_1 - u_2 \ . \tag{44}$$

We assume that α, β, c, and F are known, but that $\delta, c_1, c_2, c_3, \delta_1, \delta_2$, and δ_3 are not directly observed. We have, by (5)–(8),

$$\delta = \delta_1 - \delta_2 \ , \ \alpha \neq 1 \ , \tag{45}$$

$$c^\alpha = c_1^\alpha + c_2^\alpha \ , \tag{46}$$

$$\beta c^\alpha = c_1^\alpha - c_2^\alpha \ . \tag{47}$$

We will return to the case $\alpha = 1$, but for the moment assume $\alpha \neq 1$.

Equations (46) and (47) may be solved for

$$c_1 = ((1+\beta)/2)^{1/\alpha}c\ ,$$
$$c_2 = ((1-\beta)/2)^{1/\alpha}c\ . \tag{48}$$

Using Zolotarev's formula (9) and setting $\theta = \pi\alpha/2$, we have

$$\mathrm{E}U_i = e^{-\delta_i-\delta_3-(c_1^\alpha+c_3^\alpha)\sec\theta}\ , \quad i = 1,2\ , \tag{49}$$

so that (41) gives us

$$F = e^{\delta_1-\delta_2+(c_1^\alpha-c_2^\alpha)\sec\theta} = e^{\delta+\beta c^\alpha\sec\theta} \tag{50}$$

If $\beta = 0$ (because $c_1 = c_2$), (50) implies $\log F \doteq \mathrm{E}\log S_T$. This special case does not require logarithmic utility, but only that U_1 and U_2 make equal contributions to the uncertainty of S_T.

4.2. Option pricing

Let C be the value, in units of A_1 to be delivered unconditionally at time 0, of a European call on 1 unit of asset A_2 to be exercised at time T, with exercise (striking) price X. Let r_1 be the default-free interest rate on loans denominated in A_1 with maturity T. C units of A_1 at time 0 are thus marginally equivalent to C $\exp(r_1T)$ units at T.

If $S_T > X$ at time T, the option will be exercised. Its owner will receive 1 unit of A_2, in exchange for X units of A_1. If $S_T \le X$, the option will not be exercised. In either event, its owner will be out the interest-augmented $C\exp(r_1T)$ units of A_1 originally paid for the option. In order for the expected utility gain from a small position in this option to be zero, we must have

$$\int_{S_T>X}(U_2 - XU_1)dP(U_1,U_2) - Ce^{r_1T}\int_{\text{all } S_T}U_1dp(U_1,U_2) = 0 \tag{51}$$

or, using (41),

$$C = e^{-r_1T}\left[\frac{F}{\mathrm{E}U_2}\int_{S_T>X}U_2dP(U_1,U_2) - \frac{X}{\mathrm{E}U_1}\int_{S_T>X}U_1dP(U_1,U_2)\right]\ . \tag{52}$$

In the above, $P(U_1,U_2)$ represents the joint probability distribution for U_1 and U_2. (52) is valid for any joint distribution for which the expectations exist.

It is shown in the Appendix that for our stable model with $\alpha \ne 1$, (52) becomes

$$C = Fe^{-r_1T+c_2^\alpha\sec\theta}I_1 - Xe^{-r_1T+c_1^\alpha\sec\theta}I_2\ , \tag{53}$$

where, setting $S_{\alpha 1}^c = 1 - S_{\alpha 1}$,

$$I_1 = \int_{-\infty}^{\infty}e^{-c_2z}s_{\alpha 1}(z)S_{\alpha 1}^c\left(\left(c_2z + \log\frac{X}{F} + \beta c^\alpha\sec\theta\right)/c_1\right)dz\ , \tag{54}$$

$$I_2 = \int_{-\infty}^{\infty} e^{-c_1 z} s_{\alpha 1}(z) S_{\alpha 1}\left(\left(c_1 z - \log\frac{X}{F} - \beta c^\alpha \sec\theta\right)\Big/ c_2\right) dz \ . \tag{55}$$

Eq. (53) effectively gives C as a function $C(X, F, \alpha, \beta, c, r_1, T)$, since c_1 and c_2 are determined by (48), and $\theta = \pi\alpha/2$. Note that δ is not directly required, since all we need to know about it is contained in F through (50). The common component of uncertainty, u_3, completely drops out.

Rubinstein (1976) demonstrates that (52) leads to the Black-Scholes formula when $\log U_1$ and $\log U_2$ have a general bivariate normal distribution. Eq. (53) therefore generalizes BS to the case $\alpha < 2$.

