Durable Goods Monopoly with Quality Improvements and a Continuum of Consumer Types

Andrzej Baranski and James Peck

The Ohio State University

December 10, 2013

Abstract

We consider a two-period durable-goods monopoly model with a continuum of consumer types. In period two the monopolist launches a higher quality version of the product. We evaluate both closed and open resale markets. When consumers cannot trade their old units, the subgame perfect equilibrium exhibits a rich pattern of transactions and prices as a function of the quality improvement parameter θ. Due to the inability to commit, profits are decreasing in θ for values close to the initial quality. When consumers can trade on a competitive resale market, there is a subtle interaction between the firm's pricing and the resale price. Profits are typically (but not always) higher with, rather than without, an open resale market. In the resale market the volume of transactions is not monotonic in θ yet the resale price is always falling with the quality of the new good. In contrast with the the literature, a monopolist that can commit to future sales prices will not achieve the renter's profits.

I Introduction

Smartphones, computers, cars, and many other durable goods periodically undergo quality improvements. For example, between 2007 and 2013, Apple has launched a new iPhone roughly every thirteen months. The iPhone during this period, and many other products, share the following features: (i) consumers only receive services from at most one unit of the good at a time, (ii) there is a continuum of consumer types reflecting preference for quality, (iii) consumers are forward looking, so they face a tradeoff between buying or waiting for quality improvements, (iv) the market structure can be reasonably approximated by a monopolistic seller who cannot commit to future prices, either directly or indirectly, (vi) the quality of new products improves over time, and (v) the time horizon is finite.

There is a huge literature on durable goods monopoly and quite a few papers on durable goods monopoly with either quality improvements or, what is similar but not equivalent, depreciation. This literature is reviewed in Section 2, but we are not aware of any other papers that incorporate all of the above features. The finite time horizon requires a backward induction solution and eliminates reputational equilibria. For tractability, we consider a two period model with a uniform distribution of consumer types (or a linear demand curve for services). Despite the simple model, the subgame perfect equilibrium exhibits a rich pattern of transactions and prices as a function of the quality improvement parameter, θ. We consider the model with a competitive resale market and the model in which the resale market does not exist.

When the resale market does not exist, the period-two subgame equilibrium falls into one of four regimes, depending on θ and the period-one quantity (purchased by an interval of the highest valuation consumers). In Regime A, those consumers who purchased in period one keep their old units, and an interval of the highest valuation consumers who did not purchase in period one purchase in period two. In Regime B, an interval of the highest valuation consumers junk their old unit and purchase new in period two, the remaining consumers who purchased in period one keep their old unit, and an interval of the highest

1

valuation consumers who did not purchase in period one also purchase in period two. In Regime C, an interval of the highest valuation consumers junk their old unit and purchase new in period two, and this interval constitutes all of the period-two sales. In Regime D, all consumers who purchased in period one junk their old unit and purchase new in period two, and an interval of the highest valuation consumers who did not purchase in period one also purchase in period two. Working backwards, we characterize the period one equilibrium price and corresponding quantity. Since the period one price influences the prevailing regime in period two, the monopolist is sometimes at a corner solution in which it hugs the boundary between regimes, sacrificing profits in period one to avoid an unprofitable regime in period two. Thus, period one prices and quantities are discontinuous functions of θ. Also, in the neighborhood of no-quality-improvement (θ close to one), overall monopoly profits are decreasing in θ! This illustrates a new aspect of the time inconsistency problem, because in all period two subgames, the monopolist finds it profitable to utilize the higher quality. Intuitively, higher θ gives the marginal consumer a greater incentive to delay purchase, because she will share in the greater surplus if she waits.

When a competitive resale market exists, in choosing its price in period one, the monopolist takes into account the effect on its own optimal price in the period two subgame, the resale price, and the identity of the marginal consumer who anticipates the period two subgame. For most values of θ, the efficiency-enhancing reallocation of old goods makes the monopolist better off with a resale market, as opposed to without a resale market. However, for high values of θ, we find that the monopolist is better off without a resale market.

We find that a renter monopolist will only provide units of quality θ in period two, by pricing in such a way that incentivizes consumers to only rent the new units. In contrast with previous literature, a monopolist seller, who faces a second hand market and can commit to future prices, is not able to attain the renter's profits.

In Section 2, we survey the literature. In Section 3, we solve for the subgame perfect equilibrium when the resale market is closed. In Section 4, we solve for the subgame perfect

equilibrium with an open resale market. Section 5 compares the two models and includes the analysis of rental and committment policies. Section 6 offers some brief concluding remarks and the proofs are given in the appendix.

II Literature Review

The durable goods monopoly literature began with Coase (1972), and was first formalized theoretically by Stokey (1979, 1981) and Bulow (1982). Stokey formalizes the Coase conjecture, that as the time between periods shrinks in an infinite horizon model, the monopoly produces the competitive quantity at the competitive price. Bulow (1982) constructs a two-period model. Although competitive outcomes do not obtain, there is a time inconsistency problem as indicated by the fact that renting yields higher profits than selling. Bulow (1982) also provides an example of "planned obsolescence" on which he elaborates in his (1986) paper. Obsolescence is modeled as a probability that the good disappears or breaks after period 1, like a lightbulb; all goods that do not disappear are perfect substitutes in period 2. Bulow (1986) finds that the monopolist might choose a good less durable than the socially optimal level, in contrast to what a monopolist with commitment power would choose.[1] Bulow assumes the existence of a resale market, but there is no reason for resale trade to occur if the same consumers demand services in both periods.[2]

Bond and Samuelson (1984) show that the Coase conjecture holds in an infinite horizon model with depreciation, in the sense that there is a stationary equilibrium yielding competitive outcomes when the time between periods approaches zero. New and undepreciated old output are perfect substitutes, so once again the pattern of transactions and the role of resale markets is not very interesting. Bond and Samuelson (1987) construct a nonstationary equilibrium yielding monopoly profits, again under the perfect substitutes assumption.[3]

---

[1] See Swan (1970, 1971) and Sieper and Swan (1973).

[2] This is because old and new goods are perfect substitutes in period 2. Bulow's motivation is to extend the applicability of the model to markets in which the good is durable but consumers leave the market, as in the market for baby carriages.

[3] See also Ausubel and Deneckere (1989) for an analysis of non-stationary reputational equilibria in a

Deneckere and Liang (2008) also consider depreciation of the perfect-substitutes, lightbulb variety, and demonstrate the existence of stationary equilibria ignored in the previous literature. They show that for fixed period length, no matter how short, the only stationary equilibrium yields monopoly profits when the depreciation rate is sufficiently high.

Several papers consider depreciation in which consumers must consume a single unit and an old, depreciated unit is not a perfect substitute for a new unit. This formulation, similar to our quality formulation, allows for nontrivial decisions of whether to buy new or buy used. Rust (1986) considers an infinite horizon model, with identical consumers, in which the monopolist commits to a price and durability that is constant over time. He shows that the equilibrium durability does not coincide with the efficient durability choice, and is typically too low, due to the fact that consumers decide when to scrap their old units. Anderson and Ginsburgh (1994) and Hendel and Lizzeri (1999) consider infinite horizon models with a continuum of consumer types. Goods last for two periods, and durability is modeled as the quality of the used good. Anderson and Ginsburgh (1994) show that the second hand market can be used as a scheme to price discriminate, but that for some parameters the monopolist may want to shut down the second hand market. Hendel and Lizzeri (1999) endogenize the durability, and show that the monopolist always weakly prefers to have an open second hand market. Importantly with regard to our paper, these papers assume that the monopolist can commit to a price path and avoid the time inconsistency problem.[4] Also, in their steady-state equilibria, a consumer will either buy new every period, buy used every period, or never buy. Our model finds a richer pattern of transactions.

Michael Waldman has written a series of papers modeling a depreciated good as an imperfect substitute for a new good, in a two-period setting in which the monopolist potentially faces a time inconsistency problem. He assumes two consumer types, which allows him

model without depreciation.

[4] The authors look at steady states, assuming initial conditions consistent with the steady state. They do not show convergence or otherwise study the model with the natural initial condition in which no one holds the product. Hendel and Lizzeri (1999) point to the possibility of reputational equilibria to avoid the time inconsistency problem, which is one of the reasons that we adopt a finite horizon model.

to isolate some interesting forces but creates a somewhat simplistic pattern of prices and transactions. Waldman (1993) assumes that a new product makes the old product worthless (perhaps based on network externalities), and shows that the monopolist will introduce new products too often. Interestingly, while Bulow (1986) uses planned obsolescence to counteract the time inconsistency problem, Waldman (1993) shows that obsolescence can be a manifestation of the time inconsistency problem. Waldman (1996) builds in an R&D decision, and shows that the monopolist has an incentive to overinvest in R&D that would create a new product and make the old product obsolete. Waldman (1997) models a used good as having a deteriorated quality, and shows that the monopolist can eliminate the time inconsistency problem by eliminating the secondhand market, either through long-term leases or by committing to repurchase prices.[5] Waldman (1996), shows that the monopolist has an incentive to produce goods that deteriorate too much.

Quality improvement is similar to depreciation when old goods are imperfect substitutes for new goods. Nonetheless, there are relatively few papers in the durable goods monopoly literature that specifically model quality improvement, and all assume either a representative consumer or two consumer types. Kumar (2002) assumes two consumer types, with an equal measure of consumers of each type. Rather than quality exogenously increasing over time, Kumar (2002) endogenizes the quality choice. In equilibrium without resale markets, the monopolist chooses a lower quality in period 2 than in period 1. The equilibrium without resale markets remains an equilibrium with resale markets, although there may be another equilibrium where quality increases. Inderst (2008) considers an infinite horizon model with a monopolist that offers a menu of price-quality options each period and two types of consumers who are only allowed to purchase once. Inderst (2008) shows that when the time between successive periods is small enough, the monopolist serves the entire market at the optimal quality for the low-valuation consumers. Anton and Biglaiser (2012) consider an infinite horizon model with one consumer type and network externalities. Quality exoge-

---

[5]Waldman's notion of eliminating the secondhand market is different from the notion of closing the secondhand market, as modeled in this paper and other papers in the literature.

nously improves over time, and the good is an "upgrade good" where buyers benefit from quality only if they hold a bundle of all previous versions. Anton and Biglaiser (2012) show the existence of multiple Markov perfect equilibria, ranging from the monopoly capturing all surplus to consumers retaining almost all the surplus. There are also cyclical equilibria in which consumers wait for multiple quality improvements before updating.

