AN UPGRADE FOR "AVOIDING THE PROJECTIVE HIERARCHY IN EXPANSIONS OF THE REAL FIELD BY SEQUENCES" (POST HIERONYMI)

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The reader is assumed to have [M2] at hand. I give here: (i) stronger versions of some of the results due to a remarkable result of Philipp Hieronymi; (ii) a refinement of 3.1 and an application thereof, and (iii) an alternate formulation of Theorem A and an easier proof (that given in [M2] being based on lemmas needed to prove the rather more general 3.2).

But first, I correct a minor error: In the proof of 1.7, "well ordered" should be "anti well ordered".

Hieronymi's Theorem and consequences

Theorem (Hieronymi [H2])**.** *If $E \subseteq \mathbb{R}$ is discrete, and $f\colon E^n \to \mathbb{R}$ is somewhere dense, then $(\overline{\mathbb{R}}, f)$ defines $\mathbb{N}$.*

As a fairly easy consequence [H1]: *If $\alpha, \beta > 0$ are such that $\log\alpha$ and $\log\beta$ are $\mathbb{Q}$-linearly independent, then $(\overline{\mathbb{R}}, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}}) =$ PH.* Hence also: *If $\alpha > 1$ and $r \in \mathbb{R}\setminus\mathbb{Q}$, then $(\overline{\mathbb{R}}, x^r, \alpha^{\mathbb{Z}}) =$ PH.* Consequently, in the statement of [M2, 1.5], replace "every definable subset of $\mathbb{R}$ either has interior or is nowhere dense" with "$\mathfrak{R}$ does not define $\mathbb{N}$". As pointed out in [M2, 1.6], the lack of having this knowledge on hand at the time resulted in a number of awkward formulations of results. I shall clean these up below.

Remarks. (i) As yet another consequence of Hieronymi's theorm, we have a strengthening of the first part of AEG: *An expansion of $\overline{\mathbb{R}}$ defines $\mathbb{N}$ iff it defines the range of a strictly monotone sequence $(a_k)_{k\in\mathbb{N}}$ of nonzero real numbers such that $\lim_{k\to+\infty}(a_{k+1}/a_k) = 1$.* This could probably be used to shorten some of proofs in [M2], but as far as I know, it doesn't extend any results beyond those that are implied by the two previously-mentioned consequences. (ii) Subsequent joint work with Antongiulio Fornasiero and Hieronymi [FHM] and Hieronymi [HM] might result in yet further upgrades, but I have not worked this out.

Direct changes.

The conclusion of 1.11 becomes:

Then:

(a) *There exist $c > 0$ and $F \in \mathbb{R}((x^{\mathbb{Q}}))_\omega$ such that $\operatorname{supp}(F) \subseteq (-\infty, 0]$, and either $\operatorname{supp}(F)$ is infinite and $f \sim c\log x + F$, or $\operatorname{supp}(F)$ is finite and ultimately $f = c\log x + F$.*

(b) *$(\overline{\mathbb{R}}, \langle f^{-1}\rangle)$ defines $(e^{1/c})^{\mathbb{Z}}$.*

(c) *$(\overline{\mathbb{R}}, f', \langle f^{-1}\rangle)$ has field of exponents $\mathbb{Q}$.*

Some of the text in 1.13 needs obvious updating.

The conclusion of 2.2 becomes:

> *Then*:
> (a) *There exist* $c > 0$ *and* $F \in \mathbb{R}((x^{\mathbb{Q}}))_\omega$ *such that* $\operatorname{supp}(F) \subseteq (-\infty, 0]$, *and either* $\operatorname{supp}(F)$ *is infinite and* $f \sim c \log x + F$, *or* $\operatorname{supp}(F)$ *is finite and ultimately* $f = c \log x + F$.
> (b) $(\overline{\mathbb{R}}, \sin f)$ *defines* $(e^{\pi/c})^{\mathbb{Z}}$.
> (c) $(\overline{\mathbb{R}}, \sin f)$ *has field of exponents* $\mathbb{Q}$.