If the forward price F is not directly observed, we may use the current spot price S_0 to construct a proxy for it if we know the default-free interest rate r_2 on A_2-denominated loans, since arbitrage requires

$$F = S_0 e^{(r_1 - r_2)T} \ . \tag{56}$$

The value P of a European *put* option giving one the right to *sell* 1 unit of A_2 at striking price X at future time T may be evaluated by (53), along with the put-call parity arbitrage condition

$$P = C + (X - F)e^{-r_1 T} \ . \tag{57}$$

Equations (50) and (53) are valid even for $\alpha < 1$. When $\alpha = 1$, (50) and (53) become

$$F = e^{\delta - (2/\pi)\beta c \log c} \ , \tag{58}$$

$$C = F e^{-r_1 T - (2/\pi) c_2 \log c_2} I_1 - X e^{-r_1 T - (2/\pi) c_1 \log c_1} I_2 \ , \tag{59}$$

where c_1 and c_2 are as in (48), but now,

$$I_1 = \int_{-\infty}^{\infty} e^{-c_2 z} s_{11}(z) S_{11}^c\left(\left(c_2 z + \log\frac{X}{F} + \frac{2}{\pi}(c_2 \log c_2 - c_1 \log c_1)\right)\Big/ c_1\right) dz \ , \tag{60}$$

$$I_2 = \int_{-\infty}^{\infty} e^{-c_1 z} s_{11}(z) S_{11}\left(\left(c_1 z - \log\frac{X}{F} - \frac{2}{\pi}(c_2 \log c_2 - c_1 \log c_1)\right)\Big/ c_2\right) dz \ . \tag{61}$$

4.3. *Applications*

The stable option pricing formula (53) may be applied without modification to options on commodities, stocks, bonds, and foreign exchange rates, simply by appropriately varying the interpretation of the two assets A_1 and A_2.

4.3.a. Commodities

Let A_1 and A_2 be two consumption goods, both available for consumption on some future date T. A_1 could be an aggregate of all goods other than A_2. Let r_1 be the default-free interest rate on A_1-denominated loans. Let U_1 and U_2 be the random future marginal utilities of A_1 and A_2, and suppose that $\log U_1$ and $\log U_2$ have both independent (u_1 and u_2) and common (u_3) components, as in (42) and (43). The price S_T of A_2 in terms of A_1, as determined by (40), is then log-stable as in (44), with current forward price F as in (50). The price C of a call on 1 unit of A_2 at time T is then given by (53) above.

Such a scenario might, for example, arise from an additively separable CRRA utility function

$$U(A_1, A_2) = \frac{1}{1-\eta}(A_1^{1-\eta} + A_2^{1-\eta}),\ \eta > 0,\ \eta \neq 1\ , \tag{62}$$

with the physical endowments given by $A_i = e^{v_i + v_3}, i = 1, 2$, where v_1, v_2 and v_3 are independent stable variates with a common α and $\beta = +1$.

4.3.b. Stocks

Suppose now that there is a single good G, which serves as our numeraire, A_1. Let A_2 be a share of stock in a firm that produces a random amount y of G per share at T. Let r_1 be the default-free interest rate on G-denominated loans with maturity T. The firm pays continuous dividends, in stock, at rate r_2, and its stock has no valuable voting rights before time T, so that one share for spot delivery is equivalent to $\exp(r_2 T)$ shares at T. Let U_G be the random future marginal utility of one unit of G at time T, and suppose that

$$\log U_G = -u_1 - u_3\ , \tag{63}$$

$$\log y = u_1 - u_2\ , \tag{64}$$

where the $u_i \sim S(\alpha, +1, c_i, \delta_i)$ are independent.

The marginal utility of one share is then $yU_G = \exp(-u_2 - u_3)$, and the stock price per share using unconditional claims on G as numeraire, $S_T = (yU_G)/U_G$, is as in (44) above. The forward price of one share, $F = \mathrm{E}(yU_G)/\mathrm{E}(U_G)$, is as in (50) above. The value of a European call on 1 share at exercise price X is then given by (53). If the forward price of the stock is not directly observed, it may be constructed from r_1, r_2, and the current spot stock price S_0 by (56).

Equation (64) states that to the extent there is firm-specific good news ($-u_2$), it is assumed to have no upper Paretian tail. This means that the firm will produce a fairly predictable amount if successful, but may still be highly speculative, in the sense of having a significant probability of producing much less or virtually nothing at all. To the extent there is firm non-specific good news (u_1), the marginal utility of G, given by (63), is assumed to be correspondingly reduced. Despite this admittedly restrictive scenario, the stock price S_T can take on a completely general log-stable distribution, with any permissible α, β, c, or δ.

Note that in terms of expected arithmetic returns, the population equity premium is infinite for a log-stable stock, unless $\beta = -1$.

4.3.c. Bonds[13]

Now suppose that there is a single consumption good, G, that may be available at each of two future dates, $T_2 > T_1 > 0$. Let A_1 and A_2 be unconditional claims on one unit of G at T_1 and T_2, resp., and let U_1 and U_2 be the marginal utility of G at these two dates. Let $E_1 U_2$ be the expectation of U_2 as of T_1. As of present time 0, both U_1 and $E_1 U_2$ are random. Assume $\log U_1 = -u_1 - u_3$ and $\log E_1 U_2 = -u_2 - u_3$, where the u_i are independently $S(\alpha, +1, c_i, \delta_i)$. The price at T_1 of a bond that pays 1 unit of G at T_2, $B(T_1, T_2) = E_1 U_2 / U_1$, is then given by (44) above, and the current forward price F of such a bond implicit in the term structure at present time 0, $F = B(0, T_2)/B(0, T_1) = E_0 U_2 / E_0 U_1 = E_0(E_1 U_2)/\ E_0 U_1$, is governed by (50) above.[14] The price of a European call is then given by (53) above, where r_1 is now the time 0 real interest rate on loans maturing at time T_1, and "T" is replaced by T_1.

