Measuring Tail Thickness to Estimate the Stable Index α: A Critique

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A generalized Pareto or simple Pareto tail-index estimate above 2 has frequently been cited as evidence against infinite-variance stable distributions. It is demonstrated that this inference is invalid; tail index estimates greater than 2 are to be expected for stable distributions with α as low as 1.65. The nonregular distribution of the likelihood ratio statistic for a null of normality and an alternative of symmetric stability is tabulated by Monte Carlo methods and appropriately adjusted for sampling error in repeated tests. Real stock returns yield a stable α of 1.845 and reject iid normality at the .996 level.

KEY WORDS: Bond returns; Foreign-exchange returns; Generalized Pareto distribution; Hill estimator; Infinite variance; Monte Carlo distribution of stable likelihood ratio statistic; Pareto distribution; Stable distributions; Stock returns; Tail index.

Financial asset returns are the cumulative outcome of a vast number of individual decisions occurring continuously in time. According to the generalized central limit theorem, if the sum of a large number of iid random variates has a limiting distribution after appropriate shifting and scaling, the limiting distribution must be a member of the *stable* class (Zolotarev 1986, chap. 1). It is therefore natural to assume that asset returns are at least approximately governed by a stable distribution if the accumulation is additive, or by a log-stable distribution if the accumulation is believed to be multiplicative.

The normal distribution is the most familiar stable distribution, with stable index $\alpha = 2$, and therefore either it or the lognormal is commonly postulated to be the true distribution. Financial asset returns, however, are often much more leptokurtic than is consistent with a Gaussian distribution. Mandelbrot (1963) therefore proposed the nonnormal stable distributions, which have Pareto-like tails with an exponent $\alpha < 2$, as a particularly attractive alternative model of asset returns. The basic properties of stable distributions are summarized in Section 1.

Paretian stable-distribution parameters have been estimated for stock returns (Buckle 1995; Tsionas in press), excess bond returns (McCulloch 1985; Oh 1994), foreign-exchange-rate changes (So 1987; Tsionas in press), commodity-price movements (Liu and Brorsen 1995), and real-estate returns (Young and Graff 1995), to mention only a few more recent studies. Other leptokurtic distributions, including Student's t, mixtures of normals, and the double Weibull, have also been investigated, but these do not have the attractive central limit property or divisibility of stable distributions.

In recent years, however, several studies, including notably those of DuMouchel (1983), Akgiray and Booth (1988), Jansen and de Vries (1991), Hols and de Vries (1991), and Loretan and Phillips (1994), have found what appears to be strong evidence against the stable model. Following DuMouchel (1983), these studies have estimated the Paretian tail index directly from the tail observations, using either the Pareto distribution itself or a generalization of the Pareto distribution proposed by DuMouchel, and commonly have found a tail index that appears to be significantly greater than 2, the maximum permissible value for a stable distribution. These authors, as discussed in Section 3, concluded that this evidence rejects the stable distribution.

In Section 4, it is demonstrated that this inference is invalid, overlooking as it does the implications of the intermediate value theorem. It is shown that, with samples of the size that have been used in these studies, tail-index estimates in excess of 2 are to be expected for iid stable samples with α as low as 1.65. The findings of the works cited are therefore in no way inconsistent with a stable distribution for asset returns.

Furthermore, if the distribution is truly stable, these tail-index estimators will provide a highly unreliable estimate of the stable tail index with the moderate sample sizes that are typically available. If one is willing to assume that the distribution is truly or approximately stable, it is recommended that its parameters instead be estimated using the full sample, by maximum likelihood (ML). DuMouchel (1973) showed that ML estimators of the stable parameters are asymptotically normal when the true values lie in the interior of the parameter space. DuMouchel (1983) noted, however, that when the true distribution is normal, the asymptotic distribution of the ML estimate of α is nonregular, and its small-sample distribution may not be well approximated by its normal asymptotic limit.

Section 5 corrects this deficiency by tabulating critical values of the likelihood ratio (LR) statistic under the null hypothesis that the true value of the stable index is at the Gaussian boundary value of 2, with a symmetric stable alternative hypothesis. The critical values, which are obtained by Monte Carlo simulation, are appropriately adjusted for their own sampling error when used with multiple independent test statistics. Use of the LR statistic is illustrated

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with real Center for Research in Security Prices (CRSP) stock-market returns.

Section 6 provides some concluding caveats and briefly discusses certain other, equally invalid empirical objections that have been raised against the stable hypothesis.

1. BASIC PROPERTIES

Stable distributions may be defined most concisely in terms of their log characteristic functions. Following DuMouchel (1975), these are given as

$$\log E(\exp(iXt)) = \begin{cases} i\delta t - |ct|^\alpha \left[1 - i\beta \operatorname{sgn}(t) \tan \frac{\pi\alpha}{2}\right]; \\ \quad \alpha \neq 1 \\ i\delta t - |ct| \left[1 + i\beta \frac{2}{\pi} \operatorname{sgn}(t) \log |ct|\right]; \\ \quad \alpha = 1. \end{cases} \tag{1}$$

The *characteristic exponent* $\alpha \in (0, 2]$ and the *skewness parameter* $\beta \in [-1, 1]$ together determine the shape of the distribution. The *location parameter* $\delta \in (-\infty, \infty)$ shifts the distribution to the left or right, whereas the *scale parameter* $c \in (0, \infty)$ merely expands or contracts it around δ (see McCulloch in press a). We may therefore represent the standard stable distribution (with $c = 1$ and $\delta = 0$) by $S_{\alpha,\beta}(x)$.

