A BANK RUNS MODEL WITH A CONTINUUM OF TYPES

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Abstract. We consider a bank runs model *à la* Diamond and Dybvig (1983) with a continuum of agent types, indexed by the *degree* of patience. Much of our understanding based on the two-type model must be modified. The endogenous determination of a cutoff type is central to the analysis. In the case where the bank can credibly commit to a contract, the optimal contract results in socially excessive withdrawals in period 1. Thus, even the best equilibrium exhibits features of a bank run. In the case where commitment is not possible, there are strictly more early withdrawals and strictly lower welfare than the full-commitment equilibrium.

Keywords: Bank runs; Continuum of types; Optimal bank contract; Commitment

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1. Introduction

In the extensive literature on bank runs emanating from the classic paper by Diamond and Dybvig (1983), the usual assumption is that there are two types of agents: Impatient agents receive utility only from period 1 consumption, and patient agents receive utility from period 2 consumption. In this paper, we consider the implications of having a continuum of agent types, indexed by the degree of patience, θ. Thus, instead of assuming a known fraction of impatient agents, we assume that there is a known density of impatience describing the population of agents. We maintain the usual assumption that only full withdrawals are feasible, and show that much of our understanding based on the two-type model must be modified when we consider a continuum of types.

With a continuum of types, incentive compatibility binds, and there will be a cutoff type below which agents withdraw in period 1 and above which agents wait. The endogenous determination of this cutoff type resulting from the bank's contract choice is central to the analysis and contribution of this paper. We characterize the optimal contract, which can be viewed as a constant consumption for all who withdraw early and an induced cutoff type, $(c_1^*, \bar{\theta}^*)$. We show that: (1) given c_1^*, society would be better off if more types waited and the cutoff was less than $\bar{\theta}^*$, and (2) given $\bar{\theta}^*$, society would be better off with period 1 consumption higher than c_1^*. The intuition is that, at the socially optimal cutoff, period 1 consumption balances the benefits of insurance against being impatient vs. the benefits of providing more consumption in period 2 due to the investment technology. However, the benefits of waiting accrue primarily to the most patient types, and agents near the socially optimal cutoff would actually prefer to withdraw early. To achieve incentive compatibility, the bank accepts a cutoff with some degree of socially excessive early withdrawals (point 1 above), but it also sacrifices some insurance in order to encourage waiting and shift the cutoff type towards the socially optimal level (point 2 above). Thus, even the best equilibrium exhibits some of the flavor of a bank run.

What about other, less efficient, equilibria? When the bank is restricted to simple contracts and offers c_1^* to everyone who withdraws in period 1, the post-deposit subgame has a run equilibrium in which everyone withdraws, as in Diamond and Dybvig (1983). More interestingly, the post-deposit subgame also has a partial run equilibrium, with an interior cutoff type that is more patient than $\bar{\theta}^*$. Intuitively, it can be a self-fulfilling prophecy that a chunk of types more patient than $\bar{\theta}^*$ also withdraw in period 1, because less per-capita consumption would be left for those who wait, thereby justifying the new, more patient cutoff. If the bank is not restricted to simple contracts, we show that suspending convertability when withdrawals reaches $\bar{\theta}^*$ implements the constrained-efficient allocation as the unique equilibrium.

Now consider the case in which the bank cannot commit to a contract, but instead offers consumption in period 1 as agents arrive to withdraw. Since the cutoff withdrawal type is already determined before the bank chooses consumption, the bank will choose the socially optimal c_1, given the equilibrium cutoff. It follows from point (2) above that the bank is unable to credibly promise to keep period 1 consumption at the full-commitment level. We show that the most efficient equilibrium without commitment yields strictly more early withdrawals and strictly lower welfare than the full-commitment equilibrium.

The paper is organized as follows. The next section relates our results to the existing literature. Section 3 introduces the basic set-up. In Section 4 we study the case where the bank is able to commit to a contract, and in Section 5 the case where commitment is not possible. An example that illustrates our results is presented in Section 6. Proofs that do not appear in the main text can be found in the Appendix.

2. Literature Review

For the two-type model with a known fraction of each type, Diamond and Dybvig (1983) were the first to characterize the most efficient equilibrium as the solution to a planner's problem. The incentive compatibility constraint does not bind, and the full-information first best allocation results. With a continuum of types, however, there is no hope of achieving the first-best, because the binary withdraw/wait decision cannot fully reveal each agent's type.[1] Moreover, the equilibrium is not even constrained efficient – the equilibrium accommodates more early withdrawals than a planner choosing the cutoff type but treating all agents who wait identically.

A number of papers, including Diamond and Dybvig (1983), consider two-type models in which the fraction of impatient agents is uncertain. See Wallace (1990), Green and Lin (2003), Peck and Shell (2003, 2010), Ennis and Keister (2009), Andolfatto, Nosal, and Wallace (2007), and Nosal and Wallace (2009), among others. As in our model with a continuum of types but no aggregate uncertainty, the full-information first-best is not

[1] Even if agents were to report their types, the full-information first-best requires consumption to be an increasing function of θ for those agents receiving consumption in period 2. Clearly there is no way to induce truthful revelation without sacrificing efficiency.

feasible in these papers. However, the solution to the planner's problem always induces all patient agents to receive consumption in period 2. Thus, our excessive withdrawal result at the best equilibrium is completely new to the literature, including the literature with aggregate uncertainty.

Ennis and Keister (2010) consider a two-type model with no aggregate uncertainty and without commitment. They show that partial run equilibria exist, characterized by waves of further withdrawals as the crisis deepens. However, their model always has an equilibrium in which the allocation is the full-information first-best. With a continuum of types, the situation is different in two ways. First, the full-commitment outcome is no longer first-best, and second, the most efficient equilibrium without commitment yields strictly more early withdrawals and strictly lower welfare than the full-commitment equilibrium.

An interesting paper by Lin (1996) studies a banking model similar to ours, with a continuum of types representing degrees of impatience. In Lin (1996), agents desire to consume in both period 1 and period 2, and the optimal allocation is characterized. The bank or planner offers a menu of consumptions over the two periods, with a different menu item targeted to each type, maximizing welfare subject to feasibility and incentive compatibility. Incentive compatibility entails a higher interest rate paid to more patient agents. In our model, as in Diamond and Dybvig (2003), agents cannot consume during both period 1 and period 2.[2]

Global games models also employ a continuum of types, but the structure of the types and the results are quite different. See Carlsson and van Damme (1993), Morris and Shin (1998), or for a bank-runs application, Goldstein and Pauzner (2005). In these models, each agent receives a signal about a population parameter or common value, but an agent's signal itself does not directly affect his payoff. In our model, an agent's type is a private value that directly affects his payoff. For example, in Goldstein and Pauzner (2005), patient agents also receive a signal that is correlated with the probability that long-run asset return is 0 instead of R. Unlike our model, global games models often generate a unique equilibrium when signals are not perfectly accurate.

[2]One can interpret this specification as reflecting the prohibitive complexity of running more general mechanisms, or perhaps reflecting the nature of "consumption opportunities" that motivate withdrawals. See Peck and Shell (2010) for a discussion.

3. Set up

There are three periods, $t = 0, 1, 2$. There is a continuum of potential depositors in the population, and at the beginning of period 1 each depositor privately observes his type $0 \leq \theta \leq 1$. The distribution of agents' types is given by a CDF, $F(\theta)$, which admits a continuous density function $f(\theta)$ with $f(\theta) > 0$ for $0 \leq \theta \leq 1$ and $f(\theta) = 0$ otherwise. We also assume that the hazard rate $\frac{f(\theta)}{1-F(\theta)}$ is non-decreasing on $[0, 1)$.