4.3.d. Foreign exchange rates[15]

To the extent that real exchange rates fluctuate, they may simply be modeled as real commodity price fluctuations, as in Subsection 4.3.a above. However, the purchasing power parity (PPP) model of exchange rate movements provides an instructive alternative interpretation of the stable option model, in terms of purely *nominal risks*.

Let P_1 and P_2 be the price levels in countries 1 and 2 at future time T. Price level uncertainty itself is generally positively skewed. Astronomical inflations are easily arranged, simply by throwing the printing presses into high gear, and this policy has considerable fiscal appeal. Comparable deflations would be fiscally intolerable, and are in practice unheard of. It is therefore particularly reasonable to assume that $\log P_1$ and $\log P_2$ are both maximally positively skewed.

Let u_1 and u_2 be independent country specific components of $\log P_1$ and $\log P_2$, respectively, and let u_3 be an international component of both price levels, re-

[13] McCulloch (1985a) uses the results of this section, in the short-lived limit treated below, to evaluate deposit insurance in the presence of interest-rate risk.

[14] This model leads to the Log Expectation Hypothesis $\log F = E \log B(T_1, T_2)$ when $\beta = 0$. McCulloch (1993) demonstrates with a counterexample that the 1981 claim of Cox et al., that this necessarily violates a no-arbitrage condition in continuous time with $\alpha = 2$, is invalid. The requisite forward price F may be computed as $\exp(r_1 T_1 - R_2 T_2)$, where R_2 is the time 0 real interest rate on loans maturing at T_2.

[15] The present subsection draws heavily on McCulloch (1987), q.v. for extensions. Eq. (12.18) of that paper contains an error which is corrected in Eq. (56) of the present paper.

flecting the "herd instincts" of central bankers, that is independent of both u_1 and u_2, so that $\log P_i = u_i + u_3, i = 1,2$. Let S_T be the exchange rate giving the time T value of currency 2 (A_2) in terms of currency 1 (A_1). Under PPP, $S_T = P_1/P_2$ is then as given in (44) above.

The lower Paretian tail of $\log X$ will give the density of X itself a mode (with infinite density but no mass) at 0, as well as a second mode (unless c is large relative to unity) near $\exp(\mathrm{E} \log X)$. Thus log-stable distributions achieve the bimodality sought by Krasker (1980) to explain the "peso problem," all in terms of a single story about the underlying process, requiring as few as three parameters (if log-symmetric).

Assuming that inflation uncertainty involves no systematic risk, the forward exchange rate F must equal $\mathrm{E}(1/P_2)/\mathrm{E}(1/P_1)$ in order to set expected profits in terms of purchasing power equal to zero, and will be determined by (50) above. Let r_1 and r_2 be the default-free nominal interest rates in countries 1 and 2. Then the shadow price of a European call on one unit of currency 2 that sets the expected purchasing power gain from a small position in the option equal to zero is given by (53). The forward price F may, if necessary, be inferred from the current spot price S_0 by means of covered interest arbitrage (56).

4.3.e. *Pseudo-hedge ratio*

The risk exposure from writing a call on one unit of an asset can be partially neutralized (to a first-order approximation) by simultaneously taking a long forward position on

$$\frac{\partial(C \exp(r_1 T))}{\partial F} = e^{c_2^\alpha \sec \theta} I_1 \tag{65}$$

units of the underlying asset. Unfortunately, the discontinuities leave this position imperfectly hedged if $\alpha < 2$. At the same time, this imperfect ability to hedge implies that options are not redundant financial instruments.

4.4. *Put/call inversion and in/out duality*

$C(X, F, \alpha, \beta, c, r_1, T)$ in equation (53) above may be written as

$$C(X, F, \alpha, \beta, c, r_1, T) = e^{-r_1 T} F C^* \left(\frac{X}{F}, \alpha, \beta, c \right) , \tag{66}$$

where $C^*(X/F, 1, \alpha, \beta, c) = C(X/F, \alpha, \beta, c, 0, 1)$ (cp. Merton 1976: 139). Similarly, the value of a put on 1 unit of A_2 may be written as

$$P(X, F, \alpha, \beta, c, r_1, T) = e^{-r_1 T} F P^* \left(\frac{X}{F}, \alpha, \beta, c \right) , \tag{67}$$

where, using (57),

$$P^*\left(\frac{X}{F}, \alpha, \beta, c\right) = P\left(\frac{X}{F}, 1, \alpha, \beta, c, 0, 1\right)$$
$$= C^*\left(\frac{X}{F}, \alpha, \beta, c\right) + \frac{X}{F} - 1 \ . \tag{68}$$

