DEFINABLE CHOICE IN D-MINIMAL EXPANSIONS OF ORDERED GROUPS*

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A (first-order) theory T extending the theory of dense linear orders without endpoints is **d-minimal** (short for “discrete-minimal”) if in every model of T:

- The underlying set of the model is definably connected (in the model).
- Every unary (parametrically) definable set either has interior or is a finite union of discrete sets.

The intent is, loosely speaking, to capture the notion of the next best thing to o-minimality for theories whose models may define infinite discrete sets. Note that T is o-minimal if, in the above, “discrete sets” is replaced by “points”.

An expansion of a dense linear order without endpoints is **d-minimal** if its complete theory is d-minimal. See [FM05, Mil05, MT06] for some examples of structures that are d-minimal but not o-minimal.

The main result of this note extends a useful fact from o-minimality:

Theorem. *Let* T *be a complete d-minimal theory extending the theory of dense ordered groups with a distinguished positive element. Then* T *has definable Skolem functions and elimination of imaginaries.*

Indeed:

Definable Choice. *Let* $\mathfrak{R}$ *be a d-minimal expansion of a dense ordered group* $(R, <, +, 0, 1)$ *with* $1 > 0$*. Then*:

- *If* $Y \subseteq R^{m+n}$ *is* $\emptyset$*-definable and* X *is the projection of* Y *on the first* m *coordinates, then there is a* $\emptyset$*-definable* $f\colon X \to R^n$ *such that the graph of* f *is contained in* Y*.*
- *If* E *is a* $\emptyset$*-definable equivalence relation on a* $\emptyset$*-definable* $X \subseteq R^m$*, then there is a* $\emptyset$*-definable* $f\colon X \to X$ *such that* $xEy \Leftrightarrow f(x) = f(y)$ *for all* $x, y \in X$*.*

(Of course, the above then holds with “A-definable” in place of “$\emptyset$-definable” for any $A \subseteq R$.)

The use of abelian notation is justified: Definable connectedness allows us to work in many respects as if we were over the reals—see *e.g.* [Mil01] for details—in particular, every definably connected dense ordered group is abelian and divisible.

From now on, $\mathfrak{R}$ denotes a fixed, but arbitrary, expansion of a dense linear order without endpoints $(R, <)$. Definability is with respect to $\mathfrak{R}$. Topological notions are with respect to the usual box topologies. The variables m and n range over $\mathbb{N}$ (the non-negative integers).

The definition of d-minimality given above is possibly the easiest to state, but it misses the point somewhat, especially for structures.

*This is **not** a preprint; please do not refer to it as such.

Proposition 1. *$\mathfrak{R}$ is d-minimal if and only if R is definably connected and for every definable $S \subseteq R^{m+1}$ there exists $N \in \mathbb{N}$ such that for all $x \in R^m$ the fiber*

$$S_x := \{\, t \in R : (x,t) \in S \,\}$$

either has interior or is a union of N (not necessarily distinct) discrete sets.

Proof. It is easy to see that definable connectedness is an elementary property. The rest is a routine compactness argument; *cf.* [Mil05, §8.5]. □

Note. Under the present definition, d-minimality *of structures* is trivially preserved under elementary equivalence. One of the remarkable facts about o-minimality is that this artifice is unnecessary: If $\mathfrak{R}$ is o-minimal—*i.e.*, if every unary definable set is a finite union of points and open intervals—then the same is true of every $\mathfrak{R}' \equiv \mathfrak{R}$ [KPS86]. *Open questions:* If R is definably connected and every unary definable set either has interior or is a finite union of discrete sets, is $\mathfrak{R}$ d-minimal? What if $\mathfrak{R}$ expands an ordered group or field? What if $R = \mathbb{R}$?

Proof of Definable Choice. (Cf. [Dri98, pp. 93–94] and the proof of [Mil05, Theorem 4].) Suppose that $\mathfrak{R}$ is a d-minimal expansion of a dense ordered group $(R, <, +, 0, 1)$ with $1 > 0$. For $A \subseteq R$, put:

$$\begin{aligned}
\operatorname{int}(A) &= \text{the interior of } A\\
\operatorname{cl}(A) &= \text{the closure of } A\\
\operatorname{bd}(A) &= \text{the boundary of } A \ (= \operatorname{cl}(A) \setminus \operatorname{int}(A)\,)\\
\operatorname{isol}(A) &= \text{the isolated points of } A.
\end{aligned}$$

If A is $\emptyset$-definable, then so are each of the above sets. For each n, let $\mathcal{B}_n$ be the collection of all nonempty $\emptyset$-definable $A \subseteq R$ such that $\operatorname{bd}(A)$ is a union of n discrete sets. Note if $S \subseteq R^{m+1}$ is $\emptyset$-definable, then $\{\, x \in R^m : S_x \in \mathcal{B}_n \,\}$ is $\emptyset$-definable. By the previous proposition (and a routine induction; see [Dri98, pp. 94]), it suffices to show that for every n there exists $\beta_n \colon \mathcal{B}_n \to R$ such that:

- For every $A \in \mathcal{B}_n$, $\beta_n(A) \in \operatorname{int}(A) \cup \operatorname{isol}(A)$;
- For every $\emptyset$-definable $S \subseteq R^{m+1}$, the function

  $$x \mapsto \beta_n(S_x) \colon \{\, x \in R^m : S_x \in \mathcal{B}_n \,\} \to R$$

  is $\emptyset$-definable.

