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SIMPLE CONSISTENT ESTIMATORS OF STABLE DISTRIBUTION PARAMETERS

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ABSTRACT

The four parameters of a stable distribution may be estimated consistently from five pre-determined sample quantiles with the aid of the accompanying tables, for α in the range [0.6, 2.0] and β in the range [-1, 1]. The problem of the discontinuity of the traditional location parameter in the asymmetrical cases as α passes unity is resolved. The proposed estimators of α and c are similar to those of Fama and Roll, except that the small asymptotic bias in their estimators has been eliminated, and their restrictions that α be no less than 1.0 and that the distribution be symmetrical have been relaxed. The proposed estimators can provide good initialization values for other more efficient, but computer-intensive, methods.

1. INTRODUCTION

It is often appropriate in applied statistical work to assume that observable disturbances are the cumulative effect of a large

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number of more or less independent, similarly distributed, contributing disturbances which are themselves unobservable. According to the Generalized Central Limit Theorem, if the sum of identically and independently distributed (IID) random variables has a limiting shaped distribution as the number summed approaches infinity, the limiting distribution must be a member of the stable class, of which the normal distribution is only a special case. (See Gnedenko and Kolmogorov [1968, 162], Feller [1966 II, 544], Zolotarev [1983].) It is therefore natural in such applications to assume that observable disturbances are drawn from a member of this generalized class, particularly when, as is often the case, the presence of leptokurtosis rules out the familiar normal distribution.

The full stable class is characterized by four parameters, usually designated α, β, c, and δ. The traditional location parameter δ simply shifts the distribution to the left or right. The scale parameter c compresses or extends the distribution about δ in proportion to c. If the variable x has the stable distribution $S(x; \alpha, \beta, c, \delta)$, the transformed variable $z = (x - \delta)/c$ will have the same shaped distribution, but with location parameter 0 and scale parameter 1.

The two remaining parameters completely determine the distribution's shape. The characteristic exponent α lies in the range (0, 2] and determines the rate at which the tails of the distribution taper off. When $\alpha = 2.0$, a normal distribution results, with mean δ and variance $2c^2$. When $\alpha < 2$, the variance is infinite. When $\alpha > 1$, the mean of the distribution exists and is equal to δ. However, when $\alpha \leq 1$, the tail(s) are so heavy that even the mean does not exist.

There is almost universal agreement that the fourth parameter, which determines the skewness of the distribution, should be designated β and should lie in the range [-1, +1]. Beyond that, however, there is some often confusing variation, which has led to what Hall (1981) has characterized as a "comedy of errors." We

adopt here the definition proposed by Zolotarev (1957, 441), in which the log characteristic functions of the stable distributions have the form

$$\psi(t) = \log E(e^{ixt}) = \begin{cases} i\delta t - |ct|^{\alpha} [1 - i\beta \, \text{sgn}(t) \tan \frac{\pi\alpha}{2}], & \alpha \neq 1 \\ i\delta t - |ct| [1 + i\beta \frac{2}{\pi} \text{sgn}(t) \log |t|], & \alpha = 1. \end{cases} \quad (1.1)$$

Under this definition, β has the informative property that it is the limiting value of the ratio of the difference of the tail probabilities to the sum of the tail probabilities:

$$\beta = \lim_{x \to \infty} \frac{1 - S(x;a,b,c,d) - S(-x;a,b,c,d)}{1 - S(x;a,b,c,d) + S(-x;a,b,c,d)}, \quad \alpha \neq 2. \quad (1.2)$$

When this β is positive, the distribution is skewed to the right. When it is negative, it is skewed to the left. When $\beta = 0$, the distribution is symmetrical. As α approaches 2.0, β loses its effect and the distribution approaches the symmetrical normal distribution regardless of β.

It should be noted that in 1948 Gnedenko and Kolmogorov introduced a form similar to (1.1), but with the sign on the term involving β reversed for $\alpha \neq 1$ (1968, 164). This "β" is positive when the distribution is negatively skewed, and negative when the distribution is positively skewed, except when $\alpha = 1$. This unnecessarily confusing convention continues to be widely used in important papers, including Paulson, Holcomb and Leitch (1975), and Holt and Crow (1973).

Another parameterization, that is often more convenient for analytical work than (1.1) (e.g. Zolotarev [1966]), gives the log characteristic functions for $\alpha \neq 1$ as

$$\psi(t) = i\delta t - |c^* t|^{\alpha} \exp[-i\beta^* \frac{\pi}{2} \min(\alpha, 2-\alpha) \, \text{sgn}(t)]. \quad (1.3)$$

This formulation is used initially by Chambers, Mallows, and Stuck (1976), with β^* referred to as "β," though they eventually introduce a "β'" which is equivalent to our β. Parameterization (1.3) leaves the interpretation of α and δ unchanged. The β^* of (1.3) may be converted into the β of (1.1) by means of the identity

$$\beta \tan \frac{\pi\alpha}{2} = \tan[\frac{\pi}{2} \beta^* \min(\alpha,2-\alpha)], \ \alpha \neq 1. \quad (1.4)$$

When $\alpha = 1$, the second half of (1.1) is ordinarily used in conjunction with (1.3), and therefore β^* and β are equivalent in this case. Note that for $\alpha \neq 1$ and $\beta^* \neq 0$, the c^* of (1.3) differs from the c of (1.1):

$$c^{*\alpha} = (1 + \beta^2 \tan^2 \frac{\pi\alpha}{2})^{1/2} c^\alpha. \quad (1.5)$$

(See Zolotarev [1957, 442], DuMouchel [1971, 12].)

Yet other parameterizations have been used, e.g., by Feuerverger and McDunnough (1981). In this paper we will employ (1.1), the formulation recommended as "official" by DuMouchel (1971, 6-14).

The stable distribution and density functions may be calculated most straightforwardly by means of the proper integral representation given by Zolotarev (1964/66). The distribution function has been tabulated for $\alpha = 0.5, 0.6, \ldots, 1.9, 1.95$ with $\beta = -1.00, -0.75, \ldots, 1.00$ by DuMouchel (1971, Appendix), and for $\alpha = 1.1, 1.2, \ldots, 1.9, 1.95$ with $\beta = 0$ by Fama and Roll (1968). The density function has been tabulated and graphed by Holt and Crow (1973) for $\alpha = 0.25, 0.5, 0.75, \ldots, 2.0$ and $\beta = -1.00, -.75, \ldots 1.00$ (with the sign of β reversed for $\alpha \neq 1$).