The stable distribution and density may be computed from the integral representations of Zolotarev (1986), or by evaluating the inverse Fourier transform of the characteristic function (e.g., Tsionas 1995a,b). In the symmetric case $\beta = 0$, the numerical approximation of McCulloch (in press b) greatly simplifies and speeds calculations. See Samorodnitsky and Taqqu (1994), Janicki and Weron (1994), and McCulloch (1986, 1996) for further properties of stable distributions.

Asymptotic expansions due to Bergstrøm (1952) imply that, as $x \to \infty$,

$$\begin{aligned} S_{\alpha,\beta}(-x) &\sim (1-\beta)\,\frac{\Gamma(\alpha)}{\pi}\,\sin\frac{\pi\alpha}{2}x^{-\alpha} \\ 1 - S_{\alpha,\beta}(x) &\sim (1+\beta)\,\frac{\Gamma(\alpha)}{\pi}\,\sin\frac{\pi\alpha}{2}\,x^{-\alpha}. \end{aligned} \tag{2}$$

When α is less than 2, stable distributions therefore have "Paretian" tail(s) that behave asymptotically like $x^{-\alpha}$ and give the stable distributions infinite absolute population moments of order greater than or equal to α. In this case, β indicates the limiting ratio of the upper tail probability to the sum of the two tail probabilities. When $\alpha = 2$, the coefficients on the Paretian tails vanish along with the influence of β, and a normal distribution with variance $2c^2$ results.

Equation (1) follows Zolotarev (1986, p. 9) and DuMouchel (1975) by defining β in such a way that $\beta > 0$ indicates positive skewness for all α. Holt and Crow (1973), as well as Akgiray and Booth (1988), confusingly reversed the sign on their "β" for $\alpha \neq 1$. Buckle (1995) employed a version of the "polar" parameterization, rather than the "Cartesian" parameterization of (1). See Samorodnitsky and Taqqu (1994, pp. 8–9) on the latter distinction. These issues do not affect the present discussion.

2. THE GENERALIZED PARETO DISTRIBUTION

In an influential article, DuMouchel (1983) proposed that "a more natural way of modeling the tail behavior of data [than the stable or any other complete distribution] is to let the tails 'speak for themselves' by basing the inferences on the extreme observations without making any assumptions about the center of the distribution" (p. 1025). For this purpose he proposed using what he calls the *generalized Pareto* (GP) *distribution:*

$$P(X > x | X \geq x_0) = \begin{cases} (1 + \gamma(x - x_0)/\sigma)^{-1/\gamma}; & \gamma > 0 \\ \exp(-(x - x_0)/\sigma); & \gamma = 0 \\ (1 + \gamma(x - x_0)/\sigma)^{-1/\gamma}; & \gamma < 0, x \leq x_0 + \frac{\sigma}{|\gamma|}, \\ 0 & \gamma < 0, x > x_0 + \frac{\sigma}{|\gamma|}, \end{cases} \tag{3}$$

where $\gamma \in (-\infty, \infty), \sigma \in (0, \infty)$, and $x_0 \in (0, \infty)$. For $\gamma > 0$, this upper tail probability behaves like $x^{-\alpha}$ for large x, where $\alpha = 1/\gamma$. The standard Pareto distribution with exponent $1/\gamma$ is the special case of the GP distribution corresponding to $\gamma > 0$ and $\sigma = \gamma x_0$.

Specifically, DuMouchel proposed fitting this distribution to the upper 10% of the sample by ML, using the largest nonincluded observation as x_0. When the distribution is symmetric or approximately symmetric, the absolute values of the lower tail observations (relative to the smallest nonincluded observation) may be merged with the upper tail to obtain a larger sample. DuMouchel provided a formula for the asymptotic standard error of the ML estimates $\hat{\gamma}$ and $\hat{\sigma}$. I denote $\hat{\alpha}_{GP} = 1/\hat{\gamma}$.

The generalized Pareto distribution, unlike the stable distributions, thus permits Paretian tail behavior with $\alpha \geq 2$. Any distribution with such tails lies in the domain of attraction of the normal distribution, yet may be far more leptokurtic than the normal. If $\alpha > 2$, the variance is reassuringly finite.

DuMouchel interpreted $\hat{\gamma} < 0$ as indicating thinner tail behavior than the exponential distribution. He simulated estimation of γ for a normal distribution with a tail sample of size 1,000, drawn using the theoretical normal distribution as if it were the empirical distribution, and found $\hat{\gamma} = -.151$. In a normal sample, $\hat{\gamma}$ would almost surely rise to 0 as sample size increases to infinity because of the infinite support of the normal distribution, but he found that this rise is very slow; with a simulated tail sample of 10,000, the fitted $\hat{\gamma}$ rises to only $-.145$.

