AN UPGRADE TO "EXPANSIONS OF THE REAL FIELD BY CANONICAL PRODUCTS"

CHRIS MILLER AND PATRICK SPEISSEGGER

December 4, 2023
This is not a preprint; please do not refer to it as such.

We have some updates and corrections to [5] (which we assume the reader to have at hand).

Disclaimer

The copy editor(s) of the publishers of the journal are responsible for any awkwardly repetitive statements such as:

— "Rather more difficult, and one of our main results, is the following result."
— "By combining these two results with known technology, we obtain the following result."
— "These results are sharpened by the following result."

In each case, the terminal phrase "the following result" was not our original text. (Presumably, they would edit what we say above to something like: "…for any awkwardly repetitive statements such as the following awkwardly repetitive statements".)

1. Antiderivatives in $\mathbb{R}_{\mathcal{G}}$

We first answer a question about $\mathbb{R}_{\mathcal{G}}$: *Which unary definable functions have a definable antiderivative at $+\infty$?*

Suppose that $\mathfrak{R}$ is a polynomially bounded o-minimal expansion of $\overline{\mathbb{R}}$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be definable in $\mathfrak{R}$. Then f either has an asymptotic expansion $\sum_{k\in\mathbb{N}} a_k x^{r_k}$ at $+\infty$ where each $a_k \neq 0$ and each power function x^{r_k} is definable, or f is ultimately equal to a finite sum $\sum a_k x^{r_k}$ where each $a_k \neq 0$ and each power function x^{r_k} is definable. If some $r_k = -1$, then f has no definable antiderivative at $+\infty$. Thus, the question arises as to whether f has a definable antiderivative at $+\infty$ if no such k exists. This is not true in general. (Consider $\mathfrak{R} = \overline{\mathbb{R}}$ and $f = 1/(1+x^2)$.) We now show that it is indeed true for $\mathfrak{R} = \mathbb{R}_{\mathcal{G}}$. It is more convenient to work at $0+$.

1.1. *If u is a unary primitive function of $\mathbb{R}_{\mathcal{G}}$, then there is a unary primitive function v of $\mathbb{R}_{\mathcal{G}}$ such that $v{\restriction}(0,1)$ is an antiderivative of $u{\restriction}(0,1)$.*

Proof. Let $R > 1$. Let $S(R,\phi,\kappa)$ and $\mathcal{G}(R,\phi,\kappa)$ be as on [3, p. 514]. Let $f \in \mathcal{G}(R,\phi,\kappa)$. For every $z \in S(R,\phi,\kappa)$, the complex line segment $[1,z]$ is contained in $S(R,\phi,\kappa)$. Define $g\colon S(R,\phi,\kappa) \to \mathbb{C}$ by $g(z) = \displaystyle\int_{[1,z]} f(w)\,dw$; then g is holomorphic and $g' = f$. Observe that $|g(z)| \leq R \sup |f|$, $\lim_{z\to 0} g(z) = -\int_0^1 f(t)\,dt$ and

$$\frac{g^{(n+1)}(z)}{(n+1)!} = \frac{1}{n+1} \cdot \frac{f^{(n)}(z)}{n!}, \quad n \in \mathbb{N}.$$

Hence, $g \in \mathcal{G}(R, \phi, \kappa)$. The result now follows by chasing definitions and a change of variable. $\square$

Remark. If $u\restriction[0,1]$ is analytic, then so is $v\restriction[0,1]$.