Each agent is endowed with one unit of consumption which he can deposit in the bank at period $t = 0$. Each unit of consumption invested yields a return of $R > 1$ units of consumption if held until period 2. Each unit of consumption invested yields one unit of consumption if liquidated in period 1.

Agents can consume either in period $t = 1$ or in period $t = 2$ (but not in both). The utility of an agent consuming c_1 units in period $t = 1$ is $u(c_1)$, and is independent of his type. The utility of an agent of type θ who consumes c_2 units in period $t = 2$ is $\theta u(c_2)$. This specification is a natural extension of Diamond and Dybvig (2003). With two types, one could interpret patient agents as having to consume in period 2, where a patient agent that withdraws in period 1 costlessly stores the consumption until it is consumed in period 2. With a continuum of types, this dichotomy, between patient being required to consume in period 2 and impatient being required to consume in period 1, is no longer possible. One must interpret agents as being able to consume in either period (but not both), with θ representing the degree to which future consumption is discounted.

We make the following assumptions on the utility function $u : [0, \infty) \to \mathbb{R}$:

- u is strictly increasing, concave and twice differentiable.
- $u(0) = 0$ and $u'(1) < u'(0) \cdot R \cdot \mathbb{E}(\theta)$.
- The coefficient of relative risk aversion is greater than 1 whenever $c \geq 1$. That is $\frac{-cu''(c)}{u'(c)} > 1$ whenever $c \geq 1$.

The bank offers a contract to the depositors, which specifies the amount of consumption that will be given to depositors who decide to withdraw in the first period $t = 1$. Formally, a contract is given by a (measurable) function $c_1 : [0, 1] \to \mathbb{R}_+$, where $c_1(z)$ is the consumption level given to a depositor who withdraws in period $t = 1$ after a measure z of depositors already withdrew. Thus, service is sequential. We assume that the order of arrivals is random, and that agents do not know their place in line as they decide whether to withdraw. A contract is *feasible* if $\int_0^1 c_1(z)dz \leq 1$, and we consider only feasible contracts from now on. Any consumption that was not liquidated in the first period (and the interest on this consumption) is equally divided among the agents who did not withdraw in the first period.[3]

We will have special interest in *constant contracts*. These are contracts in which the first period consumption level $c_1(z)$ is independent of z. More formally, a *constant contract* is any contract of the form

$$c_1(z) = \begin{cases} c_1 & 0 \leq z \leq \min(1/c_1, 1) \\ 0 & \textit{otherwise.} \end{cases}$$

If no confusion may result we abuse notation and denote a constant contract by the number c_1.

4. The game with commitment

In this section we consider the case where the bank can commit to a contract before the agents make their withdrawal decisions. Thus, the timing of the game is as follows. In period $t = 0$ the bank announces the contract $c_1(z)$ and the agents deposit their money in the bank. Then, in period $t = 1$, each agent learns his type and decides whether to withdraw early. Agents who do not withdraw in the first period get their consumption (which depends on the set of agents who withdrew in the first period) in period $t = 2$.

We assume that the bank's objective is the ex ante expected utility of the representative agent. Perfect competition between banks would force them to behave as if this were their objective, although we do not model this competition formally. It is easy to see that, in any equilibrium of the post-deposit subgame, the set of types that withdraw in period 1 is an interval of the form $[0, \bar{\theta}]$. We therefore restrict attention to strategy profiles of this form, and identify each such strategy profile with the cutoff type $\bar{\theta}$. In equilibrium, the bank chooses a contract $c_1(z)$ and a cutoff type $\bar{\theta}$ that maximize welfare subject to an Incentive Compatibility (IC) constraint and a resource constraint.

[3]While the bank might provide higher welfare by giving agents with higher θ more consumption in period 2 than agents with lower θ who also withdraw in period 2, incentive compatibility and consumption smoothing requires equal division at the optimal mechanism, so we simplify the exposition by requiring this condition at the outset.

Consider some contract $c_1(z)$ and some cutoff type $\bar{\theta}$. The consumption for agents with types in $(\bar{\theta}, 1]$ (those who wait until the second period) is given by

$$c_2(\bar{\theta}, c_1(z)) = \frac{\left(1 - \int_0^{F(\bar{\theta})} c_1(z)dz\right) R}{1 - F(\bar{\theta})}. \tag{1}$$

The IC constraint corresponding to a contract $c_1(z)$ is satisfied if and only if the cutoff type $\bar{\theta}$ is exactly indifferent between withdrawing early and waiting:

$$\frac{1}{F(\bar{\theta})} \int_0^{F(\bar{\theta})} u(c_1(z))dz = \bar{\theta} u(c_2(\bar{\theta}, c_1(z))). \tag{2}$$

The social welfare when the contract is $c_1(z)$ and the agents strategy profile is $\bar{\theta}$ is given by

$$W(\bar{\theta}, c_1(z)) = \int_0^{F(\bar{\theta})} u(c_1(z))dz + \int_{F(\bar{\theta})}^1 f(\theta)\theta u(c_2(\bar{\theta}, c_1(z)))d\theta. \tag{3}$$

As will become clear below, for a given contract there may be multiple equilibria of the post-deposit subgame. That is, for a given $c_1(z)$ there may be more than one $\bar{\theta}$ such that (1) and (2) are satisfied. When we talk about an optimal contract we mean that we can choose the best equilibrium - the one for which social welfare is maximized. Thus, the optimal contract solves

$$\max_{\bar{\theta}, c_1(z)} W(\bar{\theta}, c_1(z))$$
$$s.t. \text{ (1) } and \text{ (2)}.$$

4.1. **Constant contracts are optimal.** Our first result shows that limiting attention to constant contracts is without loss of generality when considering the optimal contract.

Proposition 1. *Let $c_1(z)$ be an arbitrary contract, and assume that (1) and (2) are satisfied at $\bar{\theta} \in (0, 1)$. Assume further that $c_1(z)$ is not constant in the interval $0 \leq z \leq F\left(\bar{\theta}\right)$. Then there is a constant contract c_1 such that (1) and (2) are still satisfied at $\bar{\theta}$ under c_1, and such that $W(\bar{\theta}, c_1) > W(\bar{\theta}, c_1(z))$. In particular, an optimal contract must give the same consumption to all the agents that withdraw in the first period.*

Before proving the proposition, we state a lemma that will be useful for many of the results.

Lemma 1. *For every $0 < \bar{\theta} < 1$ there is a unique constant contract c_1 such that (1) and (2) are satisfied at $\bar{\theta}$ under c_1.*

We turn now to the proof of Proposition 1.

Proof. Let c_1 be the (unique) constant contract such that (1) and (2) are satisfied at $\bar{\theta}$, whose existence is guaranteed by Lemma 1. We show that $W(\bar{\theta}, c_1) > W(\bar{\theta}, c_1(z))$.

First, we claim that the total consumption given in the first period with the contract c_1 is strictly less than that given with the contract $c_1(z)$, i.e. $c_1 F(\bar{\theta}) < \int_0^{F(\bar{\theta})} c_1(z)dz$. Indeed, concavity of u and Jensen's inequality imply that

$$\frac{1}{F(\bar{\theta})} \int_0^{F(\bar{\theta})} u(c_1(z))dz < u\left(\frac{1}{F(\bar{\theta})} \int_0^{F(\bar{\theta})} c_1(z)dz\right).$$

Thus, by smoothing the consumption given in the first period by the contract $c_1(z)$ we get a constant contract which makes the type $\bar{\theta}$ strictly prefer to withdraw early. This implies that the total consumption given in the first period, by the constant contract c_1 (constructed to make $\bar{\theta}$ indifferent) is lower than the total consumption given by $c_1(z)$.