Now a call on 1 unit of A_2 at exercise price X [units A_1/unit A_2] is the same contract as a put on X units of A_1 at exercise price $1/X$ [units A_2/unit A_1]. The value of the latter, in units of A_2 for spot delivery, is $XP(1/X, 1/F, \alpha, -\beta, c, r_2, T)$, since the forward price measured in units of A_2 is $1/F$, and since $\log 1/S_T$ has parameters $\alpha, -\beta$ and c. Multiplying by the current spot price S_0 so as to give units of A_1 for spot delivery, we have the *put-call inversion relationship,*

$$C(X, F, \alpha, \beta, c, r_1, T) = S_0 XP\left(\frac{1}{X}, \frac{1}{F}, \alpha, -\beta, c, r_2, T\right) \ . \tag{69}$$

Using (57) and (68), this implies the following *in/out of the money duality relationship:*

$$C^*\left(\frac{X}{F}, \alpha, \beta, c\right) = \frac{X}{F} P^*\left(\frac{F}{X}, \alpha, -\beta, c\right)$$
$$= \frac{X}{F} C^*\left(\frac{F}{X}, \alpha, -\beta, c\right) - \frac{X}{F} + 1 \ . \tag{70}$$

Puts and calls for all interest rates, maturities, forward prices, and exercise prices may therefore be evaluated from $C^*(X/F, \alpha, \beta, c)$ for $X/F \geq 1$.

4.5. Numerical option values

Table 1 gives illustrative values of 100 $C^*(X/F, \alpha, \beta, c)$.[16] This is the interest-incremented value, in terms of A_1, of a European call on an amount of A_2 equal in value (at the forward price) to 100 units of A_1. E.g., if A_1 is the dollar and A_2 is a stock, the table gives the value, in dollars and cents to be paid at the maturity of the option, of a call on \$100 worth of stock.

Panel a of Table 1 holds α and β fixed at 1.5 and 0.0, while c and X/F vary. The call value declines with X/F, and increases with c. The reader may confirm that the first and last columns satisfy (70).

Panels 1b–d hold c fixed at 0.1 and allow α and β to vary for three values of X/F representing "at the money" (in terms of the forward, not spot, price) with $X/F = 1.0$; "out of the money" but still on the shoulder of the distribution with $X/F = 1.1$; and "deep out of the money" with $X/F = 2.0$. When $\alpha = 2$, β has no effect

[16] The requisite skew-stable distribution and density may obtained from the tables of McCulloch and Panton (in press), though Table 1 was based on cubic interpolation off the earlier tables of DuMouchel (1971). See McCulloch (1985b) for details. Option values are tabulated extensively in McCulloch (1984).

Table 1
$100C^*(X/F, \alpha, \beta, c)$

a) $\alpha = 1.5,\ \beta = 0.0$

c	X/F 0.5	1.0	1.1	2.0
0.01	50.007	0.787	0.079	0.014
0.03	50.038	2.240	0.458	0.074
0.10	50.240	6.784	3.466	0.481
0.30	51.704	17.694	14.064	3.408
1.00	64.131	45.642	43.065	28.262

b) $c = 0.1, X/F = 1.0$

α	β −1.0	−0.5	0.0	0.5	1.0
2.0	5.637	5.637	5.637	5.637	5.637
1.8	6.029	5.993	5.981	5.993	6.029
1.6	6.670	6.523	6.469	6.523	6.670
1.4	7.648	7.300	7.157	7.300	7.648
1.2	9.115	8.455	8.137	8.455	9.115
1.0	11.319	10.200	9.558	10.200	11.319
0.8	14.685	12.893	11.666	12.893	14.685

c) $c = 0.1, X/F = 1.1$

α	β −1.0	−0.5	0.0	0.5	1.0
2.0	2.211	2.211	2.211	2.211	2.211
1.8	2.271	2.423	2.590	2.764	2.944
1.6	2.499	2.772	3.123	3.510	3.902
1.4	2.985	3.303	3.870	4.530	5.175
1.2	3.912	4.116	4.943	5.957	6.924
1.0	5.605	5.391	6.497	8.002	9.410
0.8	8.596	7.516	8.803	11.019	13.067

d) $c = 0.1, X/F = 2.0$

α	β −1.0	−0.5	0.0	0.5	1.0
2.0	0.000[a]	0.000[a]	0.000[a]	0.000[a]	0.000[a]
1.8	0.000	0.055	0.110	0.165	0.220
1.6	0.000	0.160	0.319	0.477	0.634
1.4	0.000	0.351	0.695	1.032	1.361
1.2	0.000	0.691	1.354	1.991	2.604
1.0	0.000	1.287	2.488	3.619	4.689
0.8	0.000	2.333	4.438	6.372	8.164

Note: [a]Actual value 1.803×10^{-6} rounds to 0.000.

on the option value, even though the underlying story in terms of the two marginal utilities is changing.[17]

Implicit parameter values may be numerically computed from market option values by means of the stable option formulas above. If β is assumed to be 0, this may be done by using the synchronous prices of two otherwise identical options with different striking prices. McCulloch (1987) shows, using actual quotations on the DM for 9/17/84, how this may be done graphically. The rounding error in the two quotations used accommodated a range of (1.766, 1.832) for α, and a range of (0.0345, 0.0365) for c. The market clearly did not believe the DM was log-normal on this arbitrarily chosen date. If asymmetry is not assumed away, three option values may be used to calculate implicit values of α, β, and c.