3.4 is omitted, and the conclusion of 3.2 becomes:

> *Then there exist* $0 < \alpha \neq 1$ *and* $F \in \mathbb{R}((x^{\mathbb{Q}}))_\omega$ *such that either* $\operatorname{supp}(F)$ *is infinite and* $f \sim F(\alpha^x)$, *or* $\operatorname{supp}(F)$ *is finite and exactly one of the following holds*:
> (1) $f = F(\alpha^x)$;
> (2) $f - F(\alpha^x) \notin \mathbb{R}$ *and* $\|f - F(\alpha^x)\| > \alpha^{x^n}$ *for every* $n \in \mathbb{N}$;
> (3) $f - F(\alpha^x) \notin \mathbb{R}$ *and* $\big|\|f - F(\alpha^x)\|\, \alpha^{-P} - 1\big| \leq c\alpha^{-rx}$ *for some* $c, r > 0$ *and* $P \in \mathbb{R}[x]$ *of degree at least* 2.

A refinement of 3.1 and an application to d-minimality

0.1 (a refinement of 1.4)**.** *Let* $P \in \mathbb{R}[x]$ *and* f *be infinitely increasing such that* $f \sim e^P$. *Then* $(\overline{\mathbb{R}}, \langle f \rangle)$ *defines* $e^{\beta\mathbb{Z}}$, *where* β *is the leading coefficient of* P.

Proof. Check that $a = \beta(d-1)!$ in the proof of 1.4. □

0.2**.** *Let* $\mathfrak{R}$ *be an o-minimal expansion of* $\overline{\mathbb{R}}$, $S_1, \dots, S_N \subseteq \mathbb{R}$ *be countable sets, and* $h\colon \mathbb{R}^N \to \mathbb{R}$ *be given. If every subset of* $\mathbb{R}$ *definable in* $(\mathfrak{R}, h, S_1, \dots, S_N)$ *either has interior or is nowhere dense, or if* $(\mathfrak{R}, h, S_1, \dots, S_N)$ *is d-minimal, then the same is true of the expansion of* $\mathfrak{R}$ *by all subsets of* $h(S_1 \times \cdots \times S_N)$.

Proof. Immediate from [FM, Theorem B and Claim on pg. 62]. □

0.3 (a refinement of 3.1)**.** *If* $f \in \mathcal{H}$ *is infinitely increasing, bounded above by some* e^{x^N}, *and* $(\overline{\mathbb{R}}, \langle f \rangle) \neq \mathrm{PH}$, *then there exist* $\beta, c, r > 0$ *and a monic* $P \in x.\mathbb{Q}[x] + \mathbb{R}$ *such that* $\big|fe^{-\beta P} - 1\big| \leq ce^{-rx}$.

(A refinement of Theorem B also follows easily; I leave details to the reader.)

Proof. By 3.1, there exists $P \in \mathbb{R}[x]$ such that the remaining conditions hold, so we need show only that $P - P(0) \in \mathbb{Q}[x]$. Write $P = \sum_{j=0}^d a_j x^j$, $a_d = 1$. Put $M = \min\{\, m : a_m, \dots, a_d \in \mathbb{Q} \,\}$; we must show that $M = 1$. Put $Q = \sum_{j=M}^d a_j x^j$. Note that

$$\langle f \rangle, \langle fe^{-\beta Q} \rangle \subseteq \langle f \rangle \cdot \prod_{j=M}^{d} \langle e^{\beta x} \rangle^{-a_j}.$$

By 0.1, $(\overline{\mathbb{R}}, \langle f \rangle)$ defines $\langle e^{\beta x} \rangle$. By 0.2, every subset of $\mathbb{R}$ definable in $(\overline{\mathbb{R}}, \langle f \rangle, \langle fe^{-\beta Q} \rangle)$ either has interior or is nowhere dense. Since $fe^{-\beta Q} \sim e^{\beta(P-Q)}$, we have $M = 1$ by 0.1 and 1.5. □