It follows that the second period consumption is higher under c_1 than under $c_1(z)$. In particular, the utility that an agent of type $\bar{\theta}$ obtains by waiting is higher under c_1. But since $\bar{\theta}$ is indifferent this implies that the (expected) first period utility is is also higher under c_1. Thus, both the types $[\bar{\theta}, 1]$ who wait and the types $[0, \bar{\theta})$ who withdraw early are better off under c_1, so the total social welfare has increased. □

4.2. **The IC constraint and equilibrium multiplicity.** Now that we know that an optimal contract is constant, we restrict attention to this type of contract. Note that under a constant contract the objective function W becomes

$$W(\bar{\theta}, c_1) = F(\bar{\theta})u(c_1) + \big(1 - F(\bar{\theta})\big) \cdot \mathbb{E}(\theta|\theta > \bar{\theta}) \cdot u(c_2(\bar{\theta}, c_1)), \tag{4}$$

and the constraints are

$$u(c_1) = \bar{\theta}u(c_2(\bar{\theta}, c_1)) \tag{5}$$

and

$$c_2(\bar{\theta}, c_1) = \frac{\big(1 - F(\bar{\theta})c_1\big)\, R}{1 - F(\bar{\theta})}. \tag{6}$$

In order to understand better the properties of an optimal contract we first need to know more about the shape of the IC constraint, where (6) is substituted into (5). Note that Lemma 1 implies that the IC constraint defines a function $\tilde{c}_1(\bar{\theta})$ for $\bar{\theta} \in (0, 1)$. That is, $\tilde{c}_1(\bar{\theta})$ is the unique constant contract under which type $\bar{\theta}$ is indifferent given that types with $\theta < \bar{\theta}$ withdraw and types with $\theta > \bar{\theta}$ wait. The corresponding function $\tilde{c}_2(\bar{\theta}, \tilde{c}_1(\bar{\theta}))$

is denoted for short by $\tilde{c}_2(\bar{\theta})$. We can interpret the inverse of the function $\tilde{c}_1(\bar{\theta})$ as the depositors' "best-response correspondence" to the bank's choice of contract: Given c_1, an equilibrium behavior of the depositors must result in a cutoff type $\bar{\theta}$ that satisfies $\tilde{c}_1(\bar{\theta}) = c_1$. Figure 1 illustrates the results of the following lemma.

Lemma 2. *The following hold:*

(i) $\tilde{c}_1(\bar{\theta})$ *is continuously differentiable.*

(ii) $\lim_{\bar{\theta}\to 0} \tilde{c}_1(\bar{\theta}) = 0$ *and* $\lim_{\bar{\theta}\to 1} \tilde{c}_1(\bar{\theta}) = 1$*, so we can extend* $\tilde{c}_1(\bar{\theta})$ *continuously to the closed interval* $0 \le \bar{\theta} \le 1$ *by setting* $\tilde{c}_1(0) = 0$ *and* $\tilde{c}_1(1) = 1$*.*

(iii) If $\bar{\theta} \in \left[0, \frac{u(1)}{u(R)}\right]$ *then* $0 \le \tilde{c}_1(\bar{\theta}) \le 1$*, and if* $\bar{\theta} \in (\frac{u(1)}{u(R)}, 1)$ *then* $\tilde{c}_1(\bar{\theta}) > 1$*.*

(iv) There exists $\frac{u(1)}{u(R)} < \bar{\theta}_0 < 1$ *such that* $\tilde{c}_1(\bar{\theta})$ *is strictly increasing in the interval* $[0, \bar{\theta}_0]$ *and strictly decreasing in* $[\bar{\theta}_0, 1]$*.*

The following is an immediate corollary of Lemma 2.

Proposition 2. *Let* c_1 *be a constant contract.*

(i) If $0 \le c_1 \le 1$ *then there is a unique interior equilibrium* $\bar{\theta}$ *to the post-deposit subgame.*

(ii) If $1 < c_1 < \tilde{c}_1(\bar{\theta}_0)$ *then there are exactly two interior equilibria.*

(iii) If $c_1 = \tilde{c}_1(\bar{\theta}_0)$ *then there is a unique interior equilibrium at* $\bar{\theta} = \bar{\theta}_0$*.*

(iv) If $c_1 > \tilde{c}_1(\bar{\theta}_0)$ *then there is no interior equilibrium.*

In addition, whenever $c_1 \ge 1$ *there is a "full-run equilibrium" where* $\bar{\theta} = 1$*.*

4.3. **Necessary conditions for optimality.** The purpose of this subsection is to find the part of the curve defined by the IC constraint in which an optimal contract (and corresponding equilibrium) must be located. Note that an optimal contract must exist, as W is a continuous function over the compact graph of the IC constraint.

Proposition 3. *Let* $\left(\bar{\theta}^*, c_1^* = \tilde{c}_1(\bar{\theta}^*)\right)$ *be a point on the graph of the IC constraint where* W *is maximized. Then* $c_1^* > 1$ *and* $\tilde{c}_1'(\bar{\theta}^*) \ge 0$*. Equivalently,* $\frac{u(1)}{u(R)} < \bar{\theta}^* \le \bar{\theta}_0$*.*

It follows from the proof of Proposition 3 that, whenever there are multiple interior equilibria for a given contract, the one at which welfare is maximized is the one with the fewest types withdrawing early. Thus, for a given contract, we refer to the interior equilibrium with the smaller $\bar{\theta}$ as the "best equilibrium", and to the other (interior) equilibrium as the "partial-run equilibrium". The equilibrium in which $\bar{\theta} = 1$ is referred to as the "full-run equilibrium".

4.4. **Excessive withdrawals.** An immediate corollary of Proposition 3 is that, given the optimal contract, social welfare increases if fewer agents withdraw in the first period than in the best equilibrium under this contract. The intuition is that the cutoff type is indifferent between withdrawing in period 1 and waiting. Since the bank provides insurance against being impatient by offering those who withdraw more consumption than they deposit, waiting provides a positive externality to others who wait, by reducing this drain on resources. Thus, given c_1^*, society would be better off if agents of type just below $\bar{\theta}^*$, who are nearly indifferent, decided instead to wait.

Corollary 1. *Let* $\left(\bar{\theta}^*, c_1^* = \tilde{c}_1\left(\bar{\theta}^*\right)\right)$ *be an optimal contract and the corresponding best equilibrium. Then for every* $\bar{\theta} < \bar{\theta}^*$ *which is sufficiently close to* $\bar{\theta}^*$ *we have* $W(\bar{\theta}, c_1^*) > W(\bar{\theta}^*, c_1^*)$.

Proof. We know from Proposition 3 that $c_1^* > 1$. But whenever $c_1 > 1$ and the constraint is satisfied the partial derivative $\frac{\partial W(\bar{\theta}, c_1)}{\partial \bar{\theta}}$ is strictly negative (see (7)). Thus, if $\bar{\theta} < \bar{\theta}^*$ is sufficiently close to $\bar{\theta}^*$ then $W(\bar{\theta}, c_1^*) > W(\bar{\theta}^*, c_1^*)$. □

4.5. **Suspension of convertibility.** As should be clear from the previous results, when the planner chooses the optimal constant contract there are three equilibria to the post-deposit subgame. The best equilibrium is the one with the smallest $\bar{\theta}$, but there are also the interior partial-run equilibrium and the full-run equilibrium at which social welfare is strictly worse. The question we address in this subsection is whether the planner can achieve the best equilibrium allocation as the unique equilibrium outcome of the post-deposit subgame when non-constant contracts are considered. As in the literature with only two types, a natural candidate for such a contract is one in which after the "right" amount of consumption is given in the first period the bank suspends any further withdrawals until the second period. The next proposition shows that such a contract achieves the desired goal.