A number of methods have been devised for estimating the parameters of an unknown stable distribution. DuMouchel (1971) has developed an algorithm which applies the maximum likelihood principle to bracketed data. Following Press (1972), Paulson, Holcomb and Leitch (1975) estimate the four stable parameters by fitting

the Fourier transform of the data to the characteristic function. Both these methods are computer-intensive and require initial estimates of the parameters. Arad (1980), Koutrouvelis (1980, 1981), and Feuerverger and McDunnough (1981) also exploit the sample characteristic function. Paulson and Delahanty (1985) investigate a modified weighted squared error procedure. McCulloch (1979) has developed a reasonably inexpensive, though computer-intensive, algorithm which estimates linear regressions with symmetric stable disturbances by maximum likelihood without bracketing. Zolotarev (1980) estimates the three parameters α, β, and c by the method of moments, but requires that the location parameter be known in advance. Brockwell and Brown (1981) estimate α and c with high efficiency in the special case $\alpha < 1$, $\beta = 1$, $\delta = 0$.

In spite of the great efficiency of many of the above-mentioned methods, much of the empirical work that has been done in the past two decades has been based instead on the far simpler method of Fama and Roll (1968, 1971). Using simple functions of pre-determined order statistics, they are able to estimate δ consistently and α and c almost consistently (i.e., with at most a small asymptotic bias). However, their method is restricted to the symmetrical case $\beta = 0$, and is further restricted to α values in the range [1, 2].

Fieletz and Smith (1972) and Leitch and Paulson (1975) find evidence of asymmetry in stock price returns. It is therefore desirable to relax the Fama/Roll assumption of symmetry. Furthermore, although we would ordinarily expect the population mean to exist, we cannot really test this hypothesis unless we entertain the possibility that α lies in at least the upper portion of the range (0, 1].

This paper generalizes the Fama/Roll approach to provide consistent estimators of all four parameters, with β in its full permissible range [-1, 1], and α in the range [0.6, 2]. These estimators are based on simple functions of five pre-determined sample quantiles and are, like the Fama/Roll estimators, asymptotically normal with calculable asymptotic standard errors. Our method eliminates the small asymptotic bias in the Fama/Roll estimators of α and c; at the same time it relaxes their restrictions on α and β.

2. ESTIMATION OF α AND β

Suppose we have n independent drawings x_i from the stable distribution $S(x;\alpha,\beta,c,\delta)$, whose parameters are to be estimated. Let x_p be the p-th population quantile, so that $S(x_p;\ \alpha,\ \beta,\ c,\ \delta) = p$. Let $\hat{x}_p$ be the corresponding sample quantile, suitably corrected for continuity. (If the x_i are arranged in ascending order, this correction may be performed by identifying x_i with $\hat{x}_{q(i)}$ where $q(i) = (2i-1)/(2n)$, and then interpolating linearly to p from the two adjacent q(i) values. Without such a correction, spurious skewness will appear to be present in finite samples.) Then $\hat{x}_p$ is a consistent estimator of x_p.

Define

$$\nu_\alpha = \frac{x_{.95} - x_{.05}}{x_{.75} - x_{.25}} . \tag{2.1}$$

This index in independent of both c and δ. Its values as a function $\phi_1(\alpha,\ \beta)$ are tabulated in Table I. (Tables I , II, and V - VII are derived from DuMouchel's tabulation [1971, Appendix] of the stable distributions.) Let $\hat{\nu}_\alpha$ be the corresponding sample value:

$$\hat{\nu}_\alpha = \frac{\hat{x}_{.95} - \hat{x}_{.05}}{\hat{x}_{.75} - \hat{x}_{.25}} . \tag{2.2}$$

The statistic $\hat{\nu}_\alpha$ is a consistent estimator of the index ν_α. Since ν_α is a strictly decreasing function of α, $\hat{\nu}_\alpha$ gives us a strong fix on α.

Define

TABLE I

$\nu_\alpha = \phi_1(\alpha,\ \beta)$.

α	β = 0.00	0.25	0.50	0.75	1.00
2.00	2.439	2.439	2.439	2.439	2.439
1.90	2.512	2.512	2.513	2.513	2.515
1.80	2.608	2.609	2.610	2.613	2.617
1.70	2.737	2.738	2.739	2.742	2.746
1.60	2.912	2.909	2.904	2.900	2.902
1.50	3.148	3.136	3.112	3.092	3.089
1.40	3.464	3.436	3.378	3.331	3.316
1.30	3.882	3.834	3.720	3.626	3.600
1.20	4.447	4.365	4.171	4.005	3.963
1.10	5.217	5.084	4.778	4.512	4.451
1.00	6.314	6.098	5.624	5.220	5.126
0.90	7.910	7.590	6.861	6.260	6.124
0.80	10.448	9.934	8.779	7.900	7.687
0.70	14.838	13.954	12.042	10.722	10.370
0.60	23.483	21.768	18.332	16.216	15.584
0.50	44.281	40.137	33.002	29.140	27.782

Note that $\phi_1(\alpha,\ -\beta) = \phi_1(\alpha,\ \beta)$.

$$\nu_\beta = \frac{x_{.95} + x_{.05} - 2x_{.5}}{x_{.95} - x_{.05}} , \tag{2.3}$$

and let $\hat{\nu}_\beta$ be the corresponding sample value, defined by analogy to (2.2). Like ν_α, ν_β does not depend on either c or δ. It is tabulated as a function $\phi_2(\alpha,\ \beta)$ in Table II. This function is seen to be strictly increasing in β for each α. The statistic $\hat{\nu}_\beta$, which is a consistent estimator of the index ν_β, therefore gives us a strong fix on β, given what $\hat{\nu}_\alpha$ tells us about α.

