# AN UPGRADE FOR "EXPANSIONS OF THE REAL FIELD WITH POWER FUNCTIONS"

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As the title suggests, this brief note is a follow-up to [5] (my first published paper), which the reader is assumed to have at hand. I make more readily available some results from my thesis [6, Chapter IV] that generalize some of the main results from [5], the latter being written just before the technology became available for proving more general results. Though I think these extensions are interesting, the proofs are fairly minor modifications of the material in [5], so I never published them except in my thesis. I also give some new applications, correct a few errors, and give the final data for the references of [5].

Put $\overline{\mathbb{R}} := (\mathbb{R}, <, +, -, \cdot, 0, 1)$. Let $\mathfrak{R}$ be a polynomially bounded o-minimal expansion of $\overline{\mathbb{R}}$ having field of exponents $K_0$. Let $S \subseteq \mathbb{R}$ and put $\mathfrak{R}^S = \big(\mathfrak{R}, (x^s)_{s\in S}\big)$. By [8], $\mathfrak{R}^S$ is o-minimal; indeed, so is $(\mathfrak{R}, e^x)$. Unfortunately, the method of proof does not reveal the field of exponents of $\mathfrak{R}^S$, nor even whether $\mathfrak{R}^S$ is polynomially bounded.* But the answer is known under some fairly reasonable assumptions.

Let $K$ be the subfield of $\mathbb{R}$ generated by $S$ over $K_0$.

**Theorem.** *Suppose that $\mathfrak{R}$ defines each restriction $x^s{\restriction}[1,2]$, $s \in S$. Then $\mathfrak{R}^S$ is polynomially bounded with field of exponents $K$.*

*Sketch of proof.* For every $r \in K$, $x^r{\restriction}[1,2]$ is definable in $\mathfrak{R}^S$, so we may assume that $S = K$. By definability of Skolem functions, we may assume that $\mathfrak{R}$ admits quantifier elimination and is universally axiomatizable. By using [2, Theorem C] instead of [5, 1.2], an easy modification of the proof of [5, 2.5] shows that $\mathfrak{R}^K$ admits quantifier elimination and is explicitly universally axiomatizable over $\mathfrak{R}$. (In the proof of [5, 2.4], use $K_0$ instead of $\mathbb{Q}$, and the reduct of $A$ to the language of $\mathfrak{R}$ instead of $A_{\mathrm{an}}$.) Without loss of generality, we may assume that $\mathfrak{R}$ has no relation symbols other than $<$. Hence, by [1, 5.5, 5.12], $\mathfrak{R}^K$ is o-minimal.† By [1, 5.8], regard the Hardy field $\mathcal{H}$ of $\mathfrak{R}^K$ as an elementary extension of $\mathfrak{R}^K$. In the second and third paragraphs of the proof of [5, 2.4] (again, modified as above), let $x$ be the germ of the identity function on $\mathbb{R}$, $A = \mathfrak{R}^K$ and $B = \mathcal{H}$. Then the resulting structure $C$ is a model of $\mathrm{Th}(\mathfrak{R}^K)$ containing $\mathbb{R}(x)$ as a Hardy field. On the other hand, by [1, 5.8], $\mathcal{H}$ is the smallest model of $\mathrm{Th}(\mathfrak{R}^K)$ containing $\mathbb{R}(x)$. Hence, $C = \mathcal{H}$, $\nu(\mathcal{H}) = K.\nu(x)$, and $K$ is the field of exponents of $\mathfrak{R}^K$. $\square$

**Corollary 1.** *Suppose that $\mathfrak{R}$ defines $e^x{\restriction}[0,1]$. Then $\mathfrak{R}^S$ is o-minimal and polynomially bounded with field of exponents $K$.*

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*This is Speissegger's opinion [personal communication].

†The proof relies only on Hardy field arguments and is independent of [8].

*Proof.* For every $r \in \mathbb{R}$, the restriction $x^r \restriction [1,2]$ is definable in $\big(\mathfrak{R}, e^x \restriction [0,1]\big)$. Apply the theorem. $\square$

**Corollary 2** (of the proof)**.** *If $\mathfrak{R}$ is model complete (in a language $L$ extending the language of ordered rings with unity) and defines each partial power $x^s \restriction [1,2]$, $s \in S$, then $\mathfrak{R}^S$ is model complete (in the obvious extension of $L$).*

*Proof.* Every $r \in K$ is both universally and existentially definable in $\mathfrak{R}^S$ (since $K = K_0(S)$) so it suffices to consider the case that $S = K$. Since $\mathfrak{R}^K$ admits QE after expanding by the Skolem functions definable in $\mathfrak{R}$, and $\mathfrak{R}$ is model complete, so is $\mathfrak{R}^K$ (without the added Skolem functions). $\square$

**Corollary 3.** *Suppose that $\mathfrak{R}$ is either the "Gevrey structure" $\mathbb{R}_{\mathcal{G}}$ defined in [3] or a "Denjoy-Carlemann structure" $\mathbb{R}_{\mathcal{C}(M)}$ as defined in [7]. Then $\mathfrak{R}^S$ is model complete and polynomially bounded with field of exponents $\mathbb{Q}(S)$.*

*Proof.* In either case, $\mathfrak{R}$ is model complete, o-minimal and has field of exponents $\mathbb{Q}$. $\square$

*Challenge.* Many of the results in [5] were reproved, even superceded, without using model theory in [4]. Find "standard" proofs of the above results.

*Questions.*

— Does $(\mathfrak{R}, e^x \restriction [1,2])$ have field of exponents $K_0$? In every case that we know of, the answer is "Yes". If not, is it at least polynomially bounded?

— Does $\big(\mathfrak{R}, (x^s \restriction [1,2])_{s \in S}\big)$ have field of exponents $K_0$? In every case that we know of, the answer is "Yes". If not, is it at least polynomially bounded?

**Corrections to** [5]**.**

— Ordered abelian groups should be assumed to have a distinguished positive element $1 > 0$, and be regarded in a language extending $\{<, +, 0, 1\}$. In particular, this should be reflected in 1.1 and its preceding paragraph.

— In 1.1, replace "$A \subseteq B, A' \subseteq B'$" with "$A \preccurlyeq B, A' \preccurlyeq B'$", and strike "unique".

— In the Notation preceding 2.2, use $L_{\mathrm{an}}^{\mathbb{Q}}$ and $T_{\mathrm{an}}^{\mathbb{Q}}$ instead of $L_{\mathrm{an}}$ and $T_{\mathrm{an}}$.

— In the last line of the proof of 2.2, "$u \in A$" should be "$u^r \in A$".

— In the third line of the proof of 2.4, strike "unique".

— In the first line of the last paragraph of the proof of 2.4, strike "uniquely".

— The statement of 3.2 is missing the assumption that $T$ should have QE.

— In the third line on page 89, "$\alpha$" should be "$a$".

**Final data for References of** [5]**.**


[7] **60** (1995), 74–102. And "pairs" should be "extensions".

[8] **140** (1994), 183–205.

[9] **85** (1994), 19–56.

[11] **122** (1994), 257–259.

[12] **123** (1995), 2551–2555.

[17] and [18] appeared as one paper: *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*, J. Amer. Math. Soc. **9** (1996), 1051–1094.

## References


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