Proposition 4. *Let* $\left(\bar{\theta}^*, c_1^* = \tilde{c}_1(\bar{\theta}^*)\right)$ *be an optimal (constant) contract and the corresponding best equilibrium. Consider the (non-constant) contract defined by*

$$c_1^*(z) = \begin{cases} c_1^* & 0 \leq z \leq F\left(\bar{\theta}^*\right) \\ 0 & \textit{otherwise.} \end{cases}$$

Then the unique equilibrium under $c_1^*(z)$ *is at* $\bar{\theta}^*$, *and* $W(\bar{\theta}^*, c_1^*(z)) = W(\bar{\theta}^*, c_1^*)$.

Proof. It is obvious that W is the same under these two contracts (assuming that $\bar{\theta}^*$ is played in the post-deposit subgame). It is also clear that $\bar{\theta}^*$ is an equilibrium under $c_1^*(z)$. It remains to show that $\bar{\theta}^*$ is the unique equilibrium.

First, there cannot be an equilibrium with $\bar{\theta} < \bar{\theta}^*$ since $c_1^*(z)$ and c_1^* are identical in this interval, and $\bar{\theta}^*$ is the smallest equilibrium of c_1^*. Second, if $\bar{\theta} > \bar{\theta}^*$ then it cannot be an equilibrium: When the strategy profile $\bar{\theta}^*$ is played, an agent of type $\bar{\theta} > \bar{\theta}^*$ strictly prefers to wait. Compared to that strategy profile, when the strategy profile $\bar{\theta}$ is played the utility from withdrawing early is lower and from waiting is higher. Thus, it cannot be that $\bar{\theta}$ is indifferent. $\square$

5. The no commitment case

We now move to the case in which the bank has no ability to commit to the contract that it offers. Without commitment, the situation can be thought of as a simultaneous-move game, in which the bank chooses a contract $c_1(z)$ and each depositor chooses whether to withdraw or wait. An equilibrium of this game is a profile of strategies (for the bank and the depositors) in which $c_1(z)$ is a best response (i.e., maximizes welfare) given the strategies of the depositors, and each depositor is best responding to $c_1(z)$ and the withdrawal behavior of the other depositors.

Notice first that, like in the commitment case, in any equilibrium the set of types withdrawing early is an interval $[0, \bar{\theta}]$. Further, since the agents best respond to the contract, the constraints (1) and (2) must be satisfied in equilibrium. Thus, given a constant contract c_1, $\bar{\theta}$ must satisfy $\tilde{c}_1(\bar{\theta}) = c_1$ as in the commitment case. What differs will be the bank's choice of c_1.

Moving on to the bank's side, we know from Proposition 1 that a best response to $\bar{\theta}$ must give a constant consumption in the interval $[0, F(\bar{\theta})]$. Thus, we can restrict attention to this kind of contract. The following lemma describes additional properties of the best response of the bank to a given $\bar{\theta}$.

Lemma 3. *For every fixed* $0 < \bar{\theta} < 1$ *there exits a unique maximizer* $\hat{c}_1(\bar{\theta})$ *to the function* $W(\bar{\theta}, c_1)$ *as specified in (4), with (6) substituted into the objective function. Furthermore,* $\hat{c}_1(\bar{\theta})$ *is continuously differentiable, decreasing, satisfies* $\hat{c}_1(\bar{\theta}) > 1$ *for every* $0 < \bar{\theta} < 1$, *and* $\lim_{\bar{\theta}\to 1} \hat{c}_1(\bar{\theta}) = 1$.

To summarize the discussion so far, a strategy profile for the depositors and the bank $(\bar{\theta}, c_1)$ is an equilibrium if and only if $c_1 = \tilde{c}_1(\bar{\theta}) = \hat{c}_1(\bar{\theta})$. The reason is that the function

$\hat{c}_1(\bar{\theta})$ is the bank's best response function, and given c_1 depositors are best-responding if and only if $c_1 = \tilde{c}_1(\bar{\theta})$ is satisfied for the corresponding $\bar{\theta}$.

Proposition 5. *The game without commitment admits at least one equilibrium with* $0 < \bar{\theta} < 1$.

The main result of this section compares the no-commitment to the commitment case. Proposition 6 below shows that the optimal contract without commitment yields strictly more types withdrawing in period 1 and strictly lower welfare than the optimal contract with commitment. Intuitively, in the optimal contract with commitment, $(\bar{\theta}^*, c_1^*)$, given $\bar{\theta}^*$, welfare would be higher if c_1 were increased above c_1^*. The bank commits to c_1^* in order to manipulate $\bar{\theta}$. However, without commitment, depositors have already made their withdrawal decisions, so given a cutoff $\bar{\theta}^*$, the bank would choose consumption above c_1^*; anticipating that, more types would withdraw in period 1.

Proposition 6. *Let* $(\bar{\theta}^*, c_1^*)$ *be an optimal contract in the case with commitment, and let* $(\bar{\theta}^{**}, c_1^{**})$ *be an equilibrium of the no-commitment game. Then* $\bar{\theta}^* < \bar{\theta}^{**}$ *and* $W(\bar{\theta}^*, c_1^*) > W(\bar{\theta}^{**}, c_1^{**})$.

Proof. From Proposition 3 we know that $(\bar{\theta}^*, c_1^*)$ is on the part of the IC constraint where $\tilde{c}_1(\bar{\theta}) > 1$ and $\tilde{c}_1(\bar{\theta})$ is increasing. In this part of the curve the partial derivative of W with respect to $\bar{\theta}$ is negative. Since at the optimal contract the gradient of W is orthogonal to $\tilde{c}_1'(\bar{\theta})$, it must be that the partial derivative of W with respect to c_1 is strictly positive at $(\bar{\theta}^*, c_1^*)$. Thus, it follows from the proof of Lemma 3 that $c_1^* < \hat{c}_1(\bar{\theta}^*)$, that is, $(\bar{\theta}^*, c_1^*)$ lies below the function $\hat{c}_1(\bar{\theta})$. Since, by Lemma 3, $\hat{c}_1(\bar{\theta})$ is decreasing, all intersections of $\hat{c}_1(\bar{\theta})$ and $\tilde{c}_1(\bar{\theta})$ must occur at $\bar{\theta} > \bar{\theta}^*$, which establishes that $\bar{\theta}^* < \bar{\theta}^{**}$. Finally, since $(\bar{\theta}^{**}, c_1^{**})$ cannot be an optimal contract for the commitment game, it follows that we have $W(\bar{\theta}^*, c_1^*) > W(\bar{\theta}^{**}, c_1^{**})$. □

Ennis and Keister (2010) study the game without commitment for the two-type model. In that setting, the optimal contract and allocation under commitment remains an equilibrium without commitment. With a continuum of types, Proposition 6 shows that the inability to commit entails a loss of welfare at the best equilibrium.