A conditional Pareto distribution (proposed by Hill 1975) captures the same asymptotic behavior as the GP distribution, but one might have to go quite far out into the tails before relatively pure Paretian behavior would become evident. DuMouchel's GP distribution has the advantage that it allows a larger sample size to be brought to bear on the tail behavior yet includes the Pareto as a special case.

3. EMPIRICAL TAIL BEHAVIOR

DuMouchel (1983) found that, when symmetric stable densities are fit to 304 observations on 6- and 24-week Treasury-bill rate changes by ML, apparently infinite-variance stable parameters result: $\hat{\alpha} = 1.37 \pm .11$ and $\hat{\alpha} = 1.23 \pm .08$, respectively. Yet his GP model, estimated with a combined tail sample size of 60, gives $\hat{\gamma} \doteq -.081 \pm .119$ and $.227 \pm .158$, respectively. The derived lower bounds on 95% confidence intervals for the tail index are 8.73 and 2.05, respectively. From this he concluded, "if the stable model holds for these data, the estimates of γ^{-1} should be near those of [the stable] α. On the contrary, the lower limit for γ^{-1} is greater than the upper limit for α in both data sets. We conclude that these data are much less outlier-prone than a stable law would lead us to believe" (pp. 1027–1028).

Akgiray and Booth (1988) fitted both the stable and GP distributions to returns on 200 stocks, using the empirical characteristic function approach of Koutrouvelis (1980) to estimate all four stable parameters. With samples of 1,500 daily observations, they found that, for the stable model, $\hat{\alpha} + 2(\mathrm{SE}) < 2.0$ in all 200 cases, with $\hat{\alpha}$ commonly in the range (1.65, 1.85). Yet applying DuMouchel's GP model to the outermost 300 observations, $1/(\hat{\gamma} + 2(\mathrm{SE})) > 1.99$ in all 200 cases. From this they deduced that "empirical tails are thinner than the tails of a stable distribution with the thinnest tail possible. ... [E]mpirical tails are not even close to stable tails; they are significantly thinner" (p. 55).

Their results are somewhat weaker using 300 observations on weekly returns, and therefore 60 pooled tail observations, as would be expected from the smaller sample size. Nevertheless they found that "stable distributions can be ruled out" for 179 of the 200 stocks, in the sense that the stable $\hat{\alpha}$ is significantly less than $1/\hat{\gamma}$. Even in the 21 cases in which the empirical tails are consistent with stable tails, the hypothesis that the empirical tails are exponential ($\hat{\gamma} - 2(\mathrm{SE}) < 0$) or even normal [which they test by $\hat{\gamma} - 2(\mathrm{SE}) < -.15$] cannot be rejected in 16 and 11 cases, respectively.

Akgiray and Booth (1988) concluded, "Although the distributions of stock returns appear to be homogeneous and very similar in shape to stable distributions with index $\alpha < 2$, a stable-law assumption for return distributions may still be invalid and misleading. This is because empirical tail shapes are significantly different from the tails of stable distributions and sample estimates of the stable distribution parameters (particularly α) may not be robust to such differences. Economic and statistical inferences should not be based on [stable] index α estimated from samples of stock returns. On both theoretical and empirical grounds, it seems to be safer to use other probability models (preferably long-tailed and skewed distributions that are in the domain of attraction of the normal distribution) to explain stock-price behavior" (pp. 56–57).

Several other studies have arrived at similar results, primarily using Hill's (1975) related conditional Pareto model. Loretan and Phillips (1994) thus applied this model, separately to the upper and lower tails of an index of U.S. stock returns and on exchange-rate returns for five currencies. Their total sample sizes are 1,906 for monthly stock returns, 6,404 for daily stock returns, and approximately 3,100 for daily exchange-rate returns. They reported results with several different tail sample sizes, which place about 1% to 4% of the total data in each tail sample. They found that the results are surprisingly insensitive to the value chosen within this range. None of their 70 α estimates is smaller than 2.39, and only three (which were based on only 20 or 30 data points) were larger than 3.92. They interpreted their results as being in general agreement with what they view as the general conclusion of the prior literature—namely, that "empirical distributions in economics, especially aggregate series such as stock market prices and returns, do not follow stable laws and are better modeled by finite variance distributions" (p. 236).

Jansen and de Vries (1991) and Hols and de Vries (1991) found similar results for stock returns and the Canadian/U.S. dollar exchange rate, respectively. Koedijk, Schafgans, and de Vries (1990) found, using intra-European exchange rates under the European Monetary System, that $\alpha < 2$ can never be rejected by means of the tail index. The latter weak results may simply be due to small sample size, however.

4. THE DISTRIBUTION OF $\hat{\alpha}_{GP}$ UNDER STABLE LAWS

Unfortunately, there is a flaw in the reasoning of all the studies cited in Section 3.