The relationship

$$\nu_\alpha = \phi_1(\alpha,\beta), \tag{2.4}$$
$$\nu_\beta = \phi_2(\alpha,\beta) \tag{2.5}$$

TABLE II

$\nu_\beta = \phi_2(\alpha, \beta)$.

α	β = 0.0	0.25	0.5	0.75	1.0
2.00	0.0	0.0	0.0	0.0	0.0
1.90	0.0	0.018	0.036	0.053	0.071
1.80	0.0	0.039	0.077	0.113	0.148
1.70	0.0	0.063	0.123	0.178	0.228
1.60	0.0	0.089	0.174	0.248	0.309
1.50	0.0	0.118	0.228	0.320	0.390
1.40	0.0	0.148	0.285	0.394	0.469
1.30	0.0	0.177	0.342	0.470	0.546
1.20	0.0	0.206	0.399	0.547	0.621
1.10	0.0	0.236	0.456	0.624	0.693
1.00	0.0	0.268	0.513	0.699	0.762
0.90	0.0	0.303	0.573	0.770	0.825
0.80	0.0	0.341	0.634	0.834	0.881
0.70	0.0	0.387	0.699	0.890	0.927
0.60	0.0	0.441	0.768	0.936	0.962
0.50	0.0	0.510	0.838	0.970	0.985

Note that $\phi_2(\alpha, -\beta) = -\phi_2(\alpha, \beta)$.

may be inverted to yield the relationship

$$\alpha = \psi_1(\nu_\alpha, \nu_\beta), \tag{2.6}$$

$$\beta = \psi_2(\nu_\alpha, \nu_\beta). \tag{2.7}$$

The parameters α and β may now be consistently estimated by

$$\hat{\alpha} = \psi_1(\hat{\nu}_\alpha, \hat{\nu}_\beta), \tag{2.8}$$

$$\hat{\beta} = \psi_2(\hat{\nu}_\alpha, \hat{\nu}_\beta). \tag{2.9}$$

Tables III and IV show α and β as functions of ν_α and ν_β.

TABLE III

$\alpha = \psi_1(\nu_\alpha, \nu_\beta)$

ν_α	ν_β = 0.0	0.1	0.2	0.3	0.5	0.7	1.0
2.439	2.0	2.0	2.0	2.0	2.0	2.0	2.0
2.5	1.916	1.924	1.924	1.924	1.924	1.924	1.924
2.6	1.808	1.813	1.829	1.829	1.829	1.829	1.829
2.7	1.729	1.730	1.737	1.745	1.745	1.745	1.745
2.8	1.664	1.663	1.663	1.668	1.676	1.676	1.676
3.0	1.563	1.560	1.553	1.548	1.547	1.547	1.547
3.2	1.484	1.480	1.471	1.460	1.448	1.438	1.438
3.5	1.391	1.386	1.378	1.364	1.337	1.318	1.318
4.0	1.279	1.273	1.266	1.250	1.210	1.184	1.150
5.0	1.128	1.121	1.114	1.101	1.067	1.027	0.973
6.0	1.029	1.021	1.014	1.004	0.974	0.935	0.874
8.0	0.896	0.892	0.887	0.883	0.855	0.823	0.769
10.0	0.818	0.812	0.806	0.801	0.780	0.756	0.691
15.0	0.698	0.695	0.692	0.689	0.676	0.656	0.595
25.0	0.593	0.590	0.588	0.586	0.579	0.563	0.513

Note that $\psi_1(\nu_\alpha, -\nu_\beta) = \psi_1(\nu_\alpha, \nu_\beta)$.

With finite samples, it is possible that $\hat{\nu}_\alpha$ may be less than its smallest permissible value of 2.439, and therefore be offscale in Table III. In this case $\hat{\alpha}$ should be set equal to 2.0 and $\hat{\beta}$ may be set arbitrarily to signum $(\hat{\nu}_\beta)$. It is also possible that sampling error will lead $|\hat{\nu}_\beta|$ to be too high to be consistent with $\hat{\nu}_\alpha$. In this case, $\hat{\beta}$ should be set to ± 1.0 and $\hat{\alpha}$ may consistently be placed anywhere between the highest and the lowest value of α consistent with $\hat{\nu}_\alpha$ at $\beta = \pm 1.0$.

Note that in order to allow linear interpolation in Tables III and IV without downward bias of $|\hat{\beta}|$ when skewness is nearly maximal, "virtual values" of α and β were obtained by extrapolation off Tables I and II for impermissible (ν_α, ν_β) pairs adjacent

TABLE IV

$\beta = \psi_2(\nu_\alpha, \nu_\beta)$

ν_α \ ν_β	0.0	0.1	0.2	0.3	0.5	0.7	1.0
2.439	0.0	2.160	1.0	1.0	1.0	1.0	1.0
2.5	0.0	1.592	3.390	1.0	1.0	1.0	1.0
2.6	0.0	0.759	1.800	1.0	1.0	1.0	1.0
2.7	0.0	0.482	1.048	1.694	1.0	1.0	1.0
2.8	0.0	0.360	0.760	1.232	2.229	1.0	1.0
3.0	0.0	0.253	0.518	0.823	1.575	1.0	1.0
3.2	0.0	0.203	0.410	0.632	1.244	1.906	1.0
3.5	0.0	0.165	0.332	0.499	0.943	1.560	1.0
4.0	0.0	0.136	0.271	0.404	0.689	1.230	2.195
5.0	0.0	0.109	0.216	0.323	0.539	0.827	1.917
6.0	0.0	0.096	0.190	0.284	0.472	0.693	1.759
8.0	0.0	0.082	0.163	0.243	0.412	0.601	1.596
10.0	0.0	0.074	0.147	0.220	0.377	0.546	1.482
15.0	0.0	0.064	0.128	0.191	0.330	0.478	1.362
25.0	0.0	0.056	0.112	0.167	0.285	0.428	1.274

Note that $\psi_2(\nu_\alpha, -\nu_\beta) = -\psi_2(\nu_\alpha, \nu_\beta)$. Entries in this table greater than 1.0 are required in order to permit accurate bivariate linear interpolation as β approaches 1.0 from below. As a result, sampling error in finite samples may yield an interpolated estimate of β greater than 1.0. In this case, the estimate should be truncated back to 1.0.