0.4 (an application of 0.3). *Let $\alpha > 1$, $P \in \mathbb{R}[x] \setminus \mathbb{R}$ and $\mathfrak{R}$ be an o-minimal expansion of $\overline{\mathbb{R}}$. Then $(\mathfrak{R}, \langle \alpha^P \rangle)$ is d-minimal iff $\mathfrak{R}$ has field of exponents $\mathbb{Q}$ and there exists $\beta \in \mathbb{R}$ such that $P - P(0) \in \beta.\mathbb{Q}[x]$.*

Proof. It suffices to consider the case $\alpha = e$, P is monic, and $P(0) = 0$.

The forward implication is immediate from the definition of d-minimality, 0.1, 1.5 and 0.3.

Assume that $\mathfrak{R}$ has field of exponents $\mathbb{Q}$ and $P \in \beta.\mathbb{Q}[x]$ for some $\beta \in \mathbb{R}$. Write $P = \beta \sum_{j=1}^{d} q_j x^j$, $q_d = 1$. By 0.1, $(\mathfrak{R}, \langle e^P \rangle)$ defines $\langle e^{\beta x} \rangle$. Note that $\langle e^P \rangle \subseteq \prod_{j=1}^{d} \langle e^{\beta x} \rangle^{q_j}$. By [M1, §3.4], $(\mathfrak{R}, \langle e^{\beta x} \rangle)$ is d-minimal. Apply 0.2. □

An alternate version of Theorem A

Here, $i := \sqrt{-1}$, and for $r > 0$, x^{ir} denotes the restriction to the positive real line of the complex power function z^{ir}, defined with respect to an appropriate branch of $\log z$.

Recall 2.1.

Theorem A′. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be bounded below as $x \to +\infty$ by a compositional iterate of $\log x$. If $(\overline{\mathbb{R}}, f)$ is o-minimal and $(\overline{\mathbb{R}}, e^{if})$ does not define $\mathbb{N}$, then there exist $r > 0$ and $c \in \mathbb{C} \setminus \{0\}$ such that $e^{if} \sim cx^{ir}$. Moreover, $(\overline{\mathbb{R}}, e^{if})$ defines x^{ir}, hence also the group $(e^{\pi/r})^{\mathbb{Z}}$, so $(\overline{\mathbb{R}}, e^{if})$ has field of exponents $\mathbb{Q}$.*

Proof. Note that $(\overline{\mathbb{R}}, e^{if})$ defines f' $(= (e^{if})'/ie^{if})$ and $(\overline{\mathbb{R}}, f')$ is o-minimal. We show that $(\overline{\mathbb{R}}, f')$ defines no $h \colon \mathbb{R} \to \mathbb{R}$ such that $f' \sim h'$. Suppose otherwise. Then h is infinitely increasing (by L'Hôpital), $(\overline{\mathbb{R}}, e^{if})$ defines h^{-1}, and $f \circ h^{-1} = x + g$ with $g' \to 0$. For each $t \in \mathbb{R}$, we thus have $e^{it} = \lim_{x \to +\infty} e^{if(h^{-1}(t+x))}/e^{if(h^{-1}(x))}$, so $(\overline{\mathbb{R}}, e^{if})$ defines e^{ix}, hence also $\mathbb{N}$, a contradiction. As in the proof of 1.11, $(\overline{\mathbb{R}}, f')$ is polynomially bounded and f' has an asymptotic expansion $r/x + F$, where $r > 0$ and $F \in \mathbb{R}((x^{\mathbb{R}}))$ has support lying in $(-\infty, 0]$; in particular, there exist $a \in \mathbb{R}$ and $s > 0$ such that $f = a + r \log x + o(x^{-s})$. Hence, $e^{if} = e^{ia} x^{ir} e^{io(x^{-s})}$. For each $t > 0$, we have $t^{ir} = \lim_{x \to +\infty} e^{if(tx)}/e^{if(x)}$, so x^{ir} is definable, hence also the kernel of x^{ir}. □

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