DuMouchel (1983) tabulated the GP tail-shape parameter that might be expected to be estimated from large samples (1,000 tail observations) drawn from the upper decile of several common distributions, including the uniform, triangular, normal, exponential, Student's (5 df), lognormal ($\sigma = 1$), Student's (2 df), and Cauchy. He neglected, however, with only two exceptions, to indicate what kind of results might be expected when his GP distribution is estimated from the tails of a *stable* distribution.

The two exceptions are the Cauchy, which is symmetric stable with $\alpha = 1$, and the normal, which is stable with $\alpha = 2$. His Cauchy simulation gives $\hat{\gamma} = .988$, or $\hat{\alpha}_{GP} = 1.012$, which conforms well with his expectation that $1/\hat{\gamma}$ should approximate the stable α if the distribution is truly stable. His normal simulation, noted previously, gives $\hat{\gamma} = -.151$, which is not consistent with any finite tail index. This is not surprising, however, given that the normal distribution has tails that are thinner than any Paretian distribution.

Now, in the interval (0, 2], the symmetric stable density is a continuous function of α. It follows that the distribution of any statistic computed by a continuous function from a finite sample drawn from a stable distribution must also be a continuous function of α. Thus, by the intermediate value theorem, if $\hat{\gamma}$ is roughly 1.00 when α is 1.00 and is roughly $-.15$ when α is 2.00, it must pass .50 (the value corresponding to $\hat{\alpha}_{GP} = 2.00$) *somewhere in between.* A GP $\hat{\gamma}$ value less than .50, or even less than 0, therefore *is in no way inconsistent with an infinite-variance stable distribution.*

Table 1 shows the results of a small Monte Carlo simulation of the distribution of $\hat{\gamma}$ for samples drawn from

Table 1. Monte Carlo Distribution of Generalized Pareto Estimator $\hat{\gamma}$ When True Distribution is Symmetric Stable (100 replications)

α	Min.	1Q	Median	3Q	Max.
a. Full sample 300, pooled tail sample 60					
2.00	−.698	−.305	−.227	−.100	.069
1.99	−.743	−.287	−.178	−.050	.438
1.95	−.581	−.183	−.017	.108	.603
1.90	−.523	−.054	+.118	.260	.645
1.80	−.603	.157	.303	.394	.745
1.70	−.334	.281	.399	.530	.847
1.60	−.191	.370	.528	.652	.971
1.50	−.097	.433	.600	.731	1.171
1.40	.060	.510	.648	.800	1.266
1.30	.107	.570	.734	.862	1.448
1.20	.152	.660	.786	.961	1.401
1.10	.124	.771	.883	1.061	1.523
1.00	.263	.850	.998	1.168	1.628
.80	.578	1.049	1.268	1.465	1.935
.60	.870	1.407	1.703	1.911	2.583
.40	1.562	2.187	2.530	2.852	3.570
.20	3.616	4.777	5.284	5.863	7.103
b. Full sample 1,500, pooled tail sample 300					
2.00	−.305	−.211	−.166	−.119	−.054
1.99	−.281	−.167	−.097	−.044	.134
1.95	−.251	.015	.062	.132	.265
1.90	−.088	.128	.183	.227	.375
1.80	.096	.269	.318	.362	.504
1.70	.236	.376	.424	.476	.633
1.60	.310	.460	.513	.552	.728
1.50	.363	.531	.581	.640	.801
1.40	.468	.588	.648	.704	.860
1.30	.493	.659	.713	.787	.943
1.20	.577	.730	.786	.863	1.081
1.10	.665	.812	.874	.953	1.104
1.00	.707	.906	.967	1.046	1.222
.80	.978	1.160	1.237	1.322	1.497
.60	1.372	1.583	1.693	1.791	2.021
.40	1.988	2.436	2.567	2.694	3.092
.20	4.507	5.008	5.200	5.483	6.170

a symmetric stable distribution with selected values of α. The symmetric stable pseudorandom variates were generated by the method of Chambers, Mallows, and Stuck (1976). Panel a is based on the pooled upper and lower 10% of a sample of size 300, as in DuMouchel's treasury-bill estimates and in Akgiray and Booth's (1988) weekly stock-return estimates, and Panel b is based on the pooled upper and lower 10% of a sample of size 1,500, as in Akgiray and Booth's daily estimates. Both panels are based on 100 replications and show the minimum, first-quartile, median, third-quartile, and maximum values thus obtained. The random-number generator was initialized by the same seed value for each case, so as not to mask with additional sampling error the fundamental continuity of the distribution of the estimators with respect to α.

It may be seen from Table 1 that, with either sample size, the median value of $\hat{\gamma}$ falls below .50, and therefore its reciprocal $\hat{\alpha}_{GP}$ rises above 2.00, at approximately $\alpha = 1.62$. With finite samples, $\hat{\gamma}$ is ordinarily somewhat negative at $\alpha = 2.00$ so that $\hat{\alpha}_{GP}$ actually rises to infinity before the stable index reaches 2.00. In at least one instance, this happens with α as low as 1.50.