III Closed Resale Markets

We first consider the situation in which consumers cannot resell their old units. This could happen due to the presence of high transaction costs or some other feature of the product, but we abstract from the reasons here.

The Firm

The monopolist lives for two periods and produces a perfectly durable good in period one of quality θ_1 which we normalize to one. In period two the firm has the possibility to produce a good of quality θ_2, which is exogenously given and denoted by the parameter $\theta \in (1, 2)$. Quality is public information.

Denote by p_t the price that the firm sets in period t and by q_t the quantity sold by the firm.[6] The firm's profits are given by

$$\Pi = q_1 \cdot p_1 + q_2 \cdot p_2 \quad . \tag{1}$$

---

[6] At this time we wish to explain briefly what the choice variables of the firm will be and provide the reader with a simple intuition for the reasons behind our formulation. We write the problem as the firm choosing prices, but solve it with q_1 being the choice variable in period one. It is much more intuitive to treat the quantity sold in period one as the state variable in period two. Also, we are able to show that every subgame is characterized by a cutoff type such that all consumers with a valuation above it decide to buy.

Consumers

Consumers are differentiated according to their taste for quality, z, which is distributed uniformly on the interval $[0,1]$. The taste for quality is private information. Consumers are forward-looking, live for both periods, and their objective is to maximize lifetime utility. Finally, there is no depreciation of the good.[7]

Throughout our analysis, we assume that consumers derive utility from a single unit. A consumer who owns a good in period one and buys in period two is assumed to discard her old unit without cost. Whenever a consumer of type z purchases a new good in period t, their period t utility flow is given by $z\theta_t$ and the net value of consumption for a given period is given by $z\theta_t - p_t$. In the case in which the consumer already owns a good and decides not to purchase, her period utility flow is simply z. A non-owner's utility when choosing not to consume is normalized to 0.

The following chart contains the utility level associated to each course of action taken by the consumer:

Actions	Outcome
(B, B)	$z(1 + \theta) - (p_1 + p_2)$
(B, NB)	$2z - p_1$
(NB, B)	$z\theta - p_2$
(NB, NB)	0

Timing of the Game

The timing of the game is as follows. At the beginning of period one, the monopolist chooses p_1. Then consumers simultaneously decide whether to purchase in period one, denoted by an element of the set $\{B, NB\}$, where B stands for the action of buying and NB stands for the action of not buying. Next, the firm chooses p_2 and consumers simultaneously

---

[7]Depreciation and the fact that consumers can only derive utility from a single unit at a time would make old and new units imperfect substitutes. Quality improvement play a similar role in our model.

decide whether to purchase in period two.[8] The solution concept is subgame perfect Nash equilibrium.

Solving the Game

In principle, strategies can be very complicated. For example, a consumer's action in period two can depend on her type z, on whether she purchased in period one, on p_1, on q_1, and on p_2. However, we adopt the usual approach taken in IO models of this sort, and (i) assume that a deviation by a single consumer is not observed by the market, and (ii) only specify strategies on the equilibrium path and following a deviation by the monopolist. To allow for a solution using backward induction, we first show that in any equilibrium, whenever a consumer purchases in period one all consumers with higher valuations also purchase. For any p_1, we claim that there will exist a cutoff type, z_1, such that consumers with higher types purchase in period one and consumers with lower types do not purchase.

To formally enunciate this claim, we denote by $V^B(p_1, p_2; z)$ the value to a consumer who decides to purchase in period one, and by $V^{NB}(p_1, p_2; z)$ the value to a non-buyer when the firm has chosen an arbitrary p_1 and continuation strategies result in period two price of p_2. Recall that p_2 is unaffected by the consumer's action in period one.

LEMMA 1 For any arbitrary p_1 there exists z_1 such that $\forall z \geq z_1$ we have that $V^B(p_1, p_2; z) \geq V^{NB}(p_1, p_2; z)$ with equality when $z = z_1$.

The details of the proof can be found in Appendix 1. In the next subsection we will consider any arbitrary subgame induced by z_1, where this cutoff solves $V^B(p_1, p_2; z) - V^{NB}(p_1, p_2; z) = 0$. It helps to point out that given our uniform distribution of types we have that $q_1 = 1 - z_1$, which we will use throughout our analysis.

---

[8]We assume that the price in period 2 offered to a consumer cannot depend on whether or not the consumer purchased in period 1.

Period 2

Based on Lemma 1, and the fact that only z_1 (and not p_1) is relevant for period two behavior, consumers in period two are segmented into owners and non-owners. Consider a consumer type $z \in [z_1, 1]$. There exists a cutoff type z_2^B such that every $z \geq z_2^B$ decides to buy in period two if the consumer already owns a good. A current owner chooses to repeat a purchase if and only if $z\theta - p_2 \geq z$. Thus we define

$$z_2^B := \max \left\{ \frac{p_2}{\theta - 1}, z_1 \right\}$$

For the case in which a consumer does not own a good, $z \in [0, z_1]$, there exists a cutoff type z_2^{NB} such that a consumer type z decides to purchase in period two if and only if $z_2^{NB} < z < z_1$. This cutoff is given by

$$z_2^{NB} := \min \left\{ \frac{p_2}{\theta}, z_1 \right\}$$

Clearly, we have $z_2^{NB} \leq z_2^B$. However, four different cases can arise in period two given θ and the choice of p_2, because this will alter the location of the cutoffs z_2^{NB} and z_2^B with respect to z_1. We will denote them *regime A* through *regime D*.

In particular, we look at the first regime, in which $z_2^{NB} < z_1 < 1 < z_2^B$. Here no previous owner is discarding the unit purchased in period one and only new consumers of lower valuation are buying in period two; we call this scenario *regime A*. We have that $\frac{p_2}{\theta} \leq z_1$ and $1 \leq \frac{p_2}{\theta - 1}$ which occurs when $p_2 \in [1 - \theta, z_1\theta]$.

In regime B, there is a segment of previous owners who junk their old unit, while lower valuation owners keep their good and concurrently, a segment of non-owners acquires the good in period two. Regime C is the case in which there are no new consumers buying in period two, only a segment of the previous owners decides to acquire the good in period two. Finally, we consider regime D, in which all previous owners repeat purchase in period two

and a new segment of lower valuation consumers also buy the good.

The following table summarizes our regimes and expresses the pertinent price interval for each case which is calculated in the same way as we previously showed for regime A. We exclude the case in which there are no sales in period two, because it is never credible that the firm will abstain from selling after period one.

Regime	Case	p_2 Interval
A	$z_2^{NB} < z_1 < 1 < z_2^B$	$[\theta - 1, z_1\theta]$
B	$z_2^{NB} < z_1 < z_2^B < 1$	$[z_1(\theta - 1), \theta - 1]$
C	$z_1 < z_2^{NB} < z_2^B < 1$	$[z_1\theta, \theta - 1]$
D	$z_2^{NB} < z_2^B < z_1 < 1$	$[0, z_1(\theta - 1)]$

Notice that regimes A and C cannot occur for the same subgames: regime A is only possible if $z_1 > \frac{\theta-1}{\theta}$ and regime C is only possible if $z_1 < \frac{\theta-1}{\theta}$. Figure 1 shows clearly which regimes can arise for the two ranges of z_1 given the choice of p_2. For any given regime $j \in \{A, B, C, D\}$, the total quantity sold in period two as a function of p_2 for an arbitrary q_1 is given by

$$q_2^j(q_1) = (z_1 - z_2^{NB}) + (1 - z_2^B) = 2 - q_1 - z_2^{NB} - z_2^B \quad .$$

In period two, the firm's profits in regime j are given by

$$\Pi_2^j(p_2, q_1) = q_2^j(q_1) \cdot p_2 \quad .$$

For the sake of clarity of exposition we will briefly discuss our notation. The superscripts $\{A, B, C, D\}$ in the demand, price, and profit functions are used to denote those functions in each regime. We use "$**$" as a superscript for an interior optimum within a regime, and "$*$" denotes the global optimum given q_1. Whenever we obtain corner solutions, we will use an underbar to denote the lower bound for p_2, and an upper bar to denote the upper bound for p_2. Furthermore, we use the notation $\Pi_2^{j*}(q_1) := \Pi_2^j(p_2^{j*}(q_1), q_1)$ for the optimal period two profits in regime j.

No Sales

1) $z_1 > \dfrac{\theta - 1}{\theta}$

D | B | A

$(\theta - 1)z_1$ $(\theta - 1)$ θz_1 p_2

No Sales

2) $z_1 < \dfrac{\theta - 1}{\theta}$

D | B | C
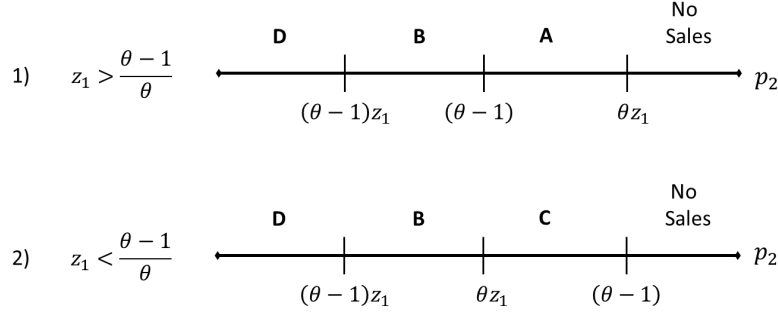
$(\theta - 1)z_1$ θz_1 $(\theta - 1)$ p_2

Figure 1: Period two Price and Corresponding Regimes

The following proposition determines the period two profit maximizing price for any subgame within each regime. The proof can be found in Appendix 2.