6. An example

We illustrate our results with the following example. Let the distribution of types in the population be uniform on $[0,1]$, and assume that the utility function is given by

$$u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma},$$

where $b > 0$ and $\gamma > 1$ are parameters.[4] Social welfare under a (constant) contract c_1 and given a threshold type $\bar{\theta}$ is given by

$$W(\bar{\theta}, c_1) = \bar{\theta} \cdot \frac{(c_1+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma} + (1-\bar{\theta}) \cdot \frac{1+\bar{\theta}}{2} \cdot \frac{(c_2+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}.$$

The incentive compatibility constraint is given by

$$\frac{(c_1+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma} = \bar{\theta} \cdot \frac{(c_2+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma},$$

where

$$c_2 = \frac{(1-\bar{\theta}c_1)R}{1-\bar{\theta}}.$$

For concreteness, assume that $R = 2$, $b = 1/2$ and $\gamma = 2$. We note that for these parameter values all of our assumptions are satisfied. Solving the incentive compatibility constraint for c_1 (as a function of $\bar{\theta}$) we get the function $\tilde{c}_1(\bar{\theta})$:

$$\tilde{c}_1(\bar{\theta}) = \frac{-2\bar{\theta}^2 + 5\bar{\theta} - 5 + \sqrt{4\bar{\theta}^4 + 12\bar{\theta}^3 + 13\bar{\theta}^2 - 50\bar{\theta} + 25}}{8\bar{\theta}(\bar{\theta}-1)}.$$

The optimal contract c_1^* and the corresponding best equilibrium $\bar{\theta}^*$ are given by ($\bar{\theta}^* = 0.856, c_1^* = 1.016$). Social welfare is given by $W(\bar{\theta}^*, c_1^*) = 1.356$.

For the no-commitment case, we first find the planner's best response function $\hat{c}_1(\bar{\theta})$. This yields

$$\hat{c}_1(\bar{\theta}) = \frac{-\bar{\theta}^3 + 3\bar{\theta}^2 - 9\bar{\theta} - 1 + \sqrt{\bar{\theta}^5 + 9\bar{\theta}^4 + 14\bar{\theta}^3 - 34\bar{\theta}^2 - 15\bar{\theta} + 25}}{2\bar{\theta}^3 - 10\bar{\theta}^2 - 2\bar{\theta} + 2}.$$

Equilibria of the no-commitment game are the intersection points of $\hat{c}_1(\bar{\theta})$ with $\tilde{c}_1(\bar{\theta})$. For this particular example, there is only one such intersection at ($\bar{\theta}^{**} = 0.883, c_1^{**} = 1.027$). Social welfare at the equilibrium is $W(\bar{\theta}^{**}, c_1^{**}) = 1.355$. Notice that more types withdraw when there is no commitment, and that social welfare decreases in this case. Figures 2 and 3 show the curves $\hat{c}_1\left(\bar{\theta}\right)$ and $\tilde{c}_1\left(\bar{\theta}\right)$, and the equilibria of the games with and without commitment.

[4] This utility function is used in Gu (2010).

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Appendix A. Proofs

A.1. **Proof of Lemma 1.**

Proof. Fix $\bar{\theta} \in (0,1)$ and consider the function $g(c_1) = u(c_1) - \bar{\theta} u(c_2(c_1, \bar{\theta}))$ defined for $c_1 \in (0, 1/F(\bar{\theta})]$. When c_1 is close to 0 g is negative, and at $c_1 = 1/F(\bar{\theta})$ it is positive. Further, it is easy to see that g is continuous and strictly increasing on its domain. Thus, there is a unique c_1 such that $g(c_1) = 0$, as required. □

A.2. **Proof of Lemma 2.**

Proof. Property (i) follows immediately from the implicit function theorem (see, e.g., de la Fuente (2000), Theorem 2.1 on page 207). As for (ii), the right-hand side of the IC constraint is bounded above by $\bar{\theta}u\left(\frac{R}{1-F(\bar{\theta})}\right)$, which converges to 0 when $\bar{\theta}$ goes to 0. It follows that $\lim_{\bar{\theta}\to 0} u(\tilde{c}_1(\bar{\theta})) = 0$, which implies that $\lim_{\bar{\theta}\to 0} \tilde{c}_1(\bar{\theta}) = 0$. The other limit follows from the facts that $\tilde{c}_1(\bar{\theta}) \leq 1/F(\bar{\theta})$ for every $\bar{\theta}$, and that $\tilde{c}_1(\bar{\theta}) > 1$ when $\bar{\theta}$ is close to 1 (since for $c_1 = 1$ any type θ sufficiently close to 1 prefers to wait).

We now prove (iii) and (iv). The derivative of $\tilde{c}_1(\bar{\theta})$ is obtained from the identity $u(\tilde{c}_1(\bar{\theta})) - \bar{\theta}u(\tilde{c}_2(\tilde{c}_1(\bar{\theta}), \bar{\theta})) = 0$. Taking derivative gives (the argument $\bar{\theta}$ is sometimes omitted to make the reading easier)

$$\begin{aligned} u'(\tilde{c}_1) \cdot \tilde{c}_1'(\bar{\theta}) &= u(\tilde{c}_2) + \bar{\theta} \cdot u'(\tilde{c}_2) \cdot \tilde{c}_2'(\bar{\theta}) = \\ & u(\tilde{c}_2) + \bar{\theta} \cdot u'(\tilde{c}_2) \left[\frac{-RF(\bar{\theta})\tilde{c}_1'(\bar{\theta})}{1-F(\bar{\theta})} + \frac{Rf(\bar{\theta})(1-\tilde{c}_1(\bar{\theta}))}{(1-F(\bar{\theta}))^2} \right]. \end{aligned}$$

After some algebra we get

$$\tilde{c}_1'(\bar{\theta}) \left[u'(\tilde{c}_1) + \frac{\bar{\theta}u'(\tilde{c}_2)RF(\bar{\theta})}{1-F(\bar{\theta})} \right] = u(\tilde{c}_2) + \frac{\bar{\theta}u'(\tilde{c}_2)Rf(\bar{\theta})(1-\tilde{c}_1(\bar{\theta}))}{(1-F(\bar{\theta}))^2}.$$

It follows that as long as $\tilde{c}_1(\bar{\theta}) \leq 1$ it is strictly increasing.

We know from (ii) of this lemma that $\tilde{c}_1(0) = 0$. Further, $\tilde{c}_1(\bar{\theta}) = 1$ implies that $\bar{\theta} = u(1)/u(R)$ or $\bar{\theta} = 1$. Thus, in the interval $\left[0, \frac{u(1)}{u(R)}\right]$ we have $0 \leq \tilde{c}_1(\bar{\theta}) \leq 1$ and $\tilde{c}_1(\bar{\theta})$ is strictly increasing. Now, at the point $\bar{\theta} = u(1)/u(R)$ (where $\tilde{c}_1(\bar{\theta}) = 1$) the derivative $\tilde{c}_1'(\bar{\theta})$ is still strictly positive, so $\tilde{c}_1(\bar{\theta}) > 1$ in a small interval to the right of this point. But it cannot happen that $\tilde{c}_1(\bar{\theta})$ falls to 1 at any point $u(1)/u(R) < \bar{\theta} < 1$, since the derivative $\tilde{c}_1'(\bar{\theta})$ is positive whenever $\tilde{c}_1(\bar{\theta}) = 1$. This concludes the proof of (iii).

Let $\bar{\theta}_0 > u(1)/u(R)$ be the first point where $\tilde{c}_1'(\bar{\theta}) = 0$, whose existence is guaranteed by the fact that $\tilde{c}_1(\bar{\theta})$ cannot increase on the entire interval [0,1] (recall that $\tilde{c}_1(1) = 1$ by (ii)). Then $\tilde{c}_1(\bar{\theta})$ is increasing in $[0, \bar{\theta}_0]$. To complete the proof of (iv) we need to show that $\tilde{c}_1(\bar{\theta})$ is strictly decreasing in $[\bar{\theta}_0, 1]$.