Table 2 illustrates the distribution of the Hill/Pareto estimator $\hat{\alpha}_H$ used by the other studies cited. It was constructed by performing 100 Monte Carlo replications, in each of which an upper tail sample of 50 was taken from a symmetric stable sample of 3,000. This sample size was chosen to be representative of Loretan and Phillips's (1994) foreign-exchange-rate estimates. Although the reciprocal of the Hill estimator may not go negative as in the GP case, $\hat{\alpha}_H$ is again seen to be an upward-biased estimate of the stable index when α is much greater than 1. The Loretan and Phillips result, $\hat{\alpha}_H \in (2.39, 3.92)$, is indicative of a stable α in the range (1.78, 1.92) and is by no means inconsistent with an infinite-variance stable distribution.

Tables 1 and 2 demonstrate that the GP and Hall tail-index estimators cannot be used to "reject" an infinite-variance stable distribution, as claimed by the studies cited in Section 2. The problem with using the tail index to estimate the stable characteristic exponent is that the $\sin(\pi\alpha/2)$ terms in (2) imply that the contribution of the Paretian tail(s) becomes weaker and finally vanishes altogether as $\alpha \uparrow 2$. Therefore we must go further and further out into the tails as we approach normality before the Paretian behavior becomes evident. For α near or below 1, the tail index computed as in Tables 1 and 2 captures the Paretian behavior reasonably well. But for higher values of α, the GP and Hill estimators become entirely misleading, unless we were to use so small a tail sample that they could not be meaningfully estimated. The asymptotic standard errors used by these studies to "reject" $\alpha < 2$ are indisputably valid, but only as the tail sample k and total sample n both go to infinity and at the same time k/n goes to 0. Dewachter and Gielens (1994) showed that the Hill estimator provides a similarly biased estimator of the degrees of freedom when the true distribution is Student's t.

5. THE DISTRIBUTION OF THE LR STATISTIC UNDER NORMALITY

I have shown that measuring tail thickness is not a reliable method of estimating the stable index α when the

Table 2. Monte Carlo Distribution of Hill/Pareto Estimator $\hat{\alpha}_H$ When True Distribution Is Symmetric Stable (100 replications)

α	Min.	1Q	Median	3Q	Max.
2.00	5.117	6.220	6.728	7.343	8.838
1.99	4.203	5.642	6.095	6.963	8.313
1.95	2.963	4.190	4.609	5.253	7.164
1.90	2.211	3.160	3.588	3.993	5.365
1.80	1.559	2.211	2.476	2.851	3.988
1.70	1.308	1.827	2.040	2.270	3.159
1.60	1.195	1.645	1.762	1.939	2.687
1.50	1.111	1.485	1.610	1.827	2.935
1.40	1.097	1.328	1.477	1.639	2.542
1.30	1.014	1.214	1.368	1.486	2.130
1.20	.886	1.111	1.256	1.357	1.744
1.10	.799	1.029	1.154	1.236	1.617
1.00	.706	.931	1.032	1.136	1.484
.80	.573	.739	.826	.902	1.202
.60	.431	.552	.603	.675	.840
.40	.294	.365	.406	.447	.548
.20	.158	.185	.197	.226	.282

NOTE: The full sample is 3,000; the upper tail sample is 50.

Table 3. Monte Carlo Distribution of Symmetric Stable ML Estimator $\hat{\alpha}$ When True Distribution Is Symmetric Stable (100 replications)

α	Min.	1Q	Median	3Q	Max.
		a. Sample size 300			
2.00	1.869	2.000	2.000	2.000	2.000
1.99	1.877	1.991	2.000	2.000	2.000
1.95	1.797	1.926	1.962	2.000	2.000
1.90	1.684	1.853	1.908	1.950	2.000
1.80	1.537	1.725	1.795	1.842	2.000
1.70	1.432	1.611	1.688	1.755	1.912
1.60	1.340	1.500	1.591	1.651	1.825
1.50	1.255	1.405	1.487	1.552	1.714
1.40	1.171	1.318	1.389	1.452	1.569
1.30	1.091	1.220	1.289	1.358	1.488
1.20	1.018	1.126	1.192	1.263	1.395
1.10	.944	1.034	1.092	1.159	1.286
1.00	.840*	.944	.992	1.060	1.170
.90	.840*	.850	.894	.950	1.047
		b. Sample size 1,500			
2.00	1.979	2.000	2.000	2.000	2.000
1.99	1.950	1.986	1.999	2.000	2.000
1.95	1.898	1.942	1.953	1.969	2.000
1.90	1.824	1.881	1.905	1.925	1.980
1.80	1.707	1.781	1.802	1.829	1.904
1.70	1.613	1.672	1.700	1.732	1.801
1.60	1.504	1.569	1.597	1.626	1.707
1.50	1.404	1.468	1.498	1.524	1.612
1.40	1.310	1.369	1.397	1.427	1.505
1.30	1.221	1.269	1.300	1.331	1.390
1.20	.840*	1.169	1.198	1.233	1.275
1.10	.840*	1.071	1.098	1.132	1.171
1.00	.840*	.972	.996	1.026	1.074
.90	.840*	.879	.895	.924	.971