PROPOSITION 2 The optimal period two prices in each regime are given by:

Regime A Let $q_A(\theta) := \frac{2-\theta}{\theta}$. Then

$$p_2^{A*} := \begin{cases} p_2^{A**} := \frac{\theta}{2}(1 - q_1) & \text{when} \quad q_1 \in [0, q_A(\theta)] \\ \theta - 1 & \text{when} \quad q_1 \in \left[q_A(\theta), \frac{1}{\theta}, \right] \end{cases}.$$

Regime B Let $q_B^L(\theta) := \frac{2(\theta-1)}{3\theta-2}$ and $q_B^H(\theta) := \frac{2\theta}{3\theta-1}$. Then

$$p_2^{B*} = \begin{cases} \theta(1 - q_1) & \text{when} \quad q_1 > q_B^H(\theta) \\ p_2^{B**} = \frac{1}{2}\frac{(2-q_1)(\theta-1)\theta}{2\theta-1} & \text{when} \quad q_B^L(\theta) < q_1 < q_B^H(\theta) \\ (\theta - 1)(1 - q_1) & \text{when} \quad q_1 < q_B^L(\theta) \end{cases}.$$

Regime C Let $q_C(\theta) := \frac{1}{2}\frac{\theta+1}{\theta}$. Then

$$p_2^{C*} := \begin{cases} p_2^{C**} := \frac{\theta-1}{2} & \text{when} \quad q_1 \in [q_C(\theta), 1] \\ (1 - q_1)\theta & \text{when} \quad q_1 \in \left[\frac{1}{\theta}, q_C(\theta)\right] \end{cases}.$$

Regime D

$$p_2^{D*} := (\theta - 1)(1 - q_1) .$$

11

Notice that in regime B when the amount sold in the first period is large $(q_1 > q_B^H(\theta))$, the optimal period two price is such that $z_2^{NB} = z_1$: new consumers are priced out of the market. Selling to new consumers has an advantage because their outside option is zero. However if they have a very low valuation for quality, it might pay to only cater to higher valuation consumers even when the monopolist must discount the price to induce them to junk their old units.

When a low quantity has been sold in period one in regime B, that is $q_1 < q_B^L(\theta)$, the monopolist is better off by pricing in such a way that lower valuation consumers acquire the higher quality good in period two and previous owners keep their old unit. The reason is that in order to induce higher valuation consumers to dispose of their old units, the monopolist must discount heavily the price of the new good. Since this segment of owners is small, it pays to charge a price that excludes them from the market. Thus, charging such a higher price will leave high valuation consumers out of the market for the good of higher quality. For an intermediate range of sales in regime B, both old and new consumers will find it beneficial to buy in period two within this regime.

In regime D, all previous consumers and a portion of new lower valuation buyers acquire the new good. The monopolist never chooses an interior price: it is always the case that the firm prices exactly at the upper boundary for that regime, where z_2^B is exactly equal to z_1. The reason is that a decrease in price will attract lower valuation consumers, but will collect less revenues from the previous owners, which is the dominating effect.

Now that we have obtained the optimal prices within each regime (given q_1) we evaluate period two profits. For regime A we evaluate optimal profits to be

$$
\Pi_2^{A*}(q_1) = \left\{
\begin{array}{ll}
\frac{\theta}{4}(1 - q_1)^2 & \text{when} \quad q_1 \in [0, q_A(\theta)] \\[2mm]
\frac{(\theta-1)(1-q_1\theta)}{\theta} & \text{when} \quad q_1 \in \left[q_A(\theta), \frac{1}{\theta},\right]
\end{array}
\right\}.
$$

In regime B profits are given by

$$\Pi_2^{B*}(q_1) = \begin{cases} \frac{\theta(1-q_1)(q_1\theta-1)}{\theta-1} & \text{when} \quad q_1 > q_B^H(\theta) \\ \frac{\theta}{4}\frac{(q_1-2)^2(\theta-1)}{2\theta-1} & \text{when} \quad q_B^L(\theta) < q_1 < q_B^H(\theta) \\ \frac{(\theta-1)(1-q_1)(\theta q_1-q_1+1)}{\theta} & \text{when} \quad q_1 < q_B^L(\theta) \end{cases}.$$

For regime C we have the profit level is given by

$$\Pi_2^{C*}(q_1) = \begin{cases} \frac{1}{4}(\theta-1) & \text{when} \quad q_1 \in [q_C(\theta), 1] \\ \frac{\theta(1-q_1)(\theta q_1-1)}{\theta-1} & \text{when} \quad q_1 \in \left[\frac{1}{\theta}, q_C(\theta)\right] \end{cases}.$$

Finally, for regime D we obtain

$$\Pi_2^{D*}(q_1) = \frac{(\theta-1)(1-q_1)(\theta q_1 - q_1 + 1)}{\theta} \quad .$$

Profitability Comparison and Optimal Regimes

Now that we have computed all the profit levels we can compare across regimes to determine which are the most profitable, given q_1. In the appendix we provide a fully detailed analysis of how we obtain the optimal regimes in each area of the (q_1, θ_1) space. The logic is simply to compare the profit expression provided in the previous subsection.

In doing so, there are several boundaries that arise. For $q_1 > 1/\theta$ it turns out that the curve given by

$$q_{BC} := \frac{2\theta - \sqrt{2\theta^2 - \theta}}{\theta} \tag{2}$$

divides the space above $1/\theta$ into two areas: one in which regime B is optimal and the other one in which C is optimal. The curve

$$q_{ABD} := \begin{cases} \frac{1-\sqrt{(1+2\theta^2-3\theta)}}{\theta} & \text{when} \quad 1 \le \theta < \frac{6+2\sqrt{2}}{7} \\ \frac{\theta^2-4\theta+4}{5\theta^2-8\theta+4} & \text{when} \quad \frac{6+2\sqrt{2}}{7} \le \theta \le 2 \end{cases} \tag{3}$$

is the boundary for A with respect to regimes B and D, such that for q_1 values below this curve, regime A yields the highest profits . The following proposition specifies the optimal
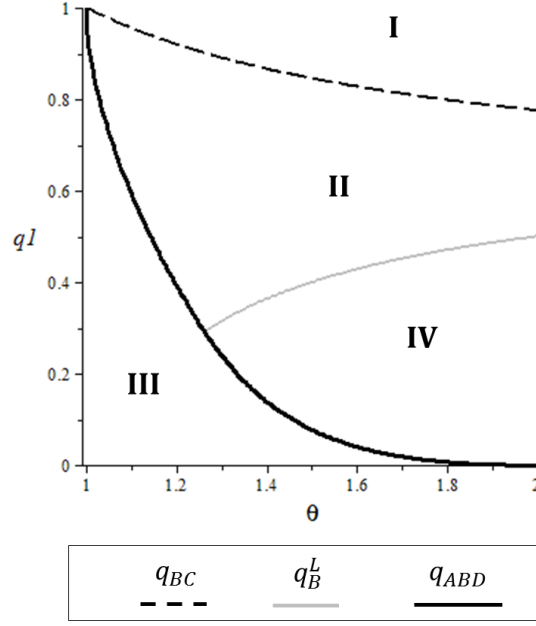
13

Figure 2: Prevailing Regime in each Subgame

regimes for each subgame, which is illustrated in figure 2.

PROPOSITION 3 The monopolist's optimal choices for each subgame are given by

Area I: p_2^{C**} for $q_1 \in [q_{BC}, 1]$.

Area II: p_2^{B**} for $q_1 \in [\max\{q_B^L, q_{ABD}\}, q_{BC}]$

Area III: p_2^{A**} for $q_1 \in [0, q_{ABD}]$

Area IV: $\bar{p}_2^D = (\theta - 1)(1 - q_1)$ for $q_1 \in [q_{ABD}, q_B^L]$

The proof can be found in Appendix 3. Notice that in every area, the solution is the unconstrained maximizer except for area IV where regime D's constrained price choice yields the highest profits.

Period One Quantity Choice

Having solved the period two subgame for each q_1 (equivalently, z_1), the monopolist in choosing p_1 takes into account the induced quantity q_1 and optimal choices it will make in

period two. The solution, expressed in terms of induced quantities, is given in Proposition 4.

PROPOSITION 4 The optimal choice of q_1 is given by

$$q_1^*(\theta) := \begin{cases} \frac{2(2-\theta)}{8-3\theta} & \text{when} & 1 \leq \theta \leq \frac{9-\sqrt{17}}{4} \\ \frac{1-\sqrt{(1+2\theta^2-3\theta)}}{\theta} & \text{when} & \frac{9-\sqrt{17}}{4} < \theta \leq \frac{6+2\sqrt{2}}{7} \\ \frac{\theta^2-4\theta+4}{5\theta^2-8\theta+4} & \text{when} & \frac{6+2\sqrt{2}}{7} < \theta < \theta_{AB} \\ \frac{2(2\theta-1)(\theta-2)}{7\theta^2-19\theta+8} & \text{when} & \theta_{AB} \leq \theta < \theta_{BD} \\ \frac{1}{2}\frac{\theta^2-2\theta+2}{\theta^2-\theta+1} & \text{when} & \theta_{BD} \leq \theta \leq 2 \end{cases} \tag{4}$$

where $\theta_{AB} \approx 1.348$, $\theta_{BD} \approx 1.469$.