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to permissible pairs in Tables III and IV. If the interpolated value of $\hat{\beta}$ from Table IV is greater in magnitude than 1.0, $\hat{\beta}$ should therefore be truncated back to ±1.0.

To illustrate the use of Tables III and IV, So (1982, p. 62) finds the following quantiles for apparent forecasting errors computed by subtracting monthly observations on the log of the 30-day forward exchange rate for the British pound (in terms of dollars) from the log of the subsequent spot rate during the floating exchange rate period July 1973-December 1981:

$$\hat{x}_{.05} = -.05413$$
$$\hat{x}_{.50} = .00533$$
$$\hat{x}_{.95} = .05309$$
$$\hat{x}_{.75} - \hat{x}_{.25} = .03354.$$

From equations (2.2) and (2.3) (substituting sample quantiles for population quantiles in the latter), we have

$$\hat{\nu}_\alpha = 3.197,$$
$$\hat{\nu}_\beta = -.1091.$$

Linear interpolation to these values on Tables III and IV yields:

$$\hat{\alpha} = 1.48,$$
$$\hat{\beta} = -0.22.$$

3. ESTIMATION OF THE SCALE PARAMETER

Table V shows the behavior of

$$\nu_c = \frac{x_{.75} - x_{.25}}{c} \tag{3.1}$$

as a function $\phi_3(\alpha, \beta)$. Since $\hat{\alpha}$, $\hat{\beta}$, $\hat{x}_{.75}$ and $\hat{x}_{.25}$ are all consistent estimators of their corresponding population values, the following is a consistent estimator of c:

$$\hat{c} = \frac{\hat{x}_{.75} - \hat{x}_{.25}}{\phi_3(\hat{\alpha}, \hat{\beta})} . \tag{3.2}$$

In the example used above, ν_c interpolated to (1.48, 0.0) is 1.940, and to (1.48, -0.25) is 1.955. Interpolating these two values to $\hat{\beta} = -0.22$ gives 1.953. Thus

TABLE V

$\nu_c = \phi_3(\alpha, \beta)$.

α	β = 0.0	0.25	0.50	0.75	1.00
2.00	1.908	1.908	1.908	1.908	1.908
1.90	1.914	1.915	1.916	1.918	1.921
1.80	1.921	1.922	1.927	1.936	1.947
1.70	1.927	1.930	1.943	1.961	1.987
1.60	1.933	1.940	1.962	1.997	2.043
1.50	1.939	1.952	1.988	2.045	2.116
1.40	1.946	1.967	2.022	2.106	2.211
1.30	1.955	1.984	2.067	2.188	2.333
1.20	1.965	2.007	2.125	2.294	2.491
1.10	1.980	2.040	2.205	2.435	2.696
1.00	2.000	2.085	2.311	2.624	2.973
0.90	2.040	2.149	2.461	2.886	3.356
0.80	2.098	2.244	2.676	3.265	3.912
0.70	2.189	2.392	3.004	3.844	4.775
0.60	2.337	2.635	3.542	4.808	6.247
0.50	2.588	3.073	4.534	6.636	9.144

Note that $\phi_3(\alpha, -\beta) = \phi_3(\alpha, \beta)$.

$$\hat{c} = \frac{0.03354}{1.953} = 0.01717.$$

Fama and Roll (1968, 1971) base their estimator of c on the fortuitous observation that $(x_{.72} - x_{.28})/c$ lies within 0.4 percent of 1.654 for all values of α in the range [1.0, 2.0] when β is constrained to 0. This enables them to estimate c with less than 0.4 percent asymptotic bias by $\tilde{c} = (\hat{x}_{.72} - \hat{x}_{.28})/1.654$ without first knowing α. They then estimate α by comparing the quantity $(\hat{x}_f - \hat{x}_{1-f})/\tilde{c}$ to a tabulation of $(x_f - x_{1-f})/c$, where f is in the tail region. They find that f = .95, .96, or .97 works best for estimating α.

The Fama/Roll method just described unnecessarily compounds the small asymptotic bias in their estimator of c into their estimator of α. Furthermore, when $\beta \neq 0$, the search for an invariant range such as the one they found becomes futile. Because our present method of estimating α and c has no asymptotic bias at all, it should be used in place of the Fama/Roll method even when β is constrained to zero. This can be done by simply comparing ν_α to the first column of Table I.

Since our method for estimating the scale parameter does not depend on the constancy of the range used to estimate it, we have substituted the "round" interquartile range for the .72-.28 range used by Fama and Roll. We do follow Fama and Roll, however, in basing our estimate of α on $\hat{x}_{.95}$ and $\hat{x}_{.05}$. Of the three values cited by them as satisfactory for f, we have chosen the smallest, since it reduces the sampling error of the quantiles with limited samples. Also, being a round value, it is more likely to have been routinely collected in prior studies. Other quantiles than the ones we have chosen may be slightly more efficient, but our goal is not complete efficiency (which can be obtained with maximum likelihood) but rather convenience and simplicity.

4. ESTIMATION OF THE LOCATION PARAMETER

Table VI shows the behavior of

$$\nu_\delta = \frac{\delta - x_{.5}}{c} \qquad (3.1)$$

as a function $\phi_4(\alpha, \beta)$. The location parameter δ could easily be estimated by $\hat{c}\,\phi_4(\hat{\alpha}, \hat{\beta}) + \hat{x}_{.5}$, were it not for a double singularity in ϕ_4 as α passes 1.0 when $\beta \neq 0$. This singularity makes interpolation meaningless between $\alpha = 0.9$ and $\alpha = 1.1$, and makes linear interpolation highly inaccurate unless α is quite far from 1.0.

TABLE VI

$\nu_\delta = \phi_4(\alpha, \beta)$.