4.5. Low probability and short-lived options

Assume $X > F$ and that c is small relative to $\log(X/F)$. Holding β constant, c_1 and c_2 are then small as well. Equation (2) then implies (see McCulloch 1985b for details) that the call value C behaves like

$$Fe^{-r_1 T}c^{\alpha}(1+\beta)\Psi(\alpha, X/F) \ , \tag{71}$$

where

$$\Psi(\alpha,x) = \frac{\Gamma(\alpha)\sin\theta}{\pi}\left[(\log x)^{-\alpha} - \alpha x\int_{\log(x)}^{\infty} e^{-\zeta}\zeta^{-\alpha-1}d\zeta\right] \ . \tag{72}$$

This function is tabulated in some detail in Table 2. It becomes infinite as $x \downarrow 1$, and 0 as $\alpha \uparrow 2$. By the put/call inversion formula (69) (with the roles of C and P reversed), P behaves like

$$Xe^{-r_1 T}c^{\alpha}(1-\beta)\Psi(\alpha, F/X) \ . \tag{73}$$

In an α-Stable Lévy Motion, the scale that accumulates in T time units is $c_0 T^{1/\alpha}$. As $T \downarrow 0$, the forward price F converges on the spot price S_0. Therefore

$$\lim_{T\downarrow 0}(C/T) = S_0(1+\beta)c_0^{\alpha}\Psi(\alpha, X/S_0) \ , \tag{74}$$

$$\lim_{T\downarrow 0}(P/T) = X(1-\beta)c_0^{\alpha}\Psi(\alpha, S_0/X) \ . \tag{75}$$

Eq. (75) has been employed by McCulloch (1981, 1985a) to evaluate the put option implicit in deposit insurance for banks and thrifts that are exposed to

[17] The values for $\alpha = 2$ reported here were, as a check, computed independently by the same numerical procedure used to obtain the sub-Gaussian values, and then checked against the Black-Scholes formula, with $\sigma = c\sqrt{2}$. Using the approximation $1 - N(x) \approx n(x)/x$ for large x, the BS formula becomes $C^* = N(d_1) - XN(d_2)F \approx \sigma N(d_1)/(d_1 d_2)$ for large values of $\log(X/F)/c$, where $d_1 = -\log(X/F)/\sigma + \sigma/2, d_2 = d_1 - \sigma, n(x) = N'(x)$, and F is determined by (56).

Table 2
$\Psi(\alpha, x)$

α \ $x = X/F$	1.001	1.01	1.02	1.04	1.06	1.10	1.15	1.20	1.40	2.00	4.00	10.00
2.00	0.00	0.000	0.000	0.000	0.000	.000	.000	.000	.000	.0000	.0000	.0000
1.95	18.10	1.962	0.989	0.492	0.324	.190	.124	.091	.043	.0168	.0062	.0028
1.90	26.43	3.199	1.665	0.854	0.573	.343	.227	.169	.082	.0329	.0126	.0059
1.80	28.38	4.275	2.369	1.291	0.896	.560	.382	.291	.149	.0633	.0256	.0125
1.70	23.13	4.319	2.544	1.471	1.056	.688	.484	.376	.203	.0914	.0391	.0199
1.60	17.01	3.916	2.448	1.498	1.112	.753	.547	.434	.246	.1172	.0531	.0282
1.50	11.93	3.365	2.227	1.441	1.103	.777	.582	.471	.280	.1411	.0676	.0375
1.40	8.22	2.812	1.966	1.341	1.059	.774	.596	.492	.306	.1634	.0827	.0479
1.30	5.65	2.319	1.707	1.225	0.995	.753	.597	.503	.327	.1842	.0985	.0594
1.20	3.92	1.904	1.471	1.106	0.923	.724	.589	.505	.343	.2039	.1150	.0723
1.10	2.77	1.567	1.266	0.995	0.852	.689	.575	.502	.356	.2227	.1325	.0868
1.00	2.02	1.300	1.092	0.894	0.784	.654	.558	.496	.366	.2411	.1511	.1031
0.90	1.51	1.090	0.949	0.806	0.722	.619	.541	.489	.375	.2592	.1710	.1215

interest rate risk, using SS ML estimates of the parameters of returns on U.S. Treasury securities to quantify pure interest rate risk.

5. Parameter estimation and empirical issues

If $\alpha > 1$, OLS provides a consistent estimator of the stable location parameter δ. However, it has an infinite variance stable distribution with the same α as the observations, and has 0 efficiency. Furthermore, expectations proxies based on a false normal assumption will generate spurious evidence of irrationality if the true distribution is stable with $\alpha < 2$ (Batchelor 1981).