The proof can be found in Appendix 4 were we compute the cutoffs[9] that compose the pieces of $q_1^*(\theta)$. The first three pieces of $q_1^*(\theta)$ correspond to q_1^{A*}: the firm is better off by selling the higher quality to lower valuation consumers . For $\theta \in [\theta_{AB}, \theta_{BD}]$ we have that the optimal quantity in period one is the one that induces regime B, and the last piece gives rise to regime D.

The function $q_1^*(\theta)$ is discontinuous at θ_{AB} and θ_{BD}. Figure 3 depicts the optimal period one quantity as it enters into the different regimes according to the different levels of θ. Profits are graphed in Figure 5.

IV The Model with an Open Resale Market

We open the resale market by introducing a trading stage after the period two price of the new good has been disclosed by the firm. The firm continues to choose p_1 and p_2, but consumers face a new set of possible actions in the second period: those who bought a unit in period one can meet with non-owners in a competitive resale market.[10]

---

[9]The cutoffs θ_{AB} and θ_{BD} are computed numerically.

[10]Although we will use some of the same notation as before, it is understood in this context that we are refering to the open resale case and not the closed resale model. Only when there can be confusion we will
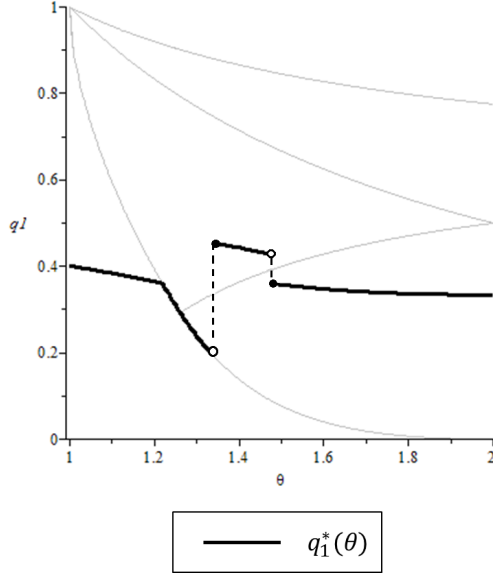
Figure 3: Optimal Period 1 Quantity (Closed Resale Market)

The timing of the game with a resale market is as follows. At the beginning of period one, the monopolist chooses p_1. Then consumers simultaneously decide whether to purchase in period one, denoted by an element of the set $\{B, NB\}$, where B stands for the action of buying and NB stands for the action of not buying. Next, the firm chooses p_2.[11] For convenience, we will assume that all of the period-one output is supplied to the resale market (since a consumer who wants to consume her old unit can always buy it back). In period two, consumers simultaneously decide either to purchase a new unit or to submit to the resale market the price below which they are willing to purchase, denoted by p_r^d. Thus, we can represent the set of period-two actions as $\{B\} \cup \{p_r^d : p_r^d \in [0,1]\}$. The resale market price, denoted by p_r, is determined by the market clearing condition which we will describe soon.

The overall utility of a consumer type z is determined from her actions and prices as follows:

---

use distinguishing marks.

[11] We assume that the price in period 2 offered to a consumer cannot depend on whether or not the consumer purchased in period 1.

16

Description	Actions in periods 1 and 2	Utility
Buy twice a new good	(B, B)	$z(1+\theta) - p_1 - p_2 + p_r$
Buy in period 1 and keep	(B, p_r^d) where $p_r^d \geq p_r$	$2z - p_1$
Buy in period 1 and sell	(B, p_r^d) where $p_r^d < p_r$	$z - p_1 + p_r$
Buy only in period 2	(NB, B)	$z\theta - p_2$
Buy only a used good	(NB, p_r^d) where $p_r^d \geq p_r$	$z - p_r$
Never buy	(NB, p_r^d) where $p_r^d < p_r$	0

Just as we proceeded with the closed resale model, we argue that each subgame in period two is characterized by a cutoff type, z_1, such that every consumer with a higher or equal valuation chooses to buy in period one. Before we formalize our claim, some additional notation will be needed.

A non-owner's period two outcome is given by $O^{NB} = \max\{z\theta - p_2 \, , \; z - p_r, \; 0\}$ assuming that the individual chooses a best response in the period two subgame. An owner's period two outcome is then $O^B = \max\{z\theta - p_2 + p_r, \; z \, , \; p_r\}$. Let $V^B(p_1, p_2; z)$ denote the lifetime utility of a consumer who chooses to buy in period one and $V^{NB}(p_1, p_2; z)$ for one who does not buy. We have that $V^B(p_1, p_2; z) = z - p_1 + O^B$ and $V^{NB}(p_1, p_2; z) = O^{NB}$.

In period one, the firm has chosen a price p_1 and consumers must weigh the benefits of owning a good in the present versus waiting and making a decision in the next period. We seek to characterize the set of types that choose to buy in period one instead of waiting, for which it is straightforward to see that $O^B - O^{NB} = p_r$.

PROPOSITION 5 There exists z_1 such that $\forall z \geq z_1$ we have that $V^B(p_1, p_2; z) \geq V^{NB}(p_1, p_2; z)$. Furthermore, $z_1 = p_1 - p_r$.

This is a mathematically simple result that has a heavy economic implication in our model because it states that all the relevant information that the consumer needs in order to make a present purchasing decision is summarized in the competitive resale price. The role of p_r in transmitting information is substantial because one might think that the consumer

takes into account future quality upgrade and the price of the new good in period two. This is true, but the all this information is embedded in the equilibrium resale price. (This result is consistent with Hendel and Lizzeri 1999)

Solving the Model with Open Resale Markets

Consider any subgame induced by a particular z_1 which uniquely determines a quantity of existing old units in period two. We proceed first to define the resale price that clears the market. Denote by $D_r(p_r)$ the Lebesgue measure of used good buyers which depends on the resale market bids (which in turn depend on q_1 and p_2). Recall that supply in the second hand market is given by q_1 and for notational purposes we denote it by S_r.

DEFINITION 6 The market clearing resale price, $p_r(q_1, p_2)$ is a solution to $S_r = D_r$. If multiple prices exist, we take the supremum of the set. If the set of used good buyers is not Lebesgue measurable then all transactions in the resale market are cancelled and the price is 0.

The previous definition confines our attention to an equilibrium with a well functioning resale market.

Period Two

We look at any subgame induced by the cutoff buying type in period one denoted by z_1 and any arbitrary choice of p_2. The decision in period two about buying a new good or reporting a demand price in the resale market occur simultaneously. We proceed first to determine the resale price in every particular subgame and then derive the demand function that the firm faces. Finally, within this subsection, we derive the optimal period two price set by the firm in period two.

Let z_2^B and z_2^{NB} denote the cutoff values in period two above which a period one owner and non-owner decide to buy the good of quality θ. A natural question to ask is if demand

Case 1: Decreasing Sales in Period 1 $(q_2 < q_1)$
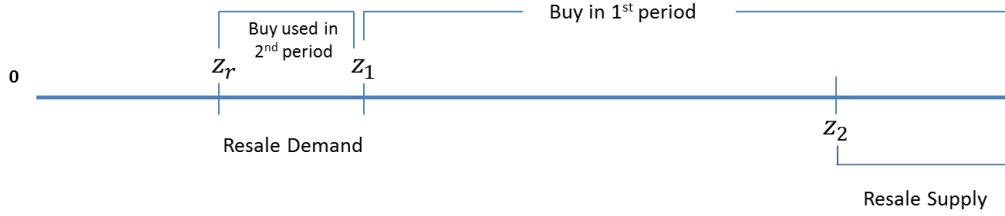
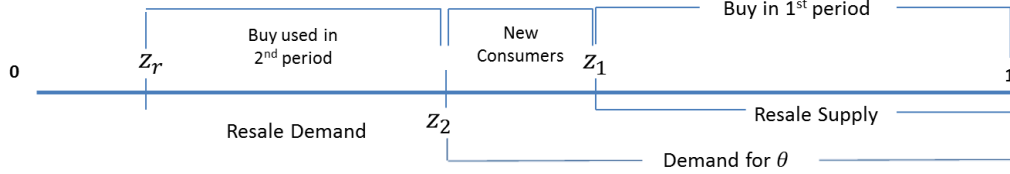**Case 2**: Increasing Sales in Period 2 $(q_2 > q_1)$

Figure 4: Possible Market Segmentations

in period two for the new good depends or not on whether the agent owns a unit bought in period one (in other words if $z_2^B = z_2^{NB}$). The following proposition shows that the cutoffs are the same. Recall that this is not the case for the model with a closed resale market.

PROPOSITION 7 In any subgame, given p_2 and the anticipated p_r we have that $z_2 := \frac{p_2 - p_r}{\theta - 1} = z_2^B = z_2^{NB}$.

Now we focus on consumers with valuations that are lower than z_2 and submit their willingness to pay for the used unit. Notice that regardless of whether an individual holds or not a used unit, the period flow of utility from consuming an old unit is $z - p_r$. It is straightforward to see that reporting one's valuation, $p_r^d = z$ is optimal since no individual report will alter the resale price and each consumer pays the equilibrium resale price, which is always less than or equal to her report.

Figure 4 provides a diagrammatic explanation of consumer type segmentations. Here, z_r represents the lowest consumer type willing to buy a used good. Thus given p_r, the quantity demanded of used units is given by $z_2 - z_r$. The condition for a consumer to buy a used good (given she is not buying a new good) is that $z - p_r \geq 0$ which determines the

19

cutoff type $z_r := p_r$. Hence, demand in the resale market is given by $D_r = z_2 - p_r$ and resale market supply. Recall that by assumption, every owner supplies her unit in the second hand market, thus it is fixed in every subgame.

