D-MINIMAL EXPANSIONS OF THE REAL FIELD HAVE THE EXCHANGE PROPERTY*

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Throughout, $\mathfrak{R}$ denotes a structure on $\mathbb{R}$. Definability is with respect to $\mathfrak{R}$. Given $A \subseteq \mathbb{R}$, $\operatorname{dcl}(A)$ denotes the definable closure of A, that is, the set of all A-definable points in $\mathbb{R}$.

$\mathfrak{R}$ **has the Exchange Property** if for all $A \subseteq \mathbb{R}$ and $x, y \in \mathbb{R}$:

$$y \in \operatorname{dcl}(A \cup \{x\}) \setminus \operatorname{dcl}(A) \Longrightarrow x \in \operatorname{dcl}(A \cup \{y\}).$$

$\mathfrak{R}$ is **d-minimal** if for every $n \in \mathbb{N}$ and definable $A \subseteq \mathbb{R}^{n+1}$, there exists $N \in \mathbb{N}$ such that for every $x \in \mathbb{R}^m$, the fiber $\{\, t \in \mathbb{R} : (x,t) \in A \,\}$ either has interior or is a union of N (not necessarily distinct) discrete sets. (Recall that $Y \subseteq \mathbb{R}^m$ is **discrete** if for every $y \in Y$ there is an open box B such that $Y \cap B = \{y\}$.)

Proposition. *If $\mathfrak{R}$ is a d-minimal expansion of $(\mathbb{R}, +, \cdot)$, then it has the Exchange Property.*

Proof. Let $A \subseteq \mathbb{R}$ and $x, y \in \mathbb{R}$ be such that $y \in \operatorname{dcl}(A \cup \{x\}) \setminus \operatorname{dcl}(A)$. Then there is an A-definable function $f \colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = y$. By arguing as in the proof of [Mil05, Theorem 3.3], there is an A-definable $S \subseteq \mathbb{R}$ such that S is closed and has no interior, and for every open interval $I \subseteq \mathbb{R} \setminus S$, the restriction $f{\restriction}I$ is either constant or strictly monotone. By d-minimality, there exists $N \in \mathbb{N}$ such that S is a union of N discrete sets. The set of all isolated points of S is A-definable. Since $\mathbb{Q} \subseteq \operatorname{dcl}(\emptyset)$, every isolated point of S is A-definable. Hence (by an easy induction on N), each $s \in S$ is A-definable. Since $y \notin \operatorname{dcl}(A)$, we have $x \notin S$, so there is an A-definable open interval I about x such that $f{\restriction}I$ is strictly monotone. Since the compositional inverse of $f{\restriction}I$ is A-definable, $x \in \operatorname{dcl}(A \cup \{y\})$. $\square$

An examination of the proof shows that we needed neither d-minimality nor the field structure *per se*:

Corollary (of the proof)**.** *Suppose that $<$ is $\emptyset$-definable and $\operatorname{dcl}(\emptyset)$ is dense in $\mathbb{R}$—in particular, if $\mathfrak{R}$ is an expansion of $(\mathbb{R}, <, +, 1)$—and every definable subset of $\mathbb{R}$ either has interior or is a finite union of discrete sets. Then $\mathfrak{R}$ has the Exchange Property.*

Warning. It may appear that no use was made of working over $\mathbb{R}$, but this is not true: I do not know how to get the conclusion of the third sentence of the proof just from d-minimality (I use the Baire Category Theorem).

REFERENCES

[Mil05] C. Miller, *Tameness in expansions of the real field*, Logic Colloquium '01 (Vienna, 2001), Lect. Notes in Logic, vol. 20, Assoc. Symbolic Logic, Urbana, IL, 2005, pp. 281–316.

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