α	β = 0.0	0.25	0.50	0.75	1.00
2.00	0.0	0.0	0.0	0.0	0.0
1.90	0.0	0.023	0.047	0.070	0.094
1.80	0.0	0.051	0.101	0.152	0.202
1.70	0.0	0.084	0.167	0.250	0.331
1.60	0.0	0.126	0.252	0.375	0.495
1.50	0.0	0.184	0.366	0.544	0.717
1.40	0.0	0.269	0.534	0.791	1.041
1.30	0.0	0.407	0.808	1.196	1.573
1.20	0.0	0.679	1.347	1.998	2.631
1.10	0.0	1.483	2.949	4.389	5.806
1.00^+	0.0	∞	∞	∞	∞
1.00	0.0	-0.098	-0.223	-0.383	-0.576
1.00^-	0.0	$-\infty$	$-\infty$	$-\infty$	$-\infty$
0.90	0.0	-1.677	-3.394	-5.159	-6.966
0.80	0.0	-0.865	-1.789	-2.777	-3.820
0.70	0.0	-0.580	-1.243	-1.992	-2.816
0.60	0.0	-0.422	-0.960	-1.613	-2.373
0.50	0.0	-0.311	-0.779	-1.409	-2.198

Note that $\phi_4(\alpha, -\beta) = -\phi_4(\alpha, \beta)$.

To solve this problem we reflect for a moment on the significance of the traditional location parameter. When n IID stable random variables with parameters $(\alpha, \beta, c, \delta)$ are averaged, the resulting mean has a stable distribution with the identical shape, i.e., with the same α and β. We define a focus of stability of a stable distribution as any quantile which remains stationary under such averaging. It can be readily shown from the characteristic function that the scale parameter of the sample mean will be the original scale parameter times $n^{1/\alpha - 1}$. This implies that for $\alpha > 1$ there will be a unique focus of stability and that the other

quantiles of the sample mean will converge toward this focus as n increases. For $\alpha < 1$ there will again be a unique focus of stability, and the other quantiles of the sample mean will all diverge from this focus. In either case, the focus of stability is the traditional location parameter δ, and this parameter is invariant under averaging.

However, when $\alpha = 1$, the scale of the sample mean is the same as the original scale. This implies either that every quantile is a focus of stability, which is true in the Cauchy case $\alpha = 1$, $\beta = 0$, or else that the distribution has no focus of stability at all, which occurs when $\beta \neq 0$. When $\alpha = 1$ and $\beta > 0$, all quantiles shift an equal amount to the right under averaging, while with $\alpha = 1$ and $\beta < 0$, all quantiles shift an equal amount to the left under averaging. In these cases, the traditional location parameter has no particular significance. It is merely an arbitrary quantile that happens to simplify the characteristic function.

The discontinuity which occurs at $\alpha = 1$ when $\beta \neq 0$ is not so much a discontinuity in the distribution as it is a discontinuity in the focus of stability. Relative to any one quantile, e.g., the median, all the other quantiles change smoothly as α passes 1. Relative to the quantiles, however, the focus of stability moves off to $+\infty$ $(-\infty)$ as α approaches unity from above, and off to $-\infty$ $(+\infty)$ as α approaches unity from below, when $\beta > 0$ (< 0). The focus of stability either does not exist (for $\beta \neq 0$) or else covers the entire real line (for $\beta = 0$) when $\alpha = 1$.

Zolotarev (1957, 454) has shown that the distribution of the random variable

$$x' = \begin{cases} x - \beta c \tan \frac{\pi\alpha}{2}, & \alpha \neq 1 \\ x, & \alpha = 1 \end{cases} \qquad (4.[illegible])$$

undergoes no discontinuity, holding δ, c, and β constant, as passes unity. This suggests that we consider the following alternative location parameter:

$$\zeta = \begin{cases} \delta + \beta c \tan \frac{\pi\alpha}{2}, & \alpha \neq 1 \\ \delta, & \alpha = 1. \end{cases} \tag{4.3}$$

Table VII shows the behavior of

$$\nu_\zeta = \frac{\zeta - x_{.5}}{c} \tag{4.4}$$

as a function $\phi_5(\alpha, \beta)$. The parameter ζ may now be consistently estimated by

$$\hat{\zeta} = \hat{x}_{.5} + \hat{c}\,\phi_5(\hat{\alpha}, \hat{\beta}). \tag{4.5}$$

If somehow we knew that α was precisely unity, $\hat{\zeta}$ would provide us with a consistent estimator of δ. If $\alpha \neq 1$, the traditional location parameter becomes the unique focus of stability and may be estimated by

$$\hat{\delta} = \hat{\zeta} - \hat{\beta}\hat{c} \tan \frac{\pi\hat{\alpha}}{2}. \tag{4.6}$$

When α is far from unity, our estimator for δ gives virtually the same result as would linear interpolation off Table VI. However, as α approaches unity, δ may become quite far from the center of the observed sample, and may even lie outside it. If α is insignificantly different from unity, the true value of δ could lie anywhere.

In general, ζ has no particular significance by itself. It is best thought of as a mere stepping stone that enables us to pass easily from the quantiles of the distribution to the focus of stability.

In the above example, ν_ζ interpolates to 0.060 at (1.48, -0.22). Therefore

$$\hat{\zeta} = 0.00533 + (0.01717)(0.060) = 0.00635$$

TABLE VII

$\nu_\zeta = \phi_5(\alpha, \beta)$.

α	β = 0.0	0.25	0.50	0.75	1.00
2.00	0.0	0.0	0.0	0.0	0.0
1.90	0.0	-0.017	-0.032	-0.049	-0.064
1.80	0.0	-0.030	-0.061	-0.092	-0.123
1.70	0.0	-0.043	-0.088	-0.132	-0.179
1.60	0.0	-0.056	-0.111	-0.170	-0.232
1.50	0.0	-0.066	-0.134	-0.206	-0.283
1.40	0.0	-0.075	-0.154	-0.241	-0.335
1.30	0.0	-0.084	-0.173	-0.276	-0.390
1.20	0.0	-0.090	-0.192	-0.310	-0.447
1.10	0.0	-0.095	-0.208	-0.346	-0.508
1.00	0.0	-0.098	-0.223	-0.383	-0.576
0.90	0.0	-0.099	-0.237	-0.424	-0.652
0.80	0.0	-0.096	-0.250	-0.469	-0.742
0.70	0.0	-0.089	-0.262	-0.520	-0.853
0.60	0.0	-0.078	-0.272	-0.581	-0.997
0.50	0.0	-0.061	-0.279	-0.659	-1.198

Note that $\phi_5(\alpha, -\beta) = -\phi_5(\alpha, \beta)$.