In Proposition 5 we showed that when a consumer decides between buying a good in period one or waiting, the resale price summarizes all the relevant information for such decision. As Figure 4 evidences, two cases may arise according to whether the quantity sold by the firm increases or decreases over time. When $z_1 < z_2$, some consumers are transacting with themselves, in the sense that they act as sellers and buyers in the used goods market. When $z_1 \geq z_2$, every period one owner is selling her unit to a lower valuation consumer. The following proposition shows that the resale price function is invariant to whether $z_1 < z_2$ or $z_1 \geq z_2$ and provides explicitly such function.

PROPOSITION 8 Consider any subgame induced by z_1 and let p_2 be fixed but arbitrary. Then, there exists a unique function $p_r : (p_1, p_2; \theta) \longrightarrow \mathbb{R}^+$ that solves $D_r = S_r$. Furthermore,

$$ p_r(q_1, q_2) = \begin{cases} \frac{(1-\theta)q_1 + p_2}{\theta} & \text{if} \quad S_r \leq D_r \\ 0 & \text{if} \quad S_r > D_r \end{cases} . \tag{5}$$

In the previous literature, the resale price is exactly the second period price whenever there is no quality upgrade which is equivalent to saying that the goods available in period two are perfect substitutes. Notice that when $\theta = 1$ we have that $p_2 = p_r$.

We are now able to define the demand function for the new good. According to Propositions 7 and 8 we have that

$$ q_2(q_1, p_2) = \begin{cases} 1 - \frac{p_2 - p_r(q_1, p_2)}{\theta - 1} & \text{when} \quad 0 \leq z_2 \leq 1 \\ 0 & \text{when} \quad z_2 > 1 \\ 1 & \text{when} \quad z_2 < 0 \end{cases} . \tag{6}$$

The firm's period two profit function depends on whether $p_r > 0$ or $p_r = 0$. For the positive resale price case, which occurs when $p_2 \geq (\theta - 1)q_1$, the firm's profit function in

period two is given by

$$\Pi_2^+ = p_2 \left(\frac{\theta - q_1 - p_2}{\theta} \right) \tag{7}$$

and when the resale price is zero the firm's profits are

$$\Pi_2^- = p_2 \left(1 - \frac{p_2}{\theta - 1} \right) . \tag{8}$$

When q_1 is relatively large, $q_1 > \frac{\theta}{2\theta - 1}$, the firm finds it beneficial to choose $p_2^-(q_1; \theta) = \frac{\theta - 1}{2}$ which maximizes (8) and gives rise to a zero resale price. This happens because the offered $p_2^-(q_1, \theta)$ is low enough as to induce a large portion of previous owners to repeat their purchase which in turn generates a surplus of supply in the resale market. Notice that this can occur when the upgrade is relatively small and the amount of used units in the market is relatively high.

We find that in subgames such that $q_1 \leq \frac{\theta}{2\theta - 1}$ the firm is better off by choosing $p_2^+(q_1; \theta) = \frac{\theta - q_1}{2}$ which maximizes (7). When $\theta > 1/3$ it is always the case that choosing $p_2^+(q_1; \theta)$ yields higher profits. This means that when the quality of the new good is high enough, the firm is better off by charging a high price that curtails the quantity demanded of the new good and increases the quantity demanded of used units up to a point in which there is no surplus of supply in the resale market.

Period One

In the previous subsection we described the optimal strategies for a buyer in period two as well as the firm's optimal choice of p_2 both for the positive and zero resale price cases. As it turns out, the firm will always find it beneficial to sell an amount in period one that is low enough so that it never pays to induce a zero resale price. The reason is that the firm internalizes the negative effect that inducing old costumers to discard or give away their used units has on their willingness to pay for both period one and period two units. In this section we focus on the equilibrium path (when $p_r > 0$), and the appendix contains the details that

21

show why inducing a zero resale price is never optimal.

Demand in period one is derived from Proposition 5 which implies that $q_1 = 1 - p_1 + p_r(q_1, p_2)$. After substituting in for $p_r(q_1, p_2)$ from equation (5) and solving for p_1 we obtain the inverse demand function to be

$$p_1(q_1) = \frac{3\theta + q_1 - 4q_1\theta}{2\theta} \quad . \tag{9}$$

We solve for the inverse demand function in period one so that the firm's total profit function can be written in terms of q_1. The monopolist's total profit function is then given by

$$\Pi^+ = q_1 \left(\frac{3\theta + q_1 - 4q_1\theta}{2\theta} \right) + \frac{(\theta - q_1)^2}{4\theta} \quad . \tag{10}$$

The following proposition characterizes the equilibrium choices by the firm.

PROPOSITION 9 In equilibrium, the firm's choices, profits, the resale price, and the resale quantity (q_r^*) are given by

$$
\begin{aligned}
q_1^*(\theta) &= \frac{2\theta}{8\theta - 3} \\
p_1^*(\theta) &= \frac{1}{2} \left(\frac{16\theta - 7}{8\theta - 3} \right) \\
q_2^*(\theta) &= \frac{1}{2} \left(\frac{8\theta - 5}{8\theta - 3} \right) \\
p_2^*(\theta) &= \frac{\theta}{2} \left(\frac{8\theta - 5}{8\theta - 3} \right) \\
\Pi^* &= \frac{\theta}{4} \left(\frac{8\theta + 1}{8\theta - 3} \right) \\
p_r^*(\theta) &= \frac{1}{2} \frac{4\theta - 1}{8\theta - 3} \\
q_r^*(\theta) &= \begin{cases} q_1^*(\theta) & \text{if} \quad \theta \leq 5/4 \\ q_2^*(\theta) & \text{if} \quad \theta > 5/4 \end{cases}
\end{aligned}
$$

Equilibrium Analysis in Open Resale Markets

Here we discuss various features of the equilibrium when the resale market is operating: demand behavior over time, pricing dynamics, and the resale market. When solving the model, we mentioned that there could be two possible market segmentations, depending on whether the quantity sold by the firm grew or shrank over time. We find that period one sales are decreasing in θ and period two sales are increasing in θ. We have that $q_1^* > q_2^*$ if and only if $\theta < 5/4$.

In a steady state of an infinite horizon model (as in Hendel and Lizzeri (1999) and Anderson and Gingsburgh (1994)) it would be required that $q_1^* = q_2^*$ and that initial holdings of the good by consumers is indeed the steady state. We find that in the absence of commitment possibilities by the monopolist in a finite horizon, the equilibrium of the game does not entail the same group of consumers buying from the firm in both periods. The fact that consumers start without holding the good, allows the present model to analyze their strategic choice of buying in period one versus waiting for period two to make their purchase decision.

The following corollary summarizes the comparative static results for the pricing scheme by the monopolist.

COROLLARY 10 The following results hold in equilibrium:

1. Period one price is increasing in θ: $\frac{\mathrm{d}p_1^*}{\mathrm{d}\theta} = \frac{4}{(8\theta-3)^2} > 0$

2. Period two price is increasing in θ: $\frac{\mathrm{d}p_2^*}{\mathrm{d}\theta} = \frac{64\theta^2 - 48\theta + 15}{2(8\theta-3)^2} > 0$

3. $p_2^* < p_1^*$ and $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{p_2^*}{p_1^*}\right) > 0$

A striking result is that p_1^* increases as θ gets larger. Notice that an increase in θ makes the good in period two more attractive relative to the period one good. It seems contradictory that the monopolist would raise p_1, but the logic is that this would reduce the supply in the resale market (by reducing the amount of owners) and this way p_r would not

fall as much. Although this yields a loss of revenues in the first period, the period two price increases more than proportionally to the fall in p_1 and overall revenues increase.[12]

The volume of transactions in the resale market (q_r^*) is not monotonic in θ even though the resale price is falling. For a range of lower qualities, $\theta < 5/4$, the equilibrium amount of units sold in the resale market is increasing and reaches a maximum at $\theta = 5/4$, exactly where $q_1^* = q_2^*$. At this quality level, only 6.6% of period one consumers retain their old units.[13]

COROLLARY 11 The following results hold in equilibrium in the resale market:

1. The resale price is falling in θ: $\frac{\mathrm{d}p_r^*}{\mathrm{d}\theta} = -\frac{2}{(8\theta-3)^2} < 0$

2. The quantity sold in the second hand market is not monotonic in θ: For $\theta \in (1, 5/4]$ we have that $\frac{\mathrm{d}q_r^*}{\mathrm{d}\theta} > 0$ and for $\theta \in (5/4, 2]$ we have that $\frac{\mathrm{d}q_r^*}{\mathrm{d}\theta} < 0$

Our findings reflect that even in the absence of physical depreciation, the used unit loses value when higher qualities are available in period two due to the relative desirability of the new product and the firm's optimal pricing.

V Comparison of Profits

A stylized result from previous models with constant quality of the durable good over time is that the game with a rental market each period and the game in which the monopolist can commit to future prices (both with open and closed resale markets) are equivalent. This equivalence is not true in our setting. In this subsection we first look at the open and closed resale market games and compare the firm's profitability. Second, we look at what a renter can achieve. Third, we contrast the renter and the perfectly committed monopolist.

---

[12]It can be shown that $\mathrm{d}\Pi_1^*(\theta)/\mathrm{d}\theta < 0$ and that $\mathrm{d}\Pi_2^*(\theta)/\mathrm{d}\theta > 0$. Further more $|\mathrm{d}\Pi_1^*(\theta)/\mathrm{d}\theta| < \mathrm{d}\Pi_2^*(\theta)/\mathrm{d}\theta$.