Fix some number $t > 1$ and consider the function

$$h(\bar{\theta}) = \bar{\theta}u\left(\frac{(1-F(\bar{\theta})t)R}{1-F(\bar{\theta})}\right).$$

The derivative $h'(\bar{\theta})$ is

$$h'(\bar{\theta}) = u\left(\frac{(1-F(\bar{\theta})t)R}{1-F(\bar{\theta})}\right) + \bar{\theta} \cdot u'\left(\frac{(1-F(\bar{\theta})t)R}{1-F(\bar{\theta})}\right) \cdot \frac{Rf(\bar{\theta})(1-t)}{(1-F(\bar{\theta}))^2}.$$

Since $t > 1$ and because the hazard rate is non-decreasing, $h'(\bar{\theta})$ is decreasing. Thus, h is a concave function. In particular, it can obtain any specific value at most twice. It follows that for any fixed $c_1 > 1$ there can be at most two solutions to the IC constraint with the constant contract c_1. To conclude, after $\tilde{c}_1(\bar{\theta})$ reaches its pick at $\bar{\theta}_0$, it must strictly decrease from that point; otherwise there will be a fixed contract c_1 such that the IC constraint is satisfied at three different points $\bar{\theta}$. $\square$

A.3. **Proof of Proposition 3.**

Proof. We start by showing that the solution is interior, i.e. that $(\bar{\theta} = 0,\ c_1 = 0)$ and $(\bar{\theta} = 1,\ c_1 = 1)$ are not maxima. Indeed, we show that $(\bar{\theta} = u(1)/u(R),\ c_1 = 1)$ is strictly better than both of them.

We have $W(0,0) = u(R)\mathbb{E}(\theta)$ and $W(1,1) = u(1)$. Denoting $\alpha = u(1)/u(R)$ we also have $W(\alpha,1) = F(\alpha)u(1) + (1 - F(\alpha))\mathbb{E}(\theta|\theta > \alpha)u(R)$. Thus,

$$\begin{aligned} W(\alpha,1) &= u(R)[F(\alpha)\alpha + (1 - F(\alpha))\mathbb{E}(\theta|\theta > \alpha)] > \\ & \quad u(R)[F(\alpha)\mathbb{E}(\theta|\theta \le \alpha) + (1 - F(\alpha))\mathbb{E}(\theta|\theta > \alpha)] = u(R)\mathbb{E}(\theta) = W(0,0), \end{aligned}$$

and

$$\begin{aligned} W(\alpha,1) &= u(1)\left[F(\alpha) + \frac{(1 - F(\alpha))\mathbb{E}(\theta|\theta > \alpha)}{\alpha}\right] > \\ & \quad u(1)[F(\alpha) + (1 - F(\alpha))] = u(1) = W(1,1). \end{aligned}$$

Since the solution is interior we can use the necessary condition for optimality obtained from the Lagrangian. This condition says that the gradient of W should be orthogonal to the derivative of the curve $\tilde{c}_1$ at $\bar{\theta}^*$. The partial derivatives of W are given by

$$\frac{\partial W(\bar{\theta},c_1)}{\partial \bar{\theta}} = f(\bar{\theta})\left[u(c_1) - \bar{\theta}u(c_2) - \frac{(c_1 - 1)Ru'(c_2)\mathbb{E}(\theta|\theta \ge \bar{\theta})}{1 - F(\bar{\theta})}\right] \tag{7}$$

$$\frac{\partial W(\bar{\theta},c_1)}{\partial c_1} = F(\bar{\theta})[u'(c_1) - R\mathbb{E}(\theta|\theta > \bar{\theta})u'(c_2)]. \tag{8}$$

From (7), if $c_1 < 1$ and the IC constraint is satisfied then $\frac{\partial W(\bar{\theta},c_1)}{\partial \bar{\theta}} > 0$. We claim that whenever $c_1 \le 1$ it is also true that $\frac{\partial W(\bar{\theta},c_1)}{\partial c_1} > 0$. Indeed, when $c_1 = 1$ we get from (8)

$$\frac{\partial W(\bar{\theta},1)}{\partial c_1} = F(\bar{\theta})[u'(1) - Ru'(R)\mathbb{E}(\theta|\theta \ge \bar{\theta})] \ge F(\bar{\theta})[u'(1) - Ru'(R)].$$

Since the coefficient of relative risk aversion is greater than 1, the function $cu'(c)$ is decreasing. Thus, $u'(1) > Ru'(R)$, and the derivative is positive. Furthermore, the second partial derivative is

$$\frac{\partial^2 W(\bar{\theta}, c_1)}{\partial c_1^2} = F(\bar{\theta}) \left[u''(c_1) + R^2 u''(c_2) \mathbb{E}(\theta | \theta \geq \bar{\theta}) \frac{F(\bar{\theta})}{1 - F(\bar{\theta})} \right] < 0.$$

Thus, for any fixed $0 < \bar{\theta} < 1$, W is a concave function of c_1. This implies that $\frac{\partial W(\bar{\theta}, c_1)}{\partial c_1} > 0$ for every $0 < \bar{\theta} < 1$ and every $c_1 \leq 1$, as claimed.

It follows that in the interval $0 < \bar{\theta} \leq u(1)/u(R)$ the gradient of W points to the north-east (exactly north at the the point $\bar{\theta} = u(1)/u(R)$). Since we know from Lemma 2 that $\tilde{c}_1(\bar{\theta})$ is increasing in this interval the necessary condition cannot be satisfied. This proves that $\bar{\theta}^* > u(1)/u(R)$.

Finally, it is not possible that an equilibrium where $\bar{\theta}_0 < \bar{\theta} < 1$ is optimal, since the equilibrium with the smaller $\bar{\theta}$ under the same contract gives higher welfare. Indeed, at any point $(\bar{\theta}, c_1)$ which is (weakly) below the curve of the IC constraint and where $c_1 > 1$ we have $u(c_1) - \bar{\theta} u(c_2) < 0$; thus, the partial derivative $\frac{\partial W(\bar{\theta}, c_1)}{\partial \theta}$ is negative. It follows that, when moving from the equilibrium with the lower $\bar{\theta}$ to the one with the higher $\bar{\theta}$ (keeping c_1 constant), welfare strictly decreases. □

A.4. **Proof of Lemma 3.**

Proof. Fix $\bar{\theta} \in (0, 1)$ and consider $W(c_1, \bar{\theta})$ as a function of c_1 only. We know from the proof of Proposition 3 that this is a strictly concave function, and that the derivative of this function (see (8)),

$$\frac{\partial W(c_1, \bar{\theta})}{\partial c_1} = F(\bar{\theta})[u'(c_1) - R\mathbb{E}(\theta | \theta > \bar{\theta}) u'(c_2)],$$

is positive in an interval of c_1, $[0, 1 + \epsilon]$ for sufficiently small $\epsilon > 0$. Furthermore, it follows from the assumption that $u'(1) < u'(0) \cdot R \cdot \mathbb{E}(\theta)$ that the derivative is negative at $c_1 = 1/F(\bar{\theta})$, since $c_2 = 0$ at this point. Thus, the maximizer $\hat{c}_1(\bar{\theta})$ is the unique number c_1 that satisfies

$$u'(c_1) = R\mathbb{E}(\theta | \theta > \bar{\theta}) u'(c_2),$$

and $\hat{c}_1 \left(\bar{\theta} \right) > 1$. The implicit function theorem implies that $\hat{c}_1(\bar{\theta})$ is a continuously differentiable function of $\bar{\theta}$ (see de la Fuente (2000), Theorem 2.1 on page 207). Since $1 < \hat{c}_1(\bar{\theta}) < 1/F(\bar{\theta})$ it follows that $\lim_{\bar{\theta} \to 1} \hat{c}_1(\bar{\theta}) = 1$.