---

and

$$\hat{\delta} = 0.00635 - (0.22)(0.01717) \tan \frac{1.48\pi}{2} = 0.00233.$$

Fama and Roll use the .5 truncated mean as their estimator of δ. When $\beta \neq 0$ (and $\alpha > 1$), this statistic is no longer an unbiased estimator of δ. An estimator of δ, again using ζ as a stepping stone, could be constructed on the basis of the .5 truncated mean, and would undoubtedly be a little more efficient than our estimator, at least for $\alpha > 1$. In the interest of simplicity, however, we prefer to base our estimator on the more readily available median.

A call to IMSL subroutine GGSTA (based on Chambers, Mallows and Stuck [1975]) produces a quasi-random stable variate with "BPRIME" equal to our β, c = 1, and our $\zeta = 0$ (not $\delta = 0$).

5. ASYMPTOTIC VARIANCES AND COVARIANCES

Let $\hat{x}_p$ and $\hat{x}_q$ be the p-th and q-th quantiles of a sample of size n drawn from a standardized stable distribution with shape parameters α and β and with c = 1 and $\delta = 0$. Let s(x) be the probability density function for this distribution. For $p \leq q$, and n large (see e.g. Kendall and Stuart [1958, 330]),

$$\operatorname{cov}(\hat{x}_p, \hat{x}_q) = \frac{p(1-q)}{n s(x_p) s(x_q)} . \tag{5.1}$$

Let $p_1, \ldots, p_5$ equal .05, .25, .50, .75, .95 and let C be the 5 x 5 matrix whose typical element c_{ij} is $\operatorname{cov}(\hat{x}_{p_i}, \hat{x}_{p_j})$. The constants multiplying 1/n in this matrix may be evaluated for α = 0.5, 0.75, ..., 2.00, and β = 0.0, 0.25, ..., 1.0 using the stable density tables of Holt and Crow (1973). (Whenever the densities in Holt and Crow's 4-place tables dropped below .0004, as happened occasionally with α = 0.5, the first term of the Bergström expansion [1952] was substituted.)

For interior values of the parameter space, the asymptotic covariance matrix of our parameter estimates may be obtained by standard methods (see, e.g., Goldberger [1964, 125]). Let $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_6$ represent $\hat{\alpha}$, $\hat{\beta}$, $\hat{c}$, $\hat{\delta}$, $\hat{\zeta}$, $\hat{x}_{.5}$, and let B be the matrix whose typical element is

$$b_{ij} = \frac{\partial \hat{\theta}_j}{\partial \hat{x}_{p_i}} \Bigg|_{\hat{x}_{p_i} = x_{p_i}} . \tag{5.2}$$

We have evaluated this matrix by means of small perturbations of the population quantiles. The 6 x 6 matrix

$$V = nB^T CB \tag{5.3}$$

gives the normalized asymptotic variances and covariances for c = 1 and sample size of 1. The 6 x 6 asymptotic variance-covariance matrix Σ may be calculated from these constants by means of

$$\Sigma = n^{-1} DVD, \tag{5.4}$$

where $D = (d_{ij})$, d_{11} and d_{22} (corresponding to α and β) both equal 1, d_{33}, d_{44}, d_{55}, and d_{66} (corresponding to c, δ, ζ, and $x_{.5}$) all equal c, and $d_{ij} = 0$ otherwise.

The normalized asymptotic standard deviations of the parameter estimates, which are the square roots of the main diagonal elements of V, are tabulated in Table VIII. For comparison, panels a), c), and f) of Table VIII show in parentheses the approximate asymptotic standard deviation of the Fama/Roll estimators of α (f = .95), c, and δ. Since our estimators of α and c are based on very nearly the same information as Fama and Roll's, the asymptotic variances of these two estimators are virtually unchanged. However, our estimator of δ ($\equiv \hat{x}_{.5}$ when $\beta = 0$) undergoes a 5 to 19% loss of efficiency.

Selected correlation coefficients

$$\rho_{ij} = \frac{v_{ij}}{\sqrt{v_{ii} v_{jj}}} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii} \sigma_{jj}}} \tag{5.5}$$

are tabulated in Table IX for $\beta \geq 0$. Correlation coefficients involving $\hat{\alpha}$ or $\hat{c}$ on the one hand and $\hat{\beta}$, $\hat{\delta}$, $\hat{\zeta}$, or $\hat{x}_{.5}$ on the other hand are odd functions of β. All other correlation coefficients are even functions of β.

In the example given above, the sample size was 100. The asymptotic standard deviations $\sigma_i = \sigma_{ii}^{1/2}$ of the parameter estimates are then

$$\sigma_{\hat{\alpha}} = 2.04(100)^{-1/2} = .204$$
$$\sigma_{\hat{\beta}} = 3.16(100)^{-1/2} = .316$$
$$\sigma_{\hat{c}} = 1.319(.01717)(100)^{-1/2} = .00226$$
$$\sigma_{\hat{\delta}} = 3.42(.01717)(100)^{-1/2} = .00588.$$

TABLE VIII

Normalized Asymptotic Standard Deviations of Parameter Estimates ($v_{ii}^{1/2}$)

a. Characteristic exponent ($\hat{\alpha}$)

α	β = 0.00	(0.00)[a]	0.25	0.50	0.75	1.00^-
2.00^-	4.02	(3.56)[b]	4.02	4.02	4.02	4.02
1.75	2.81	(2.71)	2.85	2.93	3.05	3.17
1.50	1.97	(2.10)	2.07	2.33	2.64	2.85
1.25	1.65	(1.68)	1.79	2.03	2.42	2.55
1.00	1.42	(1.37)	1.56	1.87	2.16	2.16
0.75	1.13		1.41	1.53	1.77	1.65
0.50	1.32		1.54	1.73	1.70	1.75

b. Skewness parameter ($\hat{\beta}$)