1.2. *If $f\colon \mathbb{R} \to \mathbb{R}$ is definable in $\mathbb{R}_{\mathcal{G}}$ and bounded at $0+$, then f has a definable antiderivative at $0+$.*

Proof. By [3, Theorem A], there is a nonzero $d \in \mathbb{N}$ such that $f \circ x^d$ is given at $0+$ by a unary primitive of $\mathbb{R}_{\mathcal{G}}$. Put $g = dx^{1-1/d} f$; then $g \circ x^d$ is also a unary primitive. By 1.1, $g \circ x^d$ has a definable antiderivative v at $0+$; then $(v \circ x^{1/d})' = f$ at $0+$. $\square$

Another consequence of [3, Theorem A]:

1.3. *If $f\colon (0,\infty) \to \mathbb{R}$ is definable in $\mathbb{R}_{\mathcal{G}}$, then there exist a finite $A \subseteq \mathbb{R} \setminus \{0\}$ and a function $\rho\colon A \to (0,\infty)$ such that $f - \sum_{a\in A} a x^{-\rho(a)}$ is bounded at 0 and each $x^{\rho(a)}$ is definable.*

Hence, f has a definable antiderivative at $0+$ iff $1 \notin \rho(A)$. (We have now finished answering the question.)

Note. That we can take A to be finite in 1.3 is not just because $\mathbb{R}_{\mathcal{G}}$ is polynomially bounded and o-minimal: there are such structures for which this fails, *e.g.*, $\mathbb{R}_{\mathrm{an}^*}$ (as defined in [2]).

For use in the next section, we record a restatement of 1.2.

1.4. *If $f\colon \mathbb{R} \to \mathbb{R}$ is definable in $\mathbb{R}_{\mathcal{G}}$ and $x^2 f$ is bounded at $+\infty$, then f has a definable antiderivative at $+\infty$.*

2. The case $s < 1$

As in [5], we let s and x range over positive real numbers, and whether x is regarded as fixed or variable depends upon context. We did not know quite how to deal with W_s for $s < 1$. We have resolved the situation.

2.1. *If $s < 1$, then $(\mathbb{R}_{\mathcal{G}}, e^x)$ defines W_s. If moreover $1/s \notin \mathbb{N}$, then $(\mathbb{R}_{\mathcal{G}}, x^s)$ defines W_s'/W_s.*

This is essentially immediate from 1.4 and

2.2. *Let $s < 1$ and p be the integer part of $1/s$. Then there exist $R > 0$, $A, B \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ definable in $\mathbb{R}_{\mathcal{G}}$ such that $\lim_{t\to+\infty} f(t) \in \mathbb{R} \setminus \{0\}$ and*

$$(\log W_s)^{(p+1)} = \frac{A}{x^{p+1-1/s}} + \frac{B}{x^{p+1}} + \frac{f}{x^{p+2}}, \quad x > R.$$

(The holds also if $s \geq 1$, but we already knew this in [5].)

Proof of 2.1 from 2.2. First, suppose that $1/s \notin \mathbb{N}$. By 2.2 and iterating 1.4, there exist $A_1, B_1 \in \mathbb{R}$, $P \in \mathbb{R}[x]$, and $g\colon \mathbb{R} \to \mathbb{R}$ definable in $\mathbb{R}_{\mathcal{G}}$ such that

$$\log W_s = A_1 x^{1/s} + B_1 \log x + P(x) + g, \quad x > R.$$

(*Aside.* By the Barnes formulas, we know exactly what are A_1, B_1 and P.) Recall that $\log W_s \restriction (0, R]$ is definable in $\mathbb{R}_{\text{an}}$. The argument is almost the same for $1/s \in \mathbb{N}$, but instead of $A_1 x^{1/s}$ we get the product of A with a $(p+1)$-fold antiderivative of $1/x$. All iterated antiderivatives of $1/x$ are definable in $(\overline{\mathbb{R}}, e^x)$. □

