RE-EXAMINATION OF AN OLD QUESTION

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By Wilkie [6], there are o-minimal proper expansions of $(\mathbb{Q}, <, +)$. But is there an o-minimal expansion of $(\mathbb{Q}, <, +)$ that defines a unary function from a bounded set onto an unbounded set? As far as I know, this is open.* The point of this note is to document some related folklore.†

Disclaimer. There was a period of rapid growth of results about o-minimality, and publication dates of references do not necessarily reflect the actual chronology of discovery. I shall not attempt here to unravel the precise history.

Proposition. *If there exists an o-minimal expansion of* $(\mathbb{Q}, <, +)$ *that defines a unary function from a bounded set onto an unbounded set, then there exists an o-minimal expansion of the real field that is not exponentially bounded.*

This result might be viewed as uninteresting, as it is widely believed that the antecedent is false and the consequent is true. Nevertheless, I think the ingredients of the proof are interesting and potentially useful. And of course, "wide beliefs" are not always accurate.

Proof of Proposition. Let $\mathfrak{Q}$ be an o-minimal expansion of $(\mathbb{Q}, <, +)$ that defines a unary function from a bounded set onto an unbounded set. By the Monotonicity Theorem, there is a definable strictly monotone bijection between a bounded interval and an unbounded interval; as $\mathfrak{Q}$ expands $(\mathbb{Q}, <, +)$, all nonempty open intervals are thus definably homeomorphic. By Laskowski and Steinhorn [1, Corollary 4.1], there exist an open interval I and definable binary operations $\oplus_I$ and $\odot_I$ on I such that $(I, < \cap I^2, \oplus_I, \odot_I, 0_I, 1_I)$ is an archimedean ordered field. As $(I, < \cap I^2)$ is definably isomorphic to $(\mathbb{Q}, <)$, we may take $I = \mathbb{Q}$. By translation, we may take $0_I = 0$; by dilation, we may take $1_I = 1$. Let M denote the positive multiplicative ordered group of this field. It is routine that $1 \oplus 1 \oplus 1$ is not a rational power *with respect to* M of $1 \oplus 1$.‡ Thus, M is not a 1-dimensional multiplicative $\mathbb{Q}$-vector space, and so $(\mathbb{Q}, <, +)$ is not isomorphic to M. By definable completeness, there is a solution in $\mathbb{Q}$ to $x \odot x = 1 \oplus 1$. As $\sqrt{2}$ is irrational, $(\mathbb{Q}, <, +)$ is not isomorphic to $(\mathbb{Q}, <, \oplus)$. By Miller and Starchenko [3, Theorem C], $(\mathbb{Q}, <, \oplus)$ is definably isomorphic to M, that is, $\mathfrak{Q}$ is exponential as an expansion of $(\mathbb{Q}, <, \oplus, \odot)$. By [1, Theorem 2.10], there is a binary operation $*$ on $\mathbb{R}$ such that $(\mathbb{R}, <, +, e^x, *)$ is o-minimal and $(\mathbb{R}, <, *)$ is an ordered

*Coincidentally (and annoyingly), arXiv has produced an HTML5 version of a flawed proof that was withdrawn by the authors in 2017. See https://arxiv.org/abs/1704.03050, Comments.

†I gave a talk on this at the P & S Model Theory Workshop, Leeds, July 2022.

‡This elegant approach was pointed out to me by Alf Onshuus. If I ever knew it in the past, I had forgotten it.

group that is not definably isomorphic to $(\mathbb{R},<,+)$ in $(\mathbb{R},<,+,e^x,*)$. By Miller and Speissegger [2, Proposition 3][§] , $(\mathbb{R},<,+,e^x,*)$ defines a unary function f for which there is no definable unary function g such that $g' \sim f$ at $+\infty$. By [2, Corollary 2], the Pfaffian closure of $(\mathbb{R},<,+,e^x,*)$ is not exponentially bounded (and is o-minimal by Speissegger [5]). $\square$

Remark. Actually, there is a reduct of the Pfaffian closure of $(\mathbb{R},<,+,e^x,*)$ that is not exponentially bounded; see [4, Proposition 1]. But as this has not been published (nor even submitted for publication), it should also be regarded as folklore.

An examination of the proof shows that the role of $\mathbb{Q}$ is limited. I leave it to the interested reader to think about this.

A concrete case. Let $\mathfrak{P}$ be the prime submodel of $(\mathbb{R},<,+,\cdot,0,1,e^x)$, and let P be the underlying set of $\mathfrak{P}$. As P is countable and dense in $\mathbb{R}$, there is a strictly increasing surjection $\phi\colon P \to \mathbb{Q}$. Note that the "push-forward", $\phi(\mathfrak{P})$, is an o-minimal expansion of $(\mathbb{Q},<)$. Is $\big(\phi(\mathfrak{P}),+\big)$ o-minimal? As far as I know, even this is open.

References

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