It remains to show that $\hat{c}_1(\bar{\theta})$ is decreasing. For this purpose we take the derivative with respect to $\bar{\theta}$ of the identity

$$u'(\hat{c}_1(\bar{\theta})) = R\mathbb{E}(\theta | \theta > \bar{\theta}) u'(\hat{c}_2(\bar{\theta})).$$

This yields

$$u''(\hat{c}_1)\hat{c}_1'(\bar{\theta}) = \frac{Rf(\bar{\theta})(\mathbb{E}(\theta|\theta > \bar{\theta}) - \bar{\theta})u'(\hat{c}_2)}{1 - F(\bar{\theta})} + R\mathbb{E}(\theta|\theta > \bar{\theta})u''(\hat{c}_2)\hat{c}_2'(\bar{\theta}). \tag{9}$$

We claim that this equation cannot be satisfied if $\hat{c}_1'(\bar{\theta}) \geq 0$. Indeed, if this were the case then the left-hand side of (9) would be negative. The first term on the right-hand side is obviously positive. As for the second term, the derivative $\hat{c}_2'(\bar{\theta})$ is given by

$$\hat{c}_2'(\bar{\theta}) = \frac{-RF(\bar{\theta})\hat{c}_1'(\bar{\theta})}{1 - F(\bar{\theta})} + \frac{Rf(\bar{\theta})(1 - \hat{c}_1(\bar{\theta}))}{(1 - F(\bar{\theta}))^2},$$

which is negative when $\hat{c}_1(\bar{\theta}) > 1$ and $\hat{c}_1'(\bar{\theta}) \geq 0$. But we have already shown that $\hat{c}_1(\bar{\theta})$ is always greater than 1. This proves that the second term on the right-hand side must be positive as well if $\hat{c}_1'(\bar{\theta}) \geq 0$, a contradiction. This completes the proof of the lemma. □

A.5. **Proof of Proposition 5.**

Proof. To prove the proposition we need to show that the two functions $\tilde{c}_1$ and $\hat{c}_1$ intersect at some $\bar{\theta} \in (0, 1)$. Notice that, by Lemma 2, $\tilde{c}_1(\bar{\theta})$ is close to 0 when $\bar{\theta}$ is close to 0, while $\hat{c}_1(\bar{\theta})$ is always greater than 1 by Lemma 3. Thus, $\tilde{c}_1(\bar{\theta}) > \hat{c}_1(\bar{\theta})$ holds for $\bar{\theta}$ sufficiently close to 0, and the proposition will be proved if we can show that $\tilde{c}_1(\bar{\theta}) < \hat{c}_1(\bar{\theta})$ for $\bar{\theta}$ sufficiently close to 1.

We know from Lemmas 2 and 3 that $\lim_{\bar{\theta}\to 1} \tilde{c}_1(\bar{\theta}) = \lim_{\bar{\theta}\to 1} \hat{c}_1(\bar{\theta}) = 1$. From the definition of $\tilde{c}_1(\bar{\theta})$ it follows that $\lim_{\bar{\theta}\to 1} \tilde{c}_2(\bar{\theta}) = 1$, while from the definition of $\hat{c}_1(\bar{\theta})$ it follows that $\lim_{\bar{\theta}\to 1} \hat{c}_2(\bar{\theta}) > 1$ (see the proof of Lemma 3). Thus, for $\bar{\theta}$ sufficiently close to 1, we have $\hat{c}_2(\bar{\theta}) > \tilde{c}_2(\bar{\theta})$. But whenever this is the case the inequality $\hat{c}_1(\bar{\theta}) < \tilde{c}_1(\bar{\theta})$ holds. This completes the proof. □

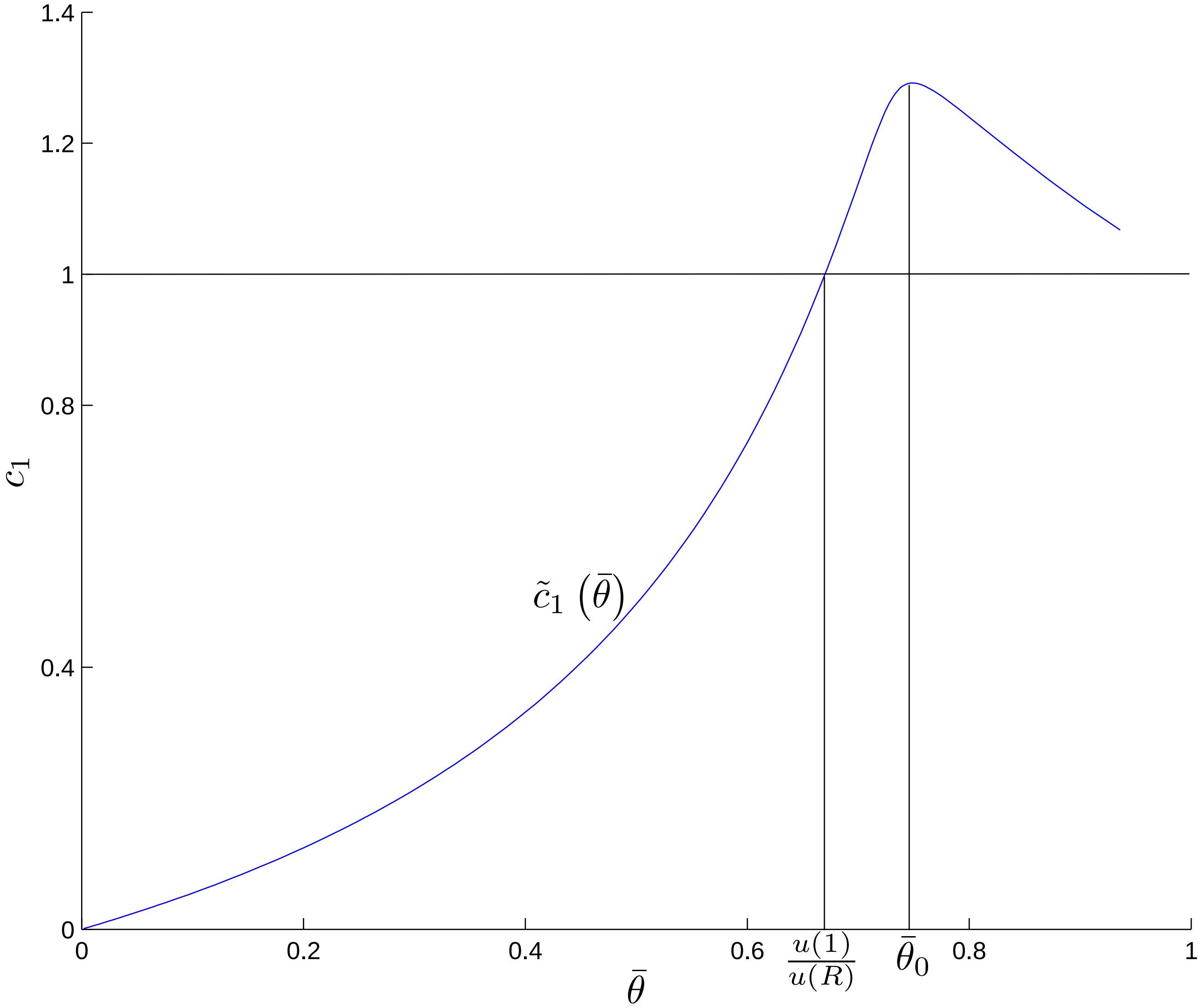

FIGURE 1. The function $\tilde{c}_1$.