[13]Total amount of buyers in the resale market is $1/3$ and $q_1^*(5/4) = 5/14$. Here were are referring to new buyers, this means we exclude previous owners who repurchase their old unit from the market.
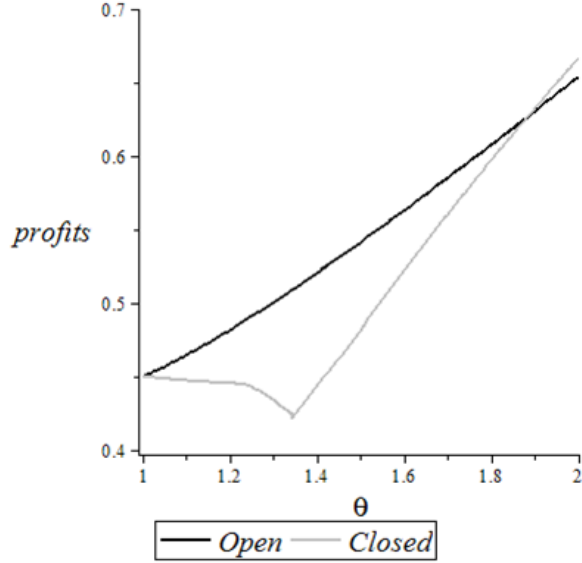
Figure 5: Profits of the Firm with Closed and Open Resale Markets

The resale market plays two roles that have opposing effects on the firm's profits. On the one hand, access to reselling an old unit can help increase the consumer's willingness to pay for a new product. On the other hand, the resale market provides access to a substitute good that induces competition with the new unit thus limiting the firm's market power. Waldman (1996) argues that the substitution effect across vintages of the used good and the presence of consumers with different tastes for quality would interfere with the firm's profits. Waldman (1996) explains that a monopolist is better off by shutting the second hand market by reducing the durability of the good to one period.[14]

Our findings below suggest that resale markets can harm or help the monopolist depending on how high is the relative improvement of the product. There exists a quality level in period two such that the aforementioned effects of the resale market balance each other.

PROPOSITION 12 The profits under an open resale market are higher than when the resale market is closed if and only if $\theta \in \left[1, \frac{7+\sqrt{65}}{8}\right]$.

The proof can be found in Appendix 5. For $\theta \in \left[\frac{7+\sqrt{65}}{8}, 2\right]$ we have that a corner

_____

[14]In other words the good is no longer durable. This result holds with quality differentiated goods and a "small" amount of low valuation consumers.

solution for the period two price in regime D is in place: all period one owners are junking their old unit and buying a new good and a segment of non-owners also acquire the good. The monopolist would wish (yet the lack of committment incentives make this action not credible) to charge a higher price and exclude a segment of owners, however this would set the market in regime B, which yields lower profits.

Selling with Commitment vs. Renting

As a benchmark, we look at the renter's problem when the firm can rent in period two both products (those of quality 1 and θ simultaneously), as to mimic the possible incentive to keep old units in circulation just as an open resale market would allow.[15] The firm's optimal choice in period two is to price as the static monopoly would do, and this implies that, in equilibrium, only new units will be rented in period two. Period one units will be priced in period two in such a way that no one will prefer them over a new unit.

The intuition behind this rental price scheme is that by allowing old units in the market, the firm introduces competition to its new unit. Charging a lower price so that lower valuation consumers are enticed to rent an old unit in period two has the countervailing incentive of making some higher valuation consumers switch to renting an old unit. It turns out the the revenues lost from those who switch to the old unit are are higher than the gain in revenues from renting to lower valuation consumers a period one unit. This result is equivalent to saying that the renter firm kills the market for used units through its pricing policy. In equilibrium, the renter monopolist's profits are given by

$$\Pi^{\text{Rent}} = \frac{1+\theta}{4} \quad . \tag{11}$$

---

[15] All the calculations for the renter and commmitted monopolist can be found in Appendix 7.

The problem that a seller with commitment power solves is given by

$$\max_{q_1, p_2} \quad q_1 \left[1 - q_1 + p_r(q_1, p_2) \right] + p_2 \left[1 - \frac{p_2 - p_r(q_1, p_2)}{\theta - 1} \right] \quad . \tag{12}$$

What we find is that a seller with commitment power serves a smaller quantity of consumers and charges a higher price in both periods than the monopolist who cannot commit. Furthermore, the resale price that arises with commitment is always greater than p_r^*. This occurs through two channels: by restricting resale supply and increasing resale demand. First, resale supply is restricted given the higher period-one price and the resulting lower q_1 served. Second, a higher p_2 makes the used unit attractive for a larger segment which increases resale demand.

Equilibrium profits for the committed monopolist are given by

$$\Pi^{\text{Commit}} = \frac{\theta^2}{2(2\theta - 1)} \quad . \tag{13}$$

A stylized conclusion in the literature (see Bulow (1982,1986)), is that a rental policy will achieve the same profits as a seller with commitment. When $\theta = 1$ we have that $\Pi^{\text{Rent}}(1) = \Pi^{\text{Commit}}(1)$. The following proposition shows that this does not hold when the monopolist upgrades the quality ($\theta > 1$) of the good over time.

PROPOSITION 13 The following inequalities hold in equilibrium for $\theta \in (1, 2)$: $\Pi^{\text{Rent}} > \Pi^{\text{Commit}} > \Pi^*$.

VI Conclusion

Our model is useful in explaining the dynamics that arise when a firm introduces a new durable good in the market and faces no competition. This could be due to the presence of patents or simply a meaningful lag in competitors to launch similar substitute products.

Previous models have claimed that the possibility to resell older units always helps the monopolist by increasing its profits, yet we find that this is not a general result in the presence of commitment problems **and** quality improvements. We also find that when the resale market is closed, the availability of a better product in the future can actually harm the monopolist when the improvement on the good is not very high. This represents a new feature of the time inconsistency problem which had not been addressed in the literature.

Competitive resale markets present interesting dynamics. An intuitive result is that the resale price falls as the quality of the period two good increases, even in the absence of physical depreciation. However, the volume of transactions is not monotonic: trading of old units increases and then falls as θ increases.

The ability to commit to future prices is not enough for the monopolist to obtain the profits that can be attained through a rental policy. The monopolist has complete control of the output at each period in time when renting instead of selling and optimally chooses to not circulate old units in period two. The firm is able to charge a price in each period that a static monopoly seller firm of a non-durable good would charge which yields the highest profits.

The complex nature of transactions in both the closed and open resale market games makes solving the model tantamount to solving each possible example separately. In this sense, assuming a stationary economy that has placed consumers and the firm in a steady state is very convenient in order to provide general solutions. However, this comes at the expense of being able to document the rich dynamics that arise with an initial product introduction. In a similar setting as ours, Hendel and Lizzeri (1999) have claimed that the possibility to resell older units always increases the monopolist's profits but we find that this is not a general result in a non-stationary environment.

References

Anderson, S. P., Ginsburgh, V. A., 1994. Price discrimination via second-hand markets. European Economic Review 38 (1), 23–44.

Anton, J. J., Biglaiser, G., 2010. Compatibility, interoperability, and market power in upgrade markets. Economics of Innovation and New Technology 19 (4), 373–385.

Bulow, J., 1986. An economic theory of planned obsolescence. The Quarterly Journal of Economics 101 (4), 729–749.

Bulow, J. I., 1982. Durable goods monopolists. The Journal of Political Economy, 314–332.

Carlton, D. W., Waldman, M., 2010. Competition, monopoly, and aftermarkets. Journal of Law, Economics, and Organization 26 (1), 54–91.

Cason, T. N., Sharma, T., 2001. Durable goods, coasian dynamics, and uncertainty: Theory and experiments. Journal of Political Economy 109 (6), 1311–1354.

Coase, R. H., 1972. Durability and monopoly. Journal of Law and Economics 15, 143.

Deneckere, R., Liang, M.-Y., 2008. Imperfect durability and the coase conjecture. The RAND Journal of Economics 39 (1), 1–19.

Hendel, I., Lizzeri, A., 1999. Interfering with secondary markets. The Rand Journal of Economics, 1–21.

Inderst, R., 2008. Durable goods with quality differentiation. Economics Letters 100 (2), 173–177.

Kumar, P., 2002. Price and quality discrimination in durable goods monopoly with resale trading. International Journal of Industrial Organization 20 (9), 1313–1339.

Morita, H., Waldman, M., 2004. Durable goods, monopoly maintenance, and time inconsistency. Journal of Economics & Management Strategy 13 (2), 273–302.

Reynolds, S. S., 2000. Durable-goods monopoly: laboratory market and bargaining experiments. The RAND Journal of Economics, 375–394.

Rust, J., 1986. When is it optimal to kill off the market for used durable goods? Econometrica: Journal of the Econometric Society, 65–86.

Smith, V. L., 1981. An empirical study of decentralized institutions of monopoly restraint. Essays in Contemporary Fields of Economics: In Honor of Emanuel T. Weiler (1914-1979). Purdue University Press, West Lafayette, IN, 83–106.

Stokey, N. L., 1981. Rational expectations and durable goods pricing. The Bell Journal of Economics, 112–128.

Swan, P. L., 1970. Durability of consumption goods. The American Economic Review 60 (5), 884–894.

Waldman, M., 1993. A new perspective on planned obsolescence. The Quarterly Journal of Economics 108 (1), 273–283.

Waldman, M., 1996a. Durable goods pricing when quality matters. Journal of Business, 489–510.

Waldman, M., 1996b. Planned obsolescence and the r&d decision. The RAND journal of Economics, 583–595.

Waldman, M., 2003. Durable goods theory for real world markets. The Journal of Economic Perspectives 17 (1), 131–154.

Appendix 1: Proof of Proposition 1

Observe that a period one owner decides to repeat a purchase if and only if $z\theta_2 - p_2 > z$ and that a player who has not consumed in period one will consume in period two if and only if $z\theta - p_2 > 0$. We then obtain the following value functions

$$V^B := z - p_1 + \begin{cases} z & \text{if } \frac{p_2}{\theta-1} \geq z \\ z\theta - p_2 & \text{if } \frac{p_2}{\theta-1} \leq z \end{cases}$$

$$V^{NB} := \begin{cases} 0 & \text{if } \frac{p_2}{\theta} \geq z \\ z\theta - p_2 & \text{if } \frac{p_2}{\theta} \leq z \end{cases}$$

in order to compute

$$V^B - V^{NB} = z - p_1 + \begin{cases} 0 & \text{if } \frac{p_2}{\theta-1} \leq z \\ z(1-\theta) + p_2 & \text{if } z \in [\frac{p_2}{\theta}, \frac{p_2}{\theta-1}] \\ z & \text{if } \frac{p_2}{\theta} \geq z \end{cases} . \tag{14}$$

We clearly observe that the difference is increasing in z. In particular, we examine $z \in [\frac{p_2}{\theta}, \frac{p_2}{\theta-1}]$ where $V^B - V^{NB} = z(2-\theta) - p_1 + p_2$, and since $\theta \in (1,2)$, that difference is also increasing in z.