5.1. Univariate stable parameter estimation

DuMouchel (1973) demonstrates that ML may be used to estimate the four stable parameters, and that the ML estimates have the usual asymptotic normality governed by the information matrix, except in the non-standard boundary cases $\alpha = 2$ and $\beta = \pm 1$. In (1975), he tabulates the information matrix, which may be used for asymptotic hypothesis testing except in the boundary cases where, as he points out, ML is actually super-efficient. Monte Carlo critical values of the likelihood ratio for the non-standard null hypothesis $\alpha = 2$ with a symmetric stable alternative have been tabulated by McCulloch (in press a). DuMouchel (1983) suggests that the ML estimator of α is biased downwards when the true α is near 2.00, but this is not borne out (apart from the effect of the $\alpha \leq 2$ boundary restriction) in larger sample simulations reported by McCulloch (in press a).

In the SS cases, the numerical approximation of McCulloch (1994b) permits fast computation of the likelihood without resorting to the bracketing procedure proposed by DuMouchel. SS ML using an early version of this approximation was applied to interest rate data in McCulloch (1981, 1985a). Asymmetric stable ML has been performed by Stuck (1976), using the Bergstrøm series, by Feuerverger and McDunnough (1981), using Fourier inversion of the log c.f., and by Brorsen and Yang (1990) and Liu and Brorsen (1995) using Zolotarev's integral representation of the stable density. See also the algorithm of Chen (1991), reported and employed by Mittnik and Rachev (1993a). ML linear regression with stable residuals has been implemented for the SS case by McCulloch (1979) and for the general case by Brorsen and Preckel (1993). Buckle (1995) and Tsionas (1995) go beyond ML to explore the Bayesian posterior distribution of stable parameters.

A much simpler, but at the same time less efficient, method of estimating SS distribution parameters from order statistics was proposed by Fama and Roll (1971), and has been widely implemented. This method has been extended to the asymmetric cases, and a small asymptotic bias in the Fama-Roll estimator of c in the SS cases removed, by McCulloch (1986).

A large body of work, following Press (1972), has focussed on fitting the empirical log c.f. to its theoretical counterpart (3), (4). See Paulson, Holcomb and Leitch (1975); Feuerverger and McDunnough (1977, 1981a,b); Arad (1980); Koutrouvelis (1980, 1981); and Paulson and Delehanty (1984, 1985). Practitioners report a high degree of efficiency relative to the ML benchmark.[18] Mantegna and Stanley (1995) implement a novel method of estimating the stable index from the modal density of returns at different sampling intervals.

Stable parameters have been estimated for stock returns by Fama (1965), Leitch and Paulson (1975), Arad (1980), McCulloch (1994b), Buckle (1995), and Manegna and Stanley (1995); for interest rate movements by Roll (1970), McCulloch (1985), Oh (1994); for foreign exchange rate changes by Bagshaw and Humpage (1987), So (1987a,b), Liu and Brorsen (1995), and Brousseau and Czarnecki (1993); for commodities price movements by Dusak (1973), Cornew, Town and Crowson (1984), and Liu and Brorsen (in press); and for real estate returns by Young and Graff (1995), to mention only a few studies.

5.2. *Empirical objections to stable distributions*

The initial interest in the stable model of financial returns has undeservedly waned, largely because of two groups of statistical tests. The first group of tests is based on the observation that if daily returns are iid stable, weekly and monthly returns must be also be stable, with the same characteristic exponent. Blattberg and Gonedes (1974), and many subsequent investigators, notably Akgiray and Booth (1988) and Hall, Brorsen and Irwin (1989), have found that weekly and monthly returns typically yield higher estimates of α than do daily returns. Such

[18] On estimation see also Blattberg and Sargent (1971), Kadiyala (1972), Brockwell and Brown (1979, 1981), Fielitz and Roselle (1981), Csörgő (1984, 1987), Zolotarev (1986: 217ff), Akgiray and Lamoureux (1987), and Klebanov, Melamed and Rachev (1994).

evidence has led even Fama (1976: 26–38) to abandon the stable model of stock prices.

However, as Diebold (1993) has pointed out, all that such evidence really rejects is the compound hypothesis of *iid* stability. It demonstrates *either* that returns are not identical, *or* that they are not independent, *or* that they are not stable. If returns are not iid, then it should come as no surprise that they are not iid stable. It is now generally acknowledged (Bollerslev, Chou and Kroner, 1992) that most time series on financial returns exhibit serial dependence of the type characterized by ARCH or GARCH models. The unconditional distribution of such disturbances will be more leptokurtic than the conditional distribution, and therefore would generate misleadingly low α estimates under a false iid stable assumption.