* A lower limit of .84 required by the stable density approximation is binding in these cases; true value may be smaller.

true distribution is stable. Statistical theory would suggest instead estimating it directly using the full sample, by ML (see Brorsen and Yang 1990; Feuerverger and McDunnough 1981; McCulloch 1979; Stuck 1976). Other methods are also available (e.g. Csörgö 1984; Koutrouvelis 1980; McCulloch 1986; Paulson, Holcomb, and Leitch 1975), but ML will be most efficient. In the symmetric stable case $\beta = 0$, the numerical approximation of McCulloch (in press b) greatly facilitates ML calculations.

DuMouchel (1973) showed that, when $\alpha \in [\varepsilon, 2)$ for some arbitrarily small $\varepsilon > 0$, the ML estimators of the stable parameters meet the standard regularity conditions and have an asymptotically normal joint distribution governed by the information matrix, which he tabulated (1975). The LR may therefore be used to perform asymptotic tests of hypotheses about α within this range by means of the usual χ^2 distribution.

DuMouchel (1983, pp. 1021–1023), noted, however, that "when the true distribution is normal, the asymptotic distribution of $\hat{\alpha}$ is nonregular, and if the true distribution is stable with index α less than but near 2, the moderate sample distribution of $\hat{\alpha}$ is not well approximated by its asymptotically normal limit." He showed with simulations that, when the true α is 1.90, the likelihood function is skewed, so as to fall off more quickly for higher values of α than for lower values of α. This in itself does not imply that the ML estimate $\hat{\alpha}$ is biased, but merely that it should be easier to reject α values somewhat above $\hat{\alpha}$ than those that are the same distance below $\hat{\alpha}$. DuMouchel pointed out, however, that, in 8 of his 10 simulations, $\hat{\alpha}$ is less than 1.90, suggesting that it indeed has some downward bias.

Table 3 shows symmetric stable ML estimates $\hat{\alpha}$, computed from the same computer-generated symmetric stable random samples as were used for the GP model in Table 2, for the cases with $\alpha \geq .9$. The location and scale parameters δ and c were estimated along with the characteristic exponent, as would ordinarily be the case in applications. The log-likelihood was maximized with a convergence criterion of .0001 for $\alpha, \log(c)$, and δ.

It may be seen by comparing Tables 1 and 2 with Table 3 that estimating the stable exponent directly by stable ML is vastly superior to estimating it indirectly by means of the tail behavior when the true distribution is stable. The expectation of $\hat{\alpha}$ is unavoidably biased downward as α approaches 2 because of the boundary $\alpha \leq 2$. If there is any bias in terms of median ($\hat{\alpha}$), however, it is not obvious from this table.

Because the case $\alpha = 2$ lies on the boundary of the parameter space, the LR statistic for the important hypothesis $\alpha = 2$ does not meet the standard regularity conditions for its distribution to be asymptotically χ^2 (Moran 1971a). Because this is not an artificially imposed boundary, the LR statistic does not meet even the modified regularity conditions for its distribution to have half the upper tail probability of the χ^2 (Moran 1971b). In fact, Michael Woodroofe, as related by DuMouchel (1983, p. 1029), demonstrated that $P(\hat{\alpha} = 2) \rightarrow 1$ as $n \rightarrow \infty$.

Table 4 tabulates the Monte Carlo distribution of the LR statistic ($2\Delta \log L$) that is computed for the null hypothesis

Table 4. Monte Carlo Critical Values of the SS Likelihood Ratio Statistic (2Δ log L) for the Null Hypothesis of Normality (10,000 replications)

	Sample size n				
	30	100	300	1,000	∞
	a. For a single test of normality				
p:					
.10	.243	.228	.214	.181	.000
.05	1.107	1.052	.988	.824	.000
.02	2.487	2.545	2.483	2.169	.000
.01	3.956	3.711	3.659	3.519	.000
.005	4.848	5.041	4.918	4.829	.000
.002	6.637	6.466	6.258	6.066	.000
.001	8.208	7.421	7.629	6.662	.000
.0005	9.047	8.478	9.282	7.890	.000
.0002	10.139	10.185	10.899	9.354	.000
.0001	10.839	11.264	11.018	10.550	.000
times LR > 0	12.94%	15.57%	16.33%	15.69%	.00%
	b. For 100 independent tests of normality				
p*:					
.10	**.289**	**.273**	**.243**	**.214**	.000
.05	**1.257**	**1.183**	**1.120**	**.949**	.000
.02	**2.939**	**2.920**	**2.925**	**2.640**	.000
.01	**4.654**	**4.764**	**4.764**	**4.574**	.000
.005	**8.249**	**7.584**	**7.664**	**6.688**	.000
.00378	**10.839**	**11.264**	**11.018**	**10.550**	.000

NOTE: Panel b should be used for routine testing.