Appendix 2: Closed Resale: Optimal Period 2 Pricing and Proof of Theorem 2

We proceed to solve for the optimal price in each regime given an arbitrary q_1.

Regime A

In this regime only new buyers acquire the good produced in period two because previous owners are priced out of the market: $z_2^{NB} < z_1 < 1 < z_2^B$. The price in period two for this regime to hold must be in the interval $[\theta - 1, z_1\theta]$, and for this interval to be non-empty the quantity sold in period one must satisfy $q_1 < 1/\theta$.

Demand is given by

$$q_2^A = 1 - q_1 - \frac{p_2}{\theta} \quad .$$

The firm maximizes

$$\Pi_2^A = \left(1 - q_1 - \frac{p_2}{\theta}\right) p_2 \quad . \tag{15}$$

The solution to the unconstrained maximization problem is given by

$$p_2^{A**} = \frac{(1 - q_1)\,\theta}{2} \quad .$$

We find that $p_2^{A**} < (1 - q_1)\theta$ if and only if $q_1 < q_A(\theta)$ where $q_A(\theta) := \frac{2-\theta}{\theta}$. Notice that $q_A(\theta) < \frac{1}{\theta}$. Hence we have to analyze which constraint binds for $q_1 \in [\,q_A(\theta), 1/\theta]$. We proceed to inspect the marginal profits at the boundaries:

$$\left.\frac{\partial \Pi_2^A}{\partial p_2}\right|_{p_2 = (\theta - 1)} = \frac{2 - \theta(1 + q_1)}{\theta}$$

$$\left.\frac{\partial \Pi_2^A}{\partial p_2}\right|_{p_2 = (\theta - 1)(1 - q_1)} = q_1 - 1 < 0 \quad .$$

One can verify that $\left.\frac{\partial \Pi_2^A}{\partial p_2}\right|_{p_2 = (\theta - 1)} < 0$ whenever $\frac{2-\theta}{\theta} < q_1$. It follows that the optimal period two price in regime A for $q_1 \in [\,q_A(\theta), 1/\theta]$ is given by $p_2^A = \theta - 1$. For $q_1 < q_A(\theta)$ we have an interior solution.

Regime B

In period two, demand stems from two segments: previous owners and non-owners that can buy if the price is suitable. For an arbitrary z_1, the measure of previous owners that will buy is given by $1 - z_2^B = 1 - \frac{p_2}{\theta - 1}$. The proportion of non-owners that will purchase is $z_1 - z_2^{NB} = z_1 - \frac{p_2}{\theta}$. Total demand in period two is then given by

$$q_2^B(q_1, p_2; \theta) = 2 - q_1 - \frac{p_2}{\theta} - \frac{p_2}{\theta - 1} \quad . \tag{16}$$

Using (16) the firm's unconstrained profit maximization problem is given by

$$\max_{p_2} \Pi_2^B = \left(2 - q_1 - \frac{p_2}{\theta} - \frac{p_2}{\theta - 1} \right) p_2 \quad . \tag{17}$$

Taking the derivative of (17) with respect to p_2 and solving for the optimal unconstrained interior price we have that

$$p_2^{B**} = \frac{1}{2} \frac{(2 - q_1)(\theta - 1)\theta}{2\theta - 1} \quad . \tag{18}$$

See Figure 6. We will inspect the optimal period two price in this regime for the two cases: $q_1 < 1/\theta$ and $q_1 > 1/\theta$.

Case 1. Consider the case in which $q_1 < 1/\theta$ or equivalently, $z_1 > \frac{\theta - 1}{\theta}$. For regime B to be in place, it must be that $p_2 \in [(\theta - 1)z_1, \theta - 1]$. We proceed to inspect if the interior solution to this case is within the boundaries of the regime. From (18), $p_2^{B**} > (\theta - 1)z_1$ if and only if $1/\theta > q_1 > \frac{2(\theta - 1)}{3\theta - 2}$. From now on we define $q_B^L(\theta) := \frac{2(\theta - 1)}{3\theta - 2}$. For the upper bound, we obtain that $p_2^{B**} < \theta - 1$ if and only if $q_1 > 2(1 - \theta)/\theta$, which is always the case since $2(1 - \theta)/\theta < 0$.
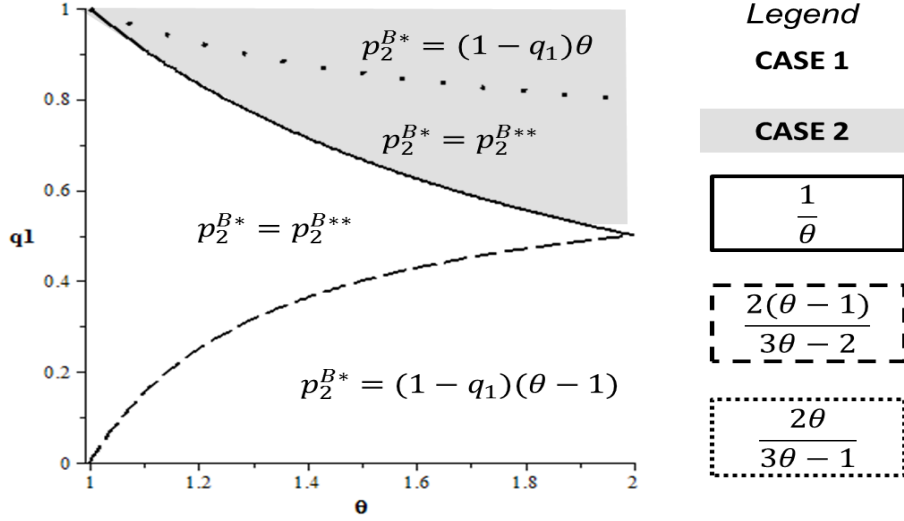
Figure 6: Optimal Period 2 Pricing in Regime B

We compute the derivative of profits with respect to p_2 and evaluate it at both corners:

$$\left.\frac{\partial \Pi_2^B}{\partial p_2}\right|_{p_2=(\theta-1)} = -\frac{q_1 + 2(\theta - 1)}{\theta} \tag{19}$$

$$\left.\frac{\partial \Pi_2^B}{\partial p_2}\right|_{p_2=(\theta-1)(1-q_1)} = \frac{q_1(3\theta - 2) - 2(\theta - 1)}{\theta} \quad . \tag{20}$$

The fact that the right hand side of equation (19) is negative indicates that for any quantity sold in period one, setting $p_2 = (\theta - 1)$ is never optimal. Next, notice that the right hand side of equation (20) positive if and only if $q_1 > q_B^L(\theta)$. Hence, for $0 < q_1 < q_B^L(\theta)$ it is optimal to set $p_2 = (\theta - 1)(1 - q_1)$.

Case 2. Let $q_1 > 1/\theta$. For regime B to occur it must be that $p_2 \in [(\theta - 1)z_1, \theta z_1]$. We verify that $p_2^{B**} < \theta(1 - q_1)$ if and only if $q_1 < \frac{2\theta}{3\theta-1}$ From now on, we define $q_B^H(\theta) := \frac{2\theta}{3\theta-1}$. Notice that $q_B^H(\theta) > \frac{1}{\theta}$. Following the same reasoning as in case 1, we compute

$$\left.\frac{\partial \Pi_2^B}{\partial p_2}\right|_{p_2=(\theta-1)q_1} = \frac{q_1(3\theta - 1) - 2\theta}{\theta - 1} \tag{21}$$

and verify that the right hand side of equation (21) is positive whenever $q_B^H(\theta) < q_1$. Hence, we have that for this range of q_1 the optimal price in period two is $\theta(1 - q_1)$.

Regime C

In this regime we have that $z_1 < z_2^{NB} < z_2^B < 1$. The period two price must be in $[z_1\theta, \theta - 1]$ and for this interval to be non-empty we require that $q_1 > 1/\theta$. Demand is given by

$$q_2^C := 1 - \frac{p_2}{\theta - 1}$$

and profits are

$$\Pi_2^C := \left(1 - \frac{p_2}{\theta - 1}\right) p_2 \quad . \tag{22}$$

Taking first order conditions and solving for the unconstrained optimal price we obtain

$$p_2^{C**} := \frac{\theta - 1}{2} \quad .$$

We verify that $p_2^{C**} < \theta - 1$. For the lower boundary, we have that $p_2^{C**} > (1 - q_1)\theta$ if and only if $q_1 > \frac{1}{2}\frac{\theta+1}{\theta}$. We define $q_C(\theta) := \frac{1}{2}\frac{\theta+1}{\theta}$. Notice that $q_C(\theta) > \frac{1}{\theta}$ for all $\theta > 1$. Hence for $q_1 \in [q_C(\theta), 1]$ the optimal period two price in regime C is given by $p_2^{C*} = p_2^{C**}$.

Now we analyze the subgames in which $q_1 \in \left[\frac{1}{\theta}, q_C(\theta)\right]$. We look at marginal profits evaluated at the lower boundary:

$$\left.\frac{\partial \Pi_2^C}{\partial p_2}\right|_{p_2=(\theta-1)(1-q_1)} = \frac{2\theta q_1 - \theta - 1}{\theta - 1} \quad . \tag{23}$$

We verify that the right hand side of equation (23) is negative whenever $q_1 < \frac{1}{2}\frac{\theta+1}{\theta}$ and conclude that the optimal period two price in this range of q_1 is given by $(1 - q_1)\theta$.