Baillie (1993) wrongly characterizes ARCH and GARCH models as "competing" with the stable hypothesis. See also Ghose and Kroner (1995), Groenendijk et al. (1995). In fact, if conditional heteroskedasticity (CH) is present, it is as desirable to remove it in the infinite variance stable case as in the Gaussian case. And if after removing it there is still leptokurtosis, it is as desirable to model the adjusted residuals correctly as it is in the iid case. McCulloch (1985b) and Oh (1994) thus fit GARCH-like and GARCH models, respectively, to monthly bond returns by symmetric stable ML, and find significant evidence of *both* CH *and* residual non-normality. Liu and Brorsen (in press) similarly find, contrary to the findings of Gribbin, Harris and Lau (1992), that a stable model for commodity and foreign exchange futures returns cannot be rejected, once GARCH effects are removed. Their observations apply also to the objections of Lau, Lau and Wingender (1990) to a stable model for stock price returns. De Vries (1991) proposes a potentially important class of GARCH-like subordinated stable processes, but this model has not yet been empirically implemented.

Day-of-the-week effects are also well known to be present in both stock market (Gibbons and Hess 1981) and foreign exchange (McFarland, Pettit and Sung 1982) data. Whether such hebdomodalities are present in the mean or the volatility, they imply that daily data is not identically distributed. It is again as important to remove these, along with any end-of-the month effects and seasonals that may be present, in the infinite variance stable case as in the normal case. Lau and Lau (1994) demonstrate that mixtures of stable distributions with different scales tend to reduce estimates of α below its true value, whereas mixtures with different locations tend to increase estimates above the true value.

A second group of tests that purport to reject a stable model of asset returns is based on estimates of the Paretian exponent of the tails, using either the Pareto distribution itself (Hill 1975), or the generalized Pareto (GP) distribution (DuMouchel 1983). Numerous investigators, including DuMouchel (1983), Akgiray and Booth (1988), Jansen and de Vries (1991), Hols and de Vries (1991), and Loretan and Phillips (1994), have applied this type of test to data that includes interest rate changes, stock returns, and foreign exchange rates. They typically have found an exponent greater than 2, and have used this to "reject" the stable model on the basis of asymptotic tests.

However, McCulloch (1994b) demonstrates that tail index estimates greater than 2 are to be expected from stable distributions with α greater than approximately 1.65 in finite samples of sizes comparable to those that have been used in these studies. These estimates may even appear to be "significantly" greater than 2 on the basis of asymptotic tests. The studies cited are therefore in no way inconsistent with a Paretian stable distribution.[19]

Several alternative distributions have been proposed to account for the conspicuously leptokurtic behavior of financial returns. Blattberg and Gonedes (1974) and Boothe and Glassman (1987) thus propose the Student's t distributions, which may be computed for fractional degrees of freedom, and which, like the stable distributions, include the Cauchy and the normal. Others (e.g. Hall, Brorsen and Irwin 1989; Durbin and Cordero 1993) consider a mixture of normals. Boothe and Glassman (1987) find somewhat higher likelihood for the Student distribution than for either the mixture of normals or stable, but these hypotheses are not nested, so that the likelihood ratio does not necessarily have a χ^2 distribution. Lee and Brorsen (1995) have had some success formally comparing such non-nested hypotheses using Cox-like tests. However, such distributions are intrinsically difficult to differentiate without extremely large samples, as noted already by DuMouchel (1973b). The choice among leptokurtic distributions may in the end depend primarily on whatever desirable properties they may have, in particular divisibility, parsimony, and central limit attributes. Csörgő (1987) constructs a formal test for one aspect of stability, and fails to reject it using selected stock price data.

Mittnik and Rachev (1993a) generalize the concept of "stability" beyond the stability under summation and multiplication that leads to the stable and log-stable distributions, respectively, to include stability under the maximum and minimum operators, as well as stability under a random repetition of these accumulation and extremum operations, with the number of repetitions governed by a geometric distribution. They find that the Weibull distribution has two of these generalized stability properties. Since it has only positive support, they propose a *double Weibull* distribution (two Weibull distributions back-to-back) as a model for asset returns. This distribution has the unfortunate property that its density is, with only one exception, either infinite or zero at the origin. The sole exception is the back-to-back exponential distribution, which still has a cusp at the origin. The stable densities, on the other hand, are finite, unimodal, absolutely differentiable, and have closed support.

5.3. *State-space models*

Stable state-space models may be estimated using the Bayesian approach of Kitagawa (1987). When there is only one state variable, the marginal retrospective posterior (filter) distribution of the state variable and the likelihood requires

[19] Mittnik and Rachev (1993b: 264–5) similarly find that the Wiebull distribution gives tail index estimators in the range 2.5–5.5, even though the Weibull distribution has no Paretian tail.

approximately mn numerical integrations with m nodes, where n is the sample size. The hyperparameters of the model may then be estimated by ML, and the marginal full sample posterior (smoother) distribution then computed by another n numerical integrations. If the disturbances are SS, the density approximation of McCulloch (1994b) makes these calculations feasible, even on a personal computer, despite the numerous iterations required by the ML step.

Oh (1994) thus estimates an AR(1) time-varying term premium (the state variable) for excess returns on U.S. Treasury securities. After also adjusting for pronounced state-space GARCH effects, he finds ML $\hat{\alpha}$ values ranging from 1.61 to 1.80 and LR statistics ($2\Delta \log L$) for the null hypothesis $\alpha = 2$ in the range 12.95 to 25.26. These all reject normality at the 0.996 level or higher, using the critical values in McCulloch (1994b). (See also Bidarkota and McCulloch (1996)).