These requirements will be realized by construction.

First, for each n, put $\mathcal{A}_n = \{\, A \in \mathcal{B}_n : \operatorname{int}(A) = \emptyset \,\}$. Note that $\mathcal{A}_0 = \emptyset$. We define functions $\alpha_n \colon \mathcal{A}_n \to R$ by induction on $n \geq 1$ such that for every $n \geq 1$ and $A \in \mathcal{A}_n$ we have $\alpha_n(A) \in \operatorname{isol}(A)$ and $\alpha_{n+1} \restriction \mathcal{A}_n = \alpha_n$.

Suppose $n = 1$. Then every $A \in \mathcal{A}$ is nonempty, closed and discrete. Define α_1 by

$$\alpha_1(A) = \begin{cases} \min A, & \text{if } \inf A \neq -\infty \\ \max A, & \text{if } \inf A = -\infty \text{ and } \sup A < +\infty \\ \min\{\, t \in A : t \geq 0 \,\}, & \text{otherwise.} \end{cases}$$

(The existence of the appropriate maxima and minima follows from definable connectedness; see [Mil01, 1.10].)

Assume the result for a certain $n \geq 1$; we show it for $n+1$. Let $A \in \mathcal{A}_{n+1}$. If $A \in \mathcal{A}_n$, then put $\alpha_{n+1}(A) = \alpha_n(A)$. Suppose that $A \notin \mathcal{A}_n$; then $\operatorname{cl}(A) \setminus \operatorname{isol}(A) \in \mathcal{A}_n$. Inductively, put $b = \alpha_n(\operatorname{cl}(A) \setminus \operatorname{isol}(A))$. Now, b is a limit point of $\operatorname{isol}(A)$—but not of $A \setminus \operatorname{isol}(A)$, *nota bene*—so it is limit point of at least one of $(-\infty, b) \cap \operatorname{isol}(A)$ or $(b, \infty) \cap \operatorname{isol}(A)$. If the former, put $a = \inf\{\, t < b : (t, b) \cap A \subseteq \operatorname{isol}(A) \,\}$ and

$$\alpha_{n+1}(A) = \begin{cases} \min\{\, t \in A : t \geq b - 1 \,\}, & \text{if } a = -\infty \\ \min\{\, t \in A : 2t \geq b - a \,\}, & \text{otherwise.} \end{cases}$$

If b is not a limit point of $(-\infty, b) \cap \operatorname{isol}(A)$, then put $c = \sup\{\, t > b : (b, t) \cap A \subseteq \operatorname{isol}(A) \,\}$ and

$$\alpha_{n+1}(A) = \begin{cases} \min\{\, t \in A : t \geq b + 1 \,\}, & \text{if } c = \infty \\ \min\{\, t \in A : 2t \geq b + a \,\}, & \text{otherwise.} \end{cases}$$

(We have finished the construction of the functions α_n.)

For $-\infty \leq a < b \leq +\infty$, put

$$\operatorname{midpt}(a, b) = \begin{cases} (a+b)/2 & \text{if } a, b \in R \\ 0 & \text{if } a = -\infty \text{ and } b = +\infty \\ a + 1 & \text{if } a \in R \text{ and } b = +\infty \\ b - 1 & \text{if } a = -\infty \text{ and } b \in R. \end{cases}$$

For $U \subseteq R$ open and definable, put

$$\operatorname{midpts}(U) = \{\, \operatorname{midpt}(a, b) : -\infty \leq a < b \leq +\infty,\ (a, b) \subseteq U,\ a, b \notin U \,\}.$$

Note that $\operatorname{midpts}(U) \in \mathcal{A}_{n+1}$ if $U \in \mathcal{B}_n \setminus \mathcal{A}_n$. Finally, for $n \in \mathbb{N}$ and $B \in \mathcal{B}_n$, put

$$\beta_n(B) = \begin{cases} \alpha_n(B), & B \in \mathcal{A}_n \\ \alpha_{n+1}(\operatorname{midpts}(\operatorname{int}(B))), & B \notin \mathcal{A}_n \end{cases} \qquad \square$$

Corollary 1 (of the proof)**.** *If $\mathfrak{R}$ expands a dense ordered group $(R, <, +, 0, 1)$ with $1 > 0$ such that R is definably connected, then for all m, n and $\emptyset$-definable $S \subseteq R^{m+1}$, the function $x \mapsto \beta_n(S_x)\colon \{\, x \in R^m : S_x \in \mathcal{B}_n \,\} \to R$ is $\emptyset$-definable.*

Corollary 2. *For all m, n and $S \subseteq \mathbb{R}^{m+1}$, $x \mapsto \beta_n(S_x)\colon \{\, x \in \mathbb{R}^m : S_x \in \mathcal{B}_n \,\} \to \mathbb{R}$ is $\emptyset$-definable in $(\mathbb{R}, <, +, 1, S)$.*

References

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