Table 5. ML Estimates of Symmetric Stable Parameters for Real CRSP Value-Weighted Stock Index Returns (including dividends)

α = 2:	
Mean	.555 (.195)
SD	4.277
c (= sd/$\sqrt{2}$)	3.024
Log L	−1.378.156
Max	14.451
Min	−24.774
SR	9.171 [$p < .0001$]
Norm. excess kurtosis	12.384 [$p < .0001$]
α unconstrained:	
$\hat{\alpha}$	1.845 (.056)
$\hat{c}$	2.712
$\hat{\delta}$	.673 (.181)
Log L	−1,364.818
2Δ log L (α = 2)	26.677 [$p \leq .00378$]
Generalized Pareto dist.: (96 pooled tail obs.)	
$\hat{\gamma}$	.080 (.110)
$\hat{\sigma}$	2.343
$\hat{\alpha}_{GP}$	12.471
$1/(\hat{\gamma}$ + 2SE)	3.326
Hill estimator: (40 pooled tail obs.)	
$\hat{\alpha}_H$	3.502 (.554)
(20 pooled tail obs.)	
$\hat{\alpha}_H$	3.653 (.817)

NOTE: Continuously compounded percent per month. 480 monthly observations. Asymptotic standard errors in parentheses.

H_0: $\alpha = 2$ when the symmetric stable parameters α, c, and δ are estimated by ML, for sample sizes $n = 30, 100, 300$, and $1,000$, using $r = 10,000$ replications. The table also shows the fraction of the time the LR statistic was greater than 0 (i.e., the fraction of the time $\hat{\alpha}$ was below the boundary 2), and the infinite-sample Woodroofe limit. The numerically approximated log-likelihood, calculated as by McCulloch (1994), has an expected error of .00003, .0001, .0003, and .001 for the four sample sizes tabulated, respectively.

Panel a of Table 4 gives critical values for a single experiment. Let LR_i be the Monte Carlo LR statistics, arranged in decreasing order, for $i = 1, \ldots r$, and let λ be the LR statistic for a single additional experiment. Under the null hypothesis, the experimental statistic has the same distribution as the Monte Carlo sample and therefore the probability that λ is greater than LR_m and the null hypothesis is falsely rejected is $p = m/(r+1)$. Accordingly, Table 4a associates LR_m with $p = m/(r+1)$ and linearly interpolates to the round p values shown in the table.

Table 4a optimally adjusts for Monte Carlo sampling error when only a single experiment is to be performed. Its critical values will be invalid, however, if it is used for more than one experiment, as I hope will be the case. If we knew the exact distribution of a test statistic for any hypothesis, and performed k independent experiments, the probability of falsely rejecting the null hypothesis k times using the true critical value for tail probability p would be p^k. But with the critical values in Table 4a, this probability would be substantially higher because each test would be contaminated by the same Monte Carlo sampling error.

Let $\lambda_1, \ldots \lambda_k$ be the LR test statistics for k independent experiments. Under the null hypothesis, these again come from the same true distribution as our Monte Carlo simulation for the appropriate sample size. For any integer m between 1 and r inclusive, the probability that $\lambda_1, \ldots \lambda_k$ are all greater than LR_m (i.e., that they are all among the $k + m - 1$ largest of the $k + r$ iid drawings) is simply $(k+m-1)^{(k)}/(k+r)^{(k)}$. Therefore,

$$p^* = \left(\frac{(k+m-1)^{(k)}}{(k+r)^{(k)}} \right)^{1/k} \tag{3}$$

is the effective single-test p value corresponding to LR_m when k tests are to be performed. Table 4b tabulates critical values corresponding to conventional values of p^* for $k = 100$. For $m = 1, p^*$ is .00378, so this is the lowest value shown. For routine testing, it is preferable to use Table 4b, so its critical values are shown in bold type.

It is noteworthy that no strong trend toward the infinite-sample Woodroofe limit is evident in Table 4 as n increases from 30 to 1,000. Even with $n = 10,000$, we find, in results not tabulated, that the proportion of times $\hat{\alpha} < 2$ is still 15.1% with $r = 1,000$. This is contrary to the finding of DuMouchel (1983, p. 1023, using $r = 205$), that the proportion at $n = 10,000$ falls to 8%.

To illustrate the use of Table 4, I have computed continuously compounded monthly real returns on the CRSP value-weighted stock-market index, including dividends, for the period 1/53–12/92 (480 monthly observations, Consumer Price Index deflation). Table 5 shows the mean standard deviation, maximum and minimum percent return per month, along with the Studentized range statistic SR = (max − min)/sd (David, Hartley, and Pearson 1954) and the normalized excess kurtosis statistic $(m_4/m_2^2 - 3)/\sqrt{(24/n)}$, where m_i is the ith sample moment about the mean (Davidson and MacKinnon 1993, p. 568). The SR and kurtosis statistics both reject iid normality at the .9999 level or higher, for the sample size used, as tabulated by the author (two-tailed test).