Sketch of proof of 2.2. By direct computation from the Weierstrass factorization of $W_s(z)$, we have

$$(\log W_s)^{(p+1)}(z) = (-1)^p p! \sum_{n>0} \frac{1}{(z+n^s)^{p+1}},$$

and the convergence is absolute and uniform on compact subsets of $\mathbb{C} \setminus \{\, -n^s : n > 0 \,\}$.[1] The rest of the proof is similar to that of the conjunction of the proofs of 2.1 and 2.2 of [5], but easier in some ways (we will not be building in the construction of an antiderivative, and the least integer greater than s is 1). Put

$$\phi_s(x) = -2 \int_0^\infty \operatorname{Im}\left[\frac{1}{(x+(it)^s)^{p+1}}\right] \frac{dt}{e^{2\pi t}-1};$$

then

$$\sum_{n>0} \frac{1}{(x+n^s)^{p+1}} = \int_0^\infty \frac{dt}{(1+t^s)^{p+1}} \cdot \frac{1}{x^{p+1-1/s}} - \frac{1}{2x^{p+1}} + \phi_s(x).$$

Let $R_s(z)$ be the rational function

$$\frac{2\sum_{k=1}^{p+1} \binom{p+1}{k} \sin(sk\pi/2) z^k}{[(1+i^s z)(1+i^{-s} z)]^{p+1}}.$$

Note that R_s has a zero of order one at the origin, poles exactly at $e^{i\pi(1 \pm s/2)}$, and for $r > 0$ we have

$$R_s(r^s) = -2 \operatorname{Im}\left[\frac{1}{(1+(ir)^s)^{p+1}}\right].$$

Thus,

$$\phi_s(1/x) = x^{p+2} \sum_{n>0} \int_0^\infty [R_s(t^s x)/x] e^{-2\pi n t}\, dt.$$

Since $[R_s(z)/z](0) \neq 0$, it suffices now to show that

$$\psi_s(x) := \sum_{n>0} \int_0^\infty [R_s(t^s x)/x] e^{-2\pi n t}\, dt$$

is given on some $(0, \epsilon)$ by a unary primitive. As in the proof of [5, 2.2], proceed by verifying that the machinery from Balser ([1]) applies to $\psi_s(1/x^s)/x^s$. Note that we may take $N = 1$. (We leave the details as an exercise.) □

[1] Thanks to Ovidiu Costin for the suggestion to differentiate the formal series $\sum_{n>0} 1/(z+n^s)$ until it becomes convergent.

3. More on W_1

In [5], we dealt with the case $s = 1$ by observing that W_1 is interdefinable over $\overline{\mathbb{R}}$ with Γ (restricted to $(0, \infty)$) and then applying the result from [3] that $(\mathbb{R}_{\mathcal{G}}, e^x)$ defines Γ, the proof of which relied on existing classical results about the Stirling series. We can now give a more direct proof that $(\mathbb{R}_{\mathcal{G}}, e^x)$ defines W_1 (hence also Γ). By arguing as before,

$$(\log W_1)'' = -\sum_{n \geq 0} \frac{1}{(x+n)^2}.$$

By the Abel-Plana formula,

$$\sum_{n \geq 0} \frac{1}{(x+n)^2} = \frac{1}{x} + \frac{1}{2x^2} - 2\int_0^\infty \operatorname{Im}\left[\frac{1}{(x+it)^2}\right] \frac{dt}{e^{2\pi t} - 1}.$$

Now argue as for the case $s < 1$ (but take $N = 2$).

Remark. $(\log W_1)''(x) = -\zeta(2, x)$, where ζ is the Hurwitz zeta function.

4. Corrections to the proof of [5, 1.5]

The constant term $\dfrac{2\pi^2 + (\log s)^2}{12 \log s}$ is missing from the statement of Littlewood's formula for $\log F_s$, but as it disappears after differentiation, it plays no role in the proof.