α	β = 0.00	0.25	0.50	0.75	1.00^-
2.00^-	∞	∞	∞	∞	∞
1.75	6.22	6.92	8.75	11.59	14.42
1.50	3.46	3.15	3.23	4.95	6.31
1.25	3.05	2.77	2.37	3.30	3.91
1.00	2.93	2.62	2.09	2.12	2.70
0.75	3.02	2.65	1.91	2.28	2.16
0.50	3.38	3.21	2.57	3.55	1.94

c. Scale parameter ($\hat{c}$)

α	β = 0.00	(0.00)[c]	0.25	0.50	0.75	1.00^-
2.00^-	1.26	(1.27)	1.26	1.26	1.26	1.26
1.75	1.25	(1.29)	1.24	1.23	1.22	1.30
1.50	1.28	(1.35)	1.31	1.38	1.27	1.36
1.25	1.38	(1.42)	1.48	1.55	1.44	1.56
1.00	1.65	(1.59)	1.86	2.37	3.18	2.08
0.75	2.15		3.29	3.36	2.63	2.95
0.50	3.36		6.30	6.46	4.95	4.89

Notes:

[a]Approximate asymptotic standard deviation of Fama/Roll $\hat{\alpha}_{.95}$, calculated as $s(\hat{\alpha})\sqrt{N}$, for N = 599 (1971, p. 333).

[b]Derived Fama/Roll value of 2.08 adjusted by $\sqrt{2\pi/(\pi-1)}$ to compensate for truncation of their estimator at $\alpha = 2$.

[c]Approximate asymptotic standard deviation of Fama/Roll $\hat{c}$, calculated as $\sigma(\hat{c})\sqrt{N}$, for N = 599 (1971, p. 332).

(continued ...)

TABLE VIII (Continued)

d. Traditional location parameter ($\hat{\delta}$)

α	β = 0.00	0.25	0.50	0.75	1.00^-
2.00^-	1.77	1.77	1.77	1.77	1.77
1.75	2.09	2.09	2.13	2.20	2.28
1.50	3.06	3.03	3.81	3.84	4.36
1.25	6.71	8.09	11.01	14.75	18.71
1.00^+	∞	∞	∞	∞	∞
1.00	1.70	1.74	1.87	2.35	2.87
1.00^-	∞	∞	∞	∞	∞
0.75	8.28	6.39	11.77	23.09	29.33
0.50	4.03	3.85	6.41	12.60	17.32

e. $\hat{\zeta}$

α	β = 0.00	0.25	0.50	0.75	1.00^-
2.00^-	1.77	1.77	1.77	1.77	1.77
1.75	2.18	2.21	2.22	2.30	2.33
1.50	1.98	1.98	2.05	2.23	2.35
1.25	1.85	1.87	1.93	2.22	2.42
1.00	1.70	1.74	1.87	2.35	2.87
0.75	1.46	1.77	2.01	3.15	4.00
0.50	0.94	1.96	2.70	5.94	6.98

f. Median ($\hat{x}_{.5}$)

α	β = 0.00	(0.00)[d]	0.25	0.50	0.75	1.00^-
2.00^-	1.77	(1.43)	1.77	1.77	1.77	1.77
1.75	1.76	(1.44)	1.77	1.77	1.79	1.80
1.50	1.74	(1.45)	1.75	1.79	1.84	1.90
1.25	1.69	(1.47)	1.72	1.79	1.89	2.01
1.00	1.57	(1.49)	1.67	1.88	2.15	2.44
0.75	1.32		1.62	2.14	2.74	3.37
0.50	0.79		1.43	2.42	3.65	5.13

Note: [d]Approximate asymptotic standard deviation of Fama/Roll .5 truncated mean, calculated as $s(E)\sqrt{n}$, for n = 101 (1968, p. 831). Note that when β is constrained to 0 as in Fama/Roll, the median becomes our estimator of δ.

TABLE IX

Asymptotic Correlation Coefficients

a. $\rho_{\hat{\alpha}\hat{c}}$

α	β = 0.0	0.25	0.50	0.75	1.00^-
2.00^-	0.77	0.77	0.77	0.77	0.77
1.75	0.64	0.64	0.64	0.64	0.69
1.50	0.46	0.49	0.60	0.54	0.63
1.25	0.32	0.42	0.52	0.55	0.67
1.00	0.35	0.45	0.71	0.88	0.71
0.75	0.37	0.63	0.70	0.62	0.72
0.50	0.49	0.71	0.76	0.53	0.75

b. $\rho_{\hat{\alpha}\hat{\delta}}$

α	β = 0.0	0.25	0.50	0.75	1.00^-
2.00^-	0.00	0.00	0.00	0.00	0.00
1.75	0.00	-0.10	-0.17	-0.27	-0.34
1.50	0.00	-0.37	-0.65	-0.63	-0.72
1.25	0.00	-0.68	-0.88	-0.93	-0.95
1.00	0.00	0.13	0.13	0.21	-0.34
0.75	0.00	-0.31	-0.89	-0.96	-0.96
0.50	0.00	-0.18	-0.77	-0.85	-0.90

c. $\rho_{\hat{\beta}\hat{c}}$

α	β = 0.0	0.25	0.50	0.75	1.00^-
1.75	0.00	0.26	0.46	0.56	0.67
1.50	0.00	0.04	0.19	0.49	0.65
1.25	0.00	-0.28	-0.10	0.39	0.67
1.00	0.00	-0.31	-0.33	-0.34	0.75
0.75	0.00	-0.66	-0.37	0.46	0.79
0.50	0.00	-0.74	-0.52	-0.29	0.85

d. $\rho_{\hat{c}\hat{\delta}}$

α	β = 0.0	0.25	0.50	0.75	1.00^-
2.00^-	0.00	0.00	0.00	0.00	0.00
1.75	0.00	0.01	0.05	0.07	0.08
1.50	0.00	-0.07	-0.18	0.01	-0.06
1.25	0.00	-0.33	-0.38	-0.36	-0.46
1.00	0.00	0.23	0.23	0.25	-0.16
0.75	0.00	0.08	-0.69	-0.73	-0.84
0.50	0.00	0.27	-0.55	-0.39	-0.87

An approximate .95 confidence interval for the characteristic exponent extends from 1.08 up to 1.88, so we may tentatively reject normality. Strictly speaking, we may only make comparisons on the interior of the parameter space, but since we may reject values of α under 2.0, we are also justified in ruling out 2.0 itself. We may reject $\alpha = 1.0$, but not by far.