When symmetric stable parameters are fit to this data by ML, the resulting characteristic exponent estimate is 1.845. The standard errors for $\hat{\alpha}$ and $\hat{\delta}$ are computed from the information matrix, as tabulated by DuMouchel (1975), with β restricted to 0. Although this is not far from 2, even a small deviation from normality can make a big difference for the pricing of out-of-the-money options (McCulloch 1985, 1996). The LR ratio statistic, $\lambda = 2\,\Delta \log L$, is 26.68. Table 4b shows that this is significant at the .996 level or better so that the data again firmly reject iid normality. An additional 99 tests on independent datasets may be performed with this table before we need to worry that significance levels are being systematically overstated due to Monte Carlo sampling error.

For comparison, Table 5 also shows the GP and Hill estimators computed from the same data with pooled tail samples comparable to those that have been used in the literature. The GP estimator $\hat{\alpha}_{\text{GP}} = 1/\hat{\gamma}$ is 12.471, roughly what we would expect, using Table 1a, from a stable sample with an α of about 1.91. The Hill estimator with 40 tail observations is 3.502. This is about what we would expect, using Table 2, from a stable sample with an α of about 1.89. Both of these purport to be significantly greater than 2. With 20 pooled tail observations, the Hill estimator is still slightly more than 2 asymptotic SE's above 2.00.

The validity of the stable LR test for iid normality does not depend on the true distribution's being stable, though its power will presumably be greatest in this case. Like the studentized range and kurtosis tests, the stable LR test is most sensitive to deviations from normality that take the form of thick tails.

6. SOME CONCLUDING CAVEATS

Of course, nothing I have said demonstrates that any one of the datasets I have mentioned really has a stable distribution. Although I have shown that tail-index estimates are not sufficient to reject stability, the converse is also true; a stable ML estimate of α "significantly less than 2" by no means rules out a nonstable distribution that has Paretian tails with index greater than 2. A nonnormal stable distribution may simply be proxying for some other leptokurtic distribution. Even though it is often easy to reject normality, alternative leptokurtic distributions are always going to be very hard to tell apart. Likelihood comparisons like those of Boothe and Glassman (1987) are not nested but perhaps can be interpreted using the methodology of Lee and Brorsen (1995). Csörgő (1987) developed explicit tests for certain aspects of stability that are also promising.

A further complication is that it is rare for an economic or financial dataset to exhibit iid errors of any type. A common "test" for stability, originated by Blattberg and Gonedes (1974) and employed also by Akgiray and Booth (1988), is to measure the stable α for daily, weekly, and monthly returns. If the daily returns are iid stable, these should all be stable with the same α, yet often the α estimates are significantly higher for weekly and monthly returns. This phenomenon has been cited as evidence rejecting stability.

In fact, as Diebold (1993) has pointed out, all that such evidence really rejects is the compound hypothesis of iid stability. It demonstrates either that returns are not identically distributed or that they are not independently distributed or that they are not stably distributed.

There are many reasons why returns, and daily returns in particular, might not be either identically or independently distributed, regardless of whether or not they are stable. These include day-of-the-week effects (Gibbons and Hess 1981; McFarland, Pettit, and Sung 1982), and autoregressive conditional heteroscedasticity (ARCH)-like behavior (Bollerslev, Chou, and Kroner 1992). If stable variates with a common mean and characteristic exponent, but different scale parameters, are mixed together into a single dataset, estimates of α will tend to be biased downward (Lau and Lau 1993). Time-aggregated data will then give a truer, and therefore higher, picture of the conditional α. At the same time, the aggregated data will represent a much smaller sample so that it will of course be harder to reject normality even if the true distribution is iid stable. The arguments of Lau, Lau, and Wingender (1990) go beyond the scope of this article, but see Liu and Brorsen (1995).

ARCH-like effects are very strong in monthly bond-return data, as documented by McCulloch (1985) and Oh (1994). Therefore DuMouchel's estimates cited previously, $\hat{\alpha} = 1.37$ and 1.23, overstate the true deviation from normality. After adjustment for generalized ARCH (GARCH) effects (and a time-varying term premium), Oh's estimates, of 1.61 to 1.69 for most maturities, are much more realistic.

The CRSP value-weighted stock-market-index returns cited previously show no conspicuous ARCH-like behavior at a monthly frequency during the postwar period. Nominal CRSP returns show much greater volatility during the period 1929–1941 than after World War II, however, so that a longer time series should definitely be modeled not as iid but with a slowly adjusting stable GARCH process. There also appear to be high-frequency ARCH-like effects in daily stock returns that do not show up in the monthly data. I made no attempt here to look for seasonals (year-end tax effects and short-month February effects are particularly likely), and there appears to be weak serial correlation in the mean of the series that could reflect variation either in the risk-free real rate of return or the risk premium.

Numerous studies (e.g., Akgiray and Booth 1988; Buckle 1995; Csörgő 1987; Tsionas in press) have found strong evidence of skewness in asset returns, and in particular in stock returns. The current lack of a numerical approximation to the skew-stable densities makes them much more difficult to work with at present than the symmetric stable densities. Undoubtedly the simple iid symmetric stable characterization employed here can be fine-tuned in future research.

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