Asserted in the proof of part (ii) is that the zero set of the function

$$f(x) = \sum_{m > 0} b_m m^2 \sin(\pi m x)$$

is equal to $\mathbb{Z}$. This is too optimistic; there will be trouble if the sequence $(b_m m^2)$ takes too long to manifest its ultimate decay rate, and this depends on s. (More on this at the end of this section.) In any case, the assertion is not actually needed. By periodicity, analyticity and that $f'(0) \neq 0$, the set of zeros of f that are neither local maxima nor local minima is a nonempty union of finitely many cosets of $2\mathbb{Z}$. With a little more calculus and basic definability, the proof of 1.5(ii) still works. Let $c \in \mathbb{R}$. We show that $\mathfrak{d}_3 \log F_s \restriction (c, \infty)$ defines $s^{\mathbb{Z}}$ over $\overline{\mathbb{R}}$. Like many of the results in [5], the argument is asymptotic and essentially independent of c, so we take $c = 1$ for convenience. (Recall that the Littlewood formulas are stated for $x > 1$.) It suffices to show that the zero set of $\mathfrak{d}_3 \log F_s \restriction (1, \infty)$ defines $s^{\mathbb{Z}}$ over $(\mathbb{R}^{>0}, <, \cdot)$, and for this, it suffices to show that the zero set of $\mathfrak{d}_3 \log F_s \circ s^{-x/2} \restriction (0, \infty)$ defines $\mathbb{Z}$ over $(\mathbb{R}, <, +)$. Put

$$g(x) = -\frac{(\log s)^3}{4\pi^3} \sum_{m > 0} a_m m^2 s^{-x/2}$$

and $X = \{\, x > 0 : g(x) = f(x) \,\}$. By the Littlewood formulas, it suffices now to show that $(\mathbb{R}, <, +, X)$ defines $\mathbb{Z}$. Observe that X is closed, discrete and unbounded above (by analyticity of $f - g$ and periodicity of f). Let σ be the successor function on X. (All we have done so far is to recapitulate the argument from [5] in more detail.) There exist positive δ and R_0 such that, for all $x \in X \cap (R_0, \infty)$, we have $\sigma(x) \leq x + \delta$ if and only if there exists $y \in (x, \sigma(x))$ such that $f(y) = 0$ and f has a local minimum at y. Put $X' := \{\, x \in X : \sigma(x) \leq x + \delta \,\}$ and $X'' = X \setminus (X' \cup \sigma(X'))$. As X'' is definable in

$(\mathbb{R}, <, +, X)$, it suffices now to show that $(\mathbb{R}, <, +, X'')$ defines $\mathbb{Z}$. Let E be the set of zeros of f that are neither local minima nor local maxima of f. Then:

$$\forall \epsilon > 0, \exists R \in \mathbb{R}, \forall x \in X'' \cap (R, \infty), \exists! y \in E, |x - y| < \epsilon.$$

As E is the union of $2\mathbb{Z}$ and finitely many cosets of $2\mathbb{Z}$, we are done by the following result.

4.1. *Let G be a cyclic subgroup of $(\mathbb{R}, +)$ and E be a nonempty finite union of cosets of G. Let $D \subseteq \mathbb{R}$ be such that*

$$(*) \qquad \forall \epsilon > 0, \exists R \in \mathbb{R}, \forall x \in D \cap (R, \infty), \exists! y \in E, |x - y| < \epsilon.$$

Then G is definable in $(\mathbb{R}, <, +, D)$.

This is known in some form or other by at least some model theorists other than us, but perhaps only indirectly or as folklore. Thus, we provide a

Proof. The result is trivial if $G = \{0\}$. For ease of notation, let $G = \mathbb{Z}$. There is an $R_0 \in \mathbb{R}$ such that $D \cap [R_0, \infty)$ is closed, discrete and unbounded above. Let $(d_n)_{n \in \mathbb{N}}$ be a strictly increasing enumeration of $D \cap [R_0, \infty)$. Let N be the number of cosets of $\mathbb{Z}$ comprising E. We proceed by induction on N.

If $N = 1$, then $\lim_{n \to \infty}(d_{n+1} - d_n) = 1$, and so the result is immediate by asymptotic extraction of groups [4, AEG] (and its proof).