Multiple state variables greatly increase the number of numerical integrals, and therefore the calculation time, required for Kitagawa's approach. However, the state variable may still be estimated in a reasonable amount of time by instead using the Posterior Mode Estimator approach of McCulloch (1994a, following Durbin and Cordero 1993). In many cases the hyperparameters may be estimated (though without the efficiency of full information ML) by applying *pooled ML* to various linear combinations of the data.

Mikosch, Gadrich, Klüppelberg and Adler (1995) consider a standard ARMA process in which the innovations belong to the domain of attraction of a SS law. Since they did not have access to a numerical density approximation, they employ the Whittle estimator, based on the sample periodogram, rather than the more readily interpretable ML.

5.4. *Estimation of multivariate stable distributions*

The estimation of multivariate stable distribution parameters is still in its infancy, despite the great importance of these distributions for financial theory and practice. Mittnik and Rachev (1993b: 365–66) propose a method of estimating the general bivariate spectral measure for a vector whose distribution lies in this domain of attraction. Cheng and Rachev (in press) apply this method to the $/DM and $/yen exchange rates, with the interesting result that there is considerable density near the center of the first and third quadrants, as would be expected if a dollar-specific factor were affecting both exchange rates equally, but very little along the axes. The latter effect seems to indicate that there are negligible DM- or yen-specific shocks.

Nolan, Panorska and McCulloch (1996) propose an alternative method based on ML, which uses the entire data set, whereas the Mittnik and Rachev method employs only a small subset of the data, drawn from the extreme tails of the sample. This method does not necessitate the often arduous task of actually computing the MV stable density (see Byczkowski et al., 1993; Nolan and Rajput, 1995), but relies only on the standard univariate stable density. This method expressly assumes that $\boldsymbol{x}$ actually has a bivariate stable distribution, rather than that it merely lies in its domain of attraction.

Appendix

Derivation of (53) from (52)

In this appendix, we let $s_i(u_i)$ and $S_i(u_i)$ represent $s(u_i;\alpha,+1,c_i,\delta_i)$ and $S(u_i;\alpha,+1,c_i,\delta_i)$, respectively, for $i=1,2,3$. We have $S_T > X$ whenever $u_2 < u_1 - \log X$. Then, setting $z=(u_2-\delta_2)/c_2$ and $S_i^c = 1 - S_i$, we have

$$\begin{aligned}\int\limits_{S_r>X} U_2 dP(U_1,U_2) &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{u_1-\log X}\int\limits_{-\infty}^{\infty} e^{-u_2-u_3}s_1(u_1)s_2(u_2)s_3(u_3)\, du_3du_2du_1\\
&= \mathrm{E}e^{-u_3}\int\limits_{-\infty}^{\infty} e^{-u_2}s_2(u_2)\int\limits_{u_2+\log X}^{\infty} s_1(u_1)\, du_1du_2\\
&= \mathrm{E}e^{-u_3}\int\limits_{-\infty}^{\infty} e^{-u_2}s_2(u_2)S_1^c(u_2+\log X)\, du_2\\
&= \mathrm{E}e^{-u_3}e^{-\delta_2}\int\limits_{-\infty}^{\infty} e^{-c_2 z}s_{\alpha 1}(z)S_1^c(c_2 z+\delta_2+\log X)\, dz\\
&= \mathrm{E}e^{-u_3}e^{-\delta_2}\int\limits_{-\infty}^{\infty} e^{-c_2 z}s_{\alpha 1}(z)S_{\alpha 1}^c\left(\frac{c_2 z-\delta+\log X}{c_1}\right) dz\\
&= \mathrm{E}e^{-u_3}e^{-\delta_2}I_1,\end{aligned}$$

where, using (50), I_1 is as given in (54) in the text. Similarly, but now setting $z=(u_1-\delta_1)/c_1$,

$$\begin{aligned}\int\limits_{S_T>X} U_1 dP(U_1,U_2) &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{u_1-\log X}\int\limits_{\infty}^{\infty} e^{-u_1-u_3}s_1(u_1)s_2(u_2)s_3(u_3)\, du_3du_2du_1\\
&= \mathrm{E}e^{-u_3}\int\limits_{-\infty}^{\infty} e^{-u_1}s_1(u_1)\int\limits_{-\infty}^{u_1-\log X} s_2(u_2)\, du_2du_1\\
&= \mathrm{E}e^{-u_3}\int\limits_{-\infty}^{\infty} e^{-u_1}s_1(u_1)S_2(u_1-\log X)\, du_1\\
&= \mathrm{E}e^{-u_3}e^{-\delta_1}\int\limits_{-\infty}^{\infty} e^{-c_1 z}s_{\alpha 1}(z)S_2(c_1 z+\delta_1-\log X)\, dz\\
&= \mathrm{E}e^{-u_3}e^{-\delta_1}I_2,\end{aligned}$$

where I_2 is as given in (55). Substituting into (52) yields (53).

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