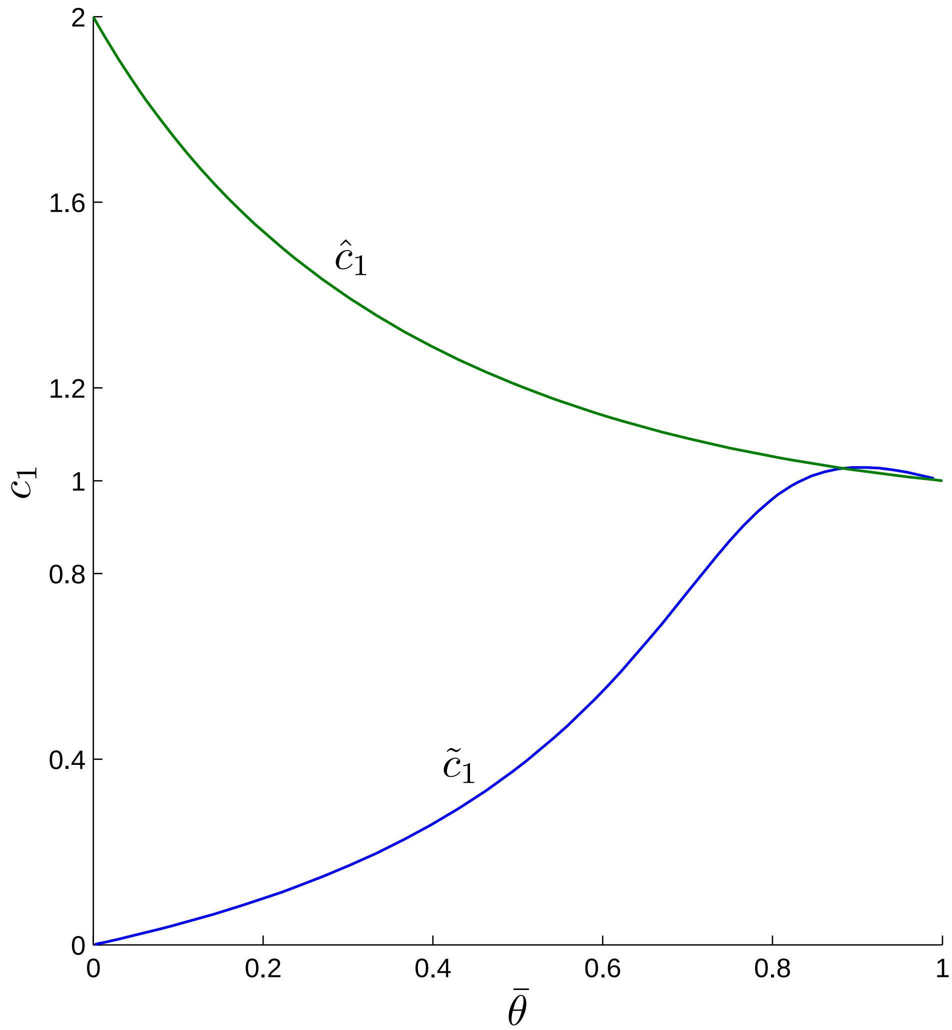

FIGURE 2. The functions $\tilde{c}_1$ and $\hat{c}_1$ from the example.

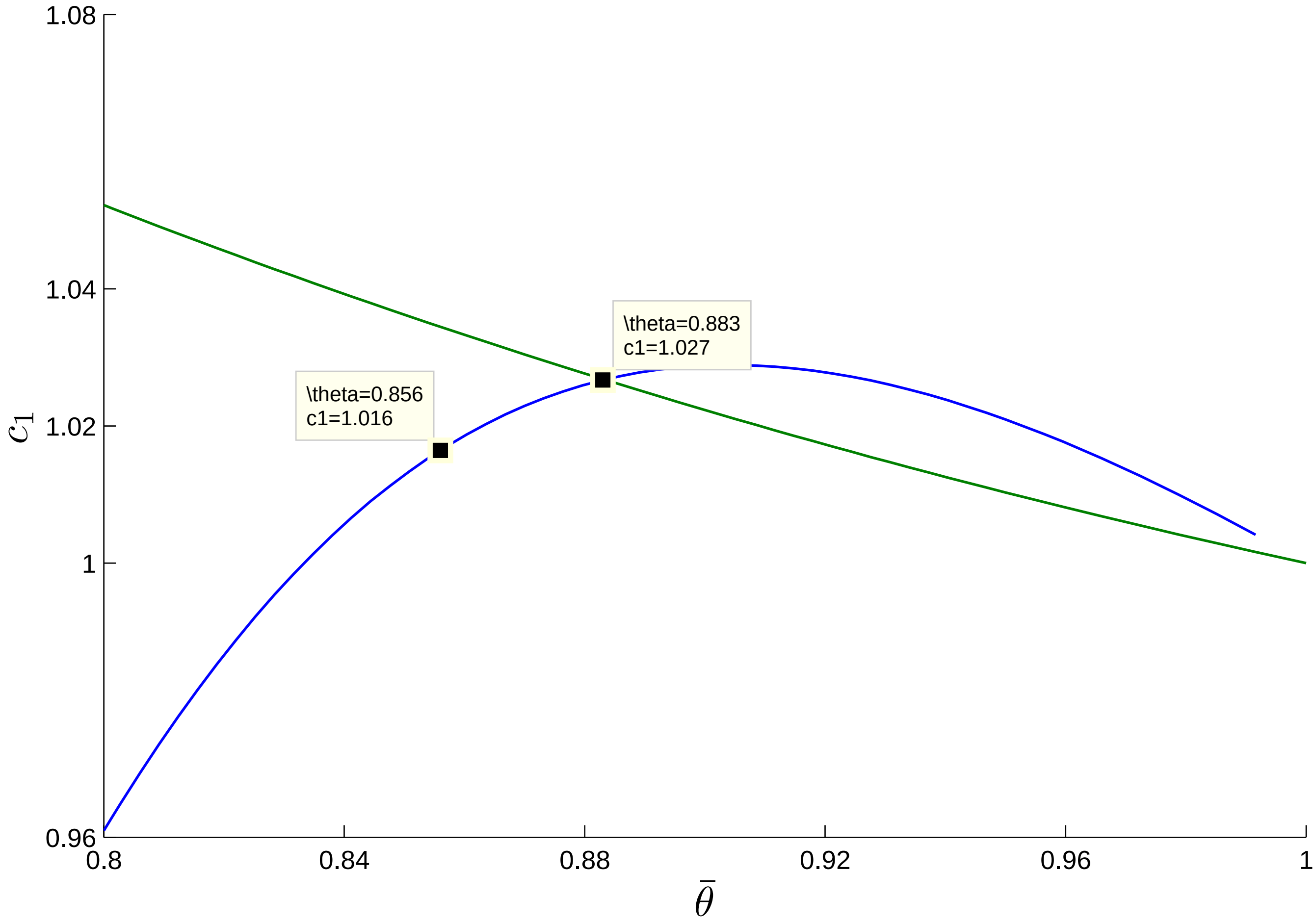

FIGURE 3. The optimal contract with commitment and the equilibrium of the game without commitment from the example.