Let $N > 1$ and assume the result for all lower values of N. Let $0 \le r_1 < \cdots < r_N < 1$ and E be the union of the $r_k + \mathbb{Z}$. Consider the set $\{ \exp(i2\pi r_k) : k = 1, \ldots, N \}$ as a subset of the unit circle. Among the arcs between these points, there are some of maximal length, and there is a maximal number J of contiguous arcs of maximal length. If all of the arcs have the same length, then $\lim_{n \to \infty}(d_{n+1} - d_n) = 1/N$. By asymptotic extraction of groups, $(\mathbb{R}, <, +, D)$ defines $(1/N)\mathbb{Z}$, hence also $\mathbb{Z}$. Assume that not all of the arcs have the same length. Let σ denote the successor function on E, and σ_j denote the j-th compositional iterate. Put $c = \max\{r_2 - r_1, \ldots, r_N - r_{N-1}, 1 + r_1 - r_N\}$. Then the set

$$E' := \left\{ x \in E : \bigwedge_{j=1}^{J} \sigma_j(x) = x + jc \right\}$$

is nonempty and the union of finitely many, but not all, of the $r_k + \mathbb{Z}$. Let τ denote the successor function on D, and τ_j denote the j-th compositional iterate. There exists $\delta > 0$ such that the set

$$D' := \left\{ x \in D : \bigwedge_{j=1}^{J} |\tau_j(x) - (x + jc)| \le \delta \right\}$$

satisfies property $(*)$ with respect to E'. Inductively, $(\mathbb{R}, <, +, D')$ defines $\mathbb{Z}$. As D' is definable in $(\mathbb{R}, <, +, D)$, we are done. □

Having now repaired the proof of Theorem 1.5, let us analyze the problematic assertion that the zero set of $\sum_{m>0} b_m m^2 \sin(\pi m x)$ is equal to $\mathbb{Z}$. Suppose that the sequence $(b_m m^2)_{m>0}$ is strictly decreasing, and at a "fast enough" rate. The sum starts with $b_1 \sin(\pi x)$, which has only integer zeros. The next term, $4b_2 \sin(2\pi x)$, contributes positivity on $(0, 1/2)$, but some negativity on $(1/2, 1)$. Nevertheless, if $2^2 b_2 < b_1$, then no new zeros arise. Continuing in this fashion yields at least a plausibility argument that no new zeros will arise in the infinite sum. But this idea cuts both ways. Recall that $b_m = \operatorname{csch}(2\pi^2 m/\log s)$. Thus, we have only *asymptotic* exponential decay for the sequence $(b_m m^2)$, and the larger is s, the longer it takes for the exponential decay to set in. Instead of starting with $b_1 \sin(\pi x)$ above, we should use $\sum_{m=1}^{M} b_m m^2 \sin(\pi m x)$ for some sufficiently large M. As this might have noninteger zeros, the same will probably be true of good approximations.

References

[1] Werner Balser, *Formal power series and linear systems of meromorphic ordinary differential equations*, Universitext, Springer-Verlag, New York, 2000. MR1722871

[2] Lou van den Dries and Patrick Speissegger, *The real field with convergent generalized power series*, Trans. Amer. Math. Soc. **350** (1998), no. 11, 4377–4421, DOI 10.1090/S0002-9947-98-02105-9. MR1458313

[3] ———, *The field of reals with multisummable series and the exponential function*, Proc. London Math. Soc. (3) **81** (2000), no. 3, 513–565, DOI 10.1112/S0024611500012648. MR1781147

[4] Chris Miller, *Avoiding the projective hierarchy in expansions of the real field by sequences*, Proc. Amer. Math. Soc. **134** (2006), no. 5, 1483–1493, DOI 10.1090/S0002-9939-05-08112-8. MR2199196

[5] Chris Miller and Patrick Speissegger, *Expansions of the real field by canonical products*, Canad. Math. Bull. **63** (2020), no. 3, 506–521, DOI 10.4153/s0008439519000572. MR4148115

Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210, USA

Email address: miller@math.osu.edu

URL: https://people.math.osu.edu/miller.1987

Department of Mathematics & Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario, Canada L8S 4K1

Email address: speisseg@math.mcmaster.ca