An approximate .95 confidence interval for β extends from -0.40 to 0.84. We may therefore reject maximal skewing, in particular negative maximal skewing, but not symmetry. The estimate of δ is less than half $\sigma_{\hat{\delta}}$, so we may not reject the hypothesis that the log forward exchange rate is an unbiased predictor of the log spot exchange rate.

Note that if α were insignificantly different from 1.0, nothing at all could be said about the location of the mean, since we could not then reject the hypothesis that the mean does not exist. Even if α differs significantly from 1.0, σ_{δ} becomes quite large as α approaches unity. The parameter ζ and the median, on the other hand, are quite well defined in this vicinity. Thus the location of the distribution may be quite clear even when very little, if anything, may be said about its "location parameter."

The correlation coefficients of greatest interest are $\rho_{\hat{\alpha}\hat{c}}$, $\rho_{\hat{\alpha}\hat{\delta}}$, $\rho_{\hat{\beta}\hat{\delta}}$, and $\rho_{\hat{c}\hat{\delta}}$. The first of these is positive because a randomly high interquartile range will tend to produce high estimates of both c and α. Thus if we are concerned about estimating the tail probabilities, the error in our estimate of α will to some extent be compensated for by an offsetting error in $\hat{c}$.

The correlation between $\hat{\alpha}$ and $\hat{\delta}$ is negative for positive β (and positive for negative β) because of the powerful effect of α on the term $-\beta c \tan(\pi\alpha/2)$ that separates δ from ζ and therefore from the vicinity of the median. The same term causes $\rho_{\hat{\beta}\hat{\delta}}$ to be positive for $\alpha > 1.0$, at least when $|\beta|$ is not too large. This correlation implies that a priori knowledge that $\beta = 0$ would greatly enhance our ability to make inferences about δ, particularly when α is just above 1.0.

TABLE X

Asymptotic Efficiency of Simple Consistent Estimators

a. Estimator of α

α	β = 0.0	0.5	1.0^{-}
2.00^{-}	0.0	0.0	0.0
1.75	0.27	0.23	0.13
1.50	0.61	0.41	0.19
1.25	0.69	0.44	0.17
1.00	0.60	0.33	0.09
0.75	0.49	0.26	0.01

b. Estimator of β

α	β = 0.0	0.5	1.0^{-}
2.00^{-}	----	----	---
1.75	0.60	0.26	0.0
1.50	0.77	0.69	0.0
1.25	0.54	0.83	0.0
1.00	0.35	0.73	0.0
0.75	0.19	0.41	0.0

c. Estimator of c

α	β = 0.0	0.5	1.0^{-}
2.00^{-}	0.31	0.31	0.31
1.75	0.57	0.56	0.41
1.50	0.71	0.58	0.44
1.25	0.81	0.59	0.40
1.00	0.77	0.35	0.31
0.75	0.67	0.25	0.23

d. Estimator of δ

α	β = 0.0	0.5	1.0^{-}
2.00^{-}	0.63	0.63	0.63
1.75	0.92	0.79	0.64
1.50	0.76	0.61	0.71
1.25	0.56	0.49	0.33
1.00	0.73	----[a]	----[a]
0.75	0.30[b]	0.52[b]	0.02[b]

Notes:

[a]DuMouchel gives no value for the asymptotic standard deviation of the maximum likelihood estimator of δ ($\equiv\zeta$) for these cases.

[b]Based on DuMouchel's values for $\alpha = 0.8$. True efficiency is actually a little smaller than indicated.

It is interesting that $\rho_{\hat{c}\hat{\delta}}$ is negative for positive β as α approaches 1.0 from above, in spite of the fact that the term separating δ from ζ is proportional to c. This is apparently more than offset by the strong positive value of $\rho_{\hat{\alpha}\hat{c}}$, coupled with the strongly negative value of $\rho_{\hat{\alpha}\hat{\delta}}$.

6. ASYMPTOTIC EFFICIENCY

Table X shows the asymptotic efficiency of our estimators of α, β, c, and δ. This is calculated as the ratio of the asymptotic variance of the maximum likelihood estimates, as reported by DuMouchel (1975, 388), to that of our estimates. DuMouchel's values were graphically interpolated (and occasionally extrapolated, from $\alpha = 0.8$ to 0.75) to the characteristic exponent values in the Holt and Crow density tabulation. DuMouchel's values for σ_α and σ_c seem to be out of line for $\beta = 0.5$ and 1.0 at $\alpha = 1.1$, and so were not employed.

For example, at $\alpha = 1.5$ and $\beta = 0$, the normalized asymptotic standard deviation of the maximum likelihood estimate of α is 1.54, while that for our estimator is 1.97. The efficiency of our estimator is therefore $(1.54/1.97)^2 = 0.61$. Using our estimator of α instead of a maximum likelihood estimate is thus equivalent, with a very large sample, to a loss of 39% of the sample, for these particular shape parameters.

The zero asymptotic efficiency of our estimators of α as α approaches 2.0 from below and of β as β approaches 1.0 from below (or -1.0 from above) reflects DuMouchel's observation that maximum likelihood is "super-efficient" in these cases, in the sense that it has a zero asymptotic standard deviation. The absence of one or both Paretian tails is apparently so distinctive in an infinite sample that these cases stand by themselves. With a finite sample, the maximum likelihood estimators of α and β would of course have a positive sampling error, even in these cases, and use of our estimators would be equivalent to only a finite loss of sample. It should be remembered that the asymptotic efficiency of least squares as an estimator of δ is zero for $\alpha < 2.0$, even though the sample mean can still provide very valuable information about δ as long as $\alpha > 1$ (McCulloch 1980).

7. CONCLUDING REMARKS

The technique we have presented in this paper consistently estimates the four stable distribution parameters with only minimal calculations. The loss of asymptotic efficiency ranges from as little as 19% to as much as 100% (in certain boundary cases). Even in the latter cases our method provides useful estimates with finite samples. We hope that the availability of these estimators facilitates future research into these important distributions.

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