Regime D

In regime D all previous owners junk their old units and a new segment may acquire the new good (if the price is interior). This regime is characterized by the following ordering of the cutoffs for buying in period two with respect to the period one cutoff consumer type :

$z_2^{NB} < z_2^B < z_1^* < 1$. For this regime to be in place, the price in period two must be selected from $[0, z_1(\theta - 1)]$.

Demand is given by

$$q_2^D = 1 - \frac{p_2}{\theta}$$

and the firm's objective is to maximize profits given by

$$\Pi_2^D := \left(1 - \frac{p_2}{\theta}\right) p_2 \quad . \tag{24}$$

Taking first order conditions with respect to p_2 and solving for the unconstrained optimal price we obtain that

$$p_2^{D**} = \frac{\theta}{2} \quad .$$

We verify that $\frac{\theta}{2} < (1 - q_1)(\theta - 1)$ if and only if $q_1 < \frac{1}{2}\frac{\theta - 2}{\theta - 1}$, which means that the interior solution only holds for negative q_1 and we conclude that the solution to the profit maximizing problem is at one of the boundaries.

We proceed to evaluate the derivative of equation (24) at both corners:

$$\left. \frac{\partial \Pi_2^D}{\partial p_2} \right|_{p_2 = (\theta - 1)} = 1 \tag{25}$$

$$\left. \frac{\partial \Pi_2^D}{\partial p_2} \right|_{p_2 = (\theta - 1)(1 - q_1)} = \frac{2q_1(\theta - 1) + 2 - \theta}{\theta} \quad . \tag{26}$$

Clearly, choosing $p_2 = 0$ is not optimal. Then, we verify that the right hand side of expression (26) is positive if and only if $\frac{1}{2}\frac{\theta - 2}{\theta - 1} < q_1$. We conclude that the optimal price in regime D is given by $p_2^{D*} := (\theta - 1)(1 - q_1)$.

Appendix 3: Closed Resale: Period Two Profits Comparison for All Subgames

We have two sections in this appendix according to whether $q_1 > 1/\theta$ or $q_1 < 1/\theta$.

Case: $q_1 > 1/\theta$

The following chart summarizes the subcases we consider for $q_1 > 1/\theta$ and which regimes are valid for in the subgame for comparison of profits.

	Subgame Region of q_1	Profits to Compare
1	$q_B^H(\theta) < q_1$	$\overline{\Pi}_2^B, \Pi_2^{C**}, \overline{\Pi}_2^D$
2	$q_C(\theta) < q_1 < q_B^H(\theta)$	$\Pi_2^{B**}, \Pi_2^{C**}, \overline{\Pi}_2^D$
3	$\frac{1}{\theta} < q_1 < q_C(\theta)$	$\Pi_2^{B**}, \underline{\Pi}_2^C, \overline{\Pi}_2^D$

Let us analyze the region where $q_B^H(\theta) < q_1$. We need to rank $\overline{\Pi}_2^B, \Pi_2^{C**}$, and $\overline{\Pi}_2^D$. We first compute $\Pi_2^{C**} - \overline{\Pi}_2^B = \frac{1}{4}\frac{(2q_1\theta - \theta - 1)^2}{\theta - 1}$ and observe that this difference is always positive. Now we examine $\overline{\Pi}_2^B - \overline{\Pi}_2^D$ which is given by:

$$\frac{(1 - q_1)(-2\theta^2 + 3q_1\theta^2 + 2\theta - 3q_1\theta - 1 + q_1)}{\theta(\theta - 1)} \quad . \tag{27}$$

Notice that expression (27) is a quadratic polynomial in q_1 with a negative quadratic term. We find that for $q_1 \in \left[\frac{2\theta^2 - 2\theta + 1}{3\theta^2 - 3\theta + 1}, 1\right]$ the difference $\overline{\Pi}_2^B - \overline{\Pi}_2^D$ is positive, and that

$$\frac{2\theta^2 - 2\theta + 1}{3\theta^2 - 3\theta + 1} \quad < \quad q_B^H(\theta) = \frac{2\theta}{3\theta - 1}$$

$$\iff \quad 0 < (2\theta - 1)(\theta - 1) \quad .$$

Hence we can conclude that $\overline{\Pi}_2^B - \overline{\Pi}_2^D$ is always positive for $q_B^H(\theta) < q_1$.

We turn to the case in which $q_C(\theta) < q_1 < q_B^H(\theta)$. First notice that $\Pi_2^{B**} > \max\{\underline{\Pi}_2^B, \overline{\Pi}_2^B\}$ which in turn implies that $\Pi_2^{B**} > \overline{\Pi}_2^D$ because $\underline{\Pi}_2^B = \overline{\Pi}_2^D$. We need to compare Π_2^{B**} with
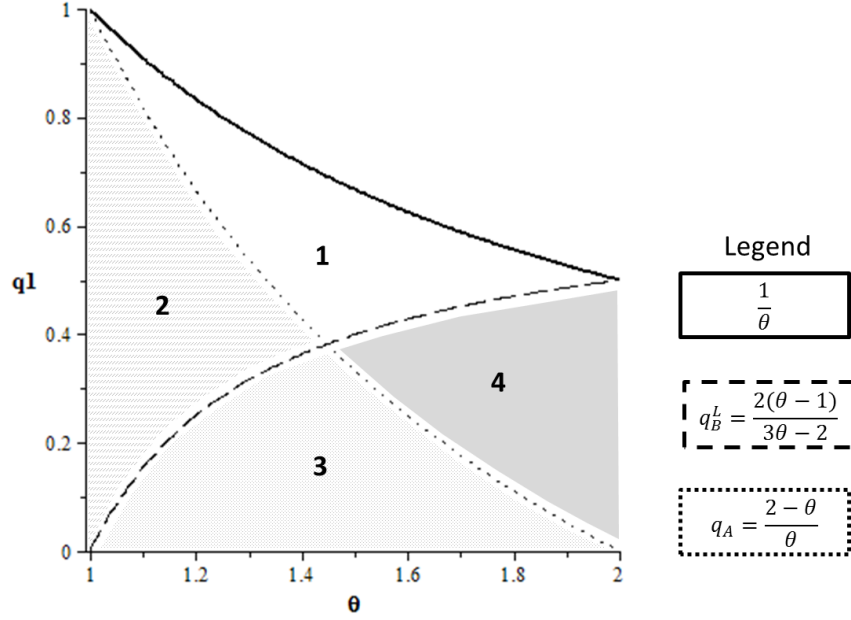
Figure 7: Subgame Regions for $q_1 < 1/\theta$

Π_2^{C**}.

$$\Pi_2^{B**} - \Pi_2^{C**} = \frac{(\theta-1)}{4} \frac{\left(q_1^2 \theta + 2\theta - 4q_1 \theta + 1\right)}{2\theta - 1}$$

We find that $\Pi_2^{B**} - \Pi_2^{C**} > 0$ whenever $q_1 < \frac{2\theta - \sqrt{2\theta^2 - \theta}}{\theta} := q_{BC}(\theta)$. One can verify that $q_{BC}(\theta) \in [q_C(\theta), q_B^H(\theta)]$.

Now we turn to the subgame in which $\frac{1}{\theta} < q_1 < q_C(\theta)$. Since we know that $\Pi_2^{B**} > \Pi_2^{C**}$, we have that $\Pi_2^{B**} > \underline{\Pi}_2^C$. Also, we have already verified that $\Pi_2^{B**} > \overline{\Pi}_2^D$ since $\underline{\Pi}_2^B = \overline{\Pi}_2^D$.

Case $q_1 < 1/\theta$

In this region of subgames, the analysis becomes easier to follow with the aid of Figure 7. Notice that the cutoff values q_A and q_B^L, together with $1/\theta$ partition the (q_1, θ) space into four disjoint areas:

- Area 1: $q_1 \in [\max\{q_B^L, q_A\}, 1/\theta]$

- Area 2: $q_1 \in [q_B^L, q_A]$

38

- Area 3: $q_1 \in [0, \min\{q_B^L, q_A\}]$

- Area 4: $q_1 \in [q_A, q_B^L]$

The following chart summarizes the areas for q_1 and the profits one should compare.

#	Boundaries of q_1 in each Area	Profits to Compare
1	$[\max\{q_B^L, q_A\}, 1/\theta]$	$\underline{\Pi}_2^A, \Pi_2^{B**}, \overline{\Pi}_2^D$
2	$[q_B^L, q_A]$	$\Pi_2^{A**}, \Pi_2^{B**}, \overline{\Pi}_2^D$
3	$[0, \min\{q_B^L, q_A\}]$	$\Pi_2^{A**}, \underline{\Pi}_2^B, \overline{\Pi}_2^D$
4	$[q_A, q_B^L]$	$\underline{\Pi}_2^A, \underline{\Pi}_2^B, \overline{\Pi}_2^D$

In area 1 we compute

$$\Pi_2^{B**} - \overline{\Pi}_2^D = \frac{1}{4}\frac{\theta - 1}{(2\theta - 1)\theta}(3q_1\theta - 2\theta + 2 - 2q_1)^2 \tag{28}$$

and also

$$\Pi_2^{B**} - \underline{\Pi}_2^A = \frac{1}{4}\frac{\theta - 1}{(2\theta - 1)\theta}(q_1\theta + 2\theta - 2)^2 \quad . \tag{29}$$

Notice that both equations (28) and (29) are always positive, and hence we conclude that in area 1 the monopolist is better off by pricing in such a way as to induce regime B interior profits Π_2^{B**}.

Now we turn to region 2. We have already shown that $\Pi_2^{B**} > \overline{\Pi}_2^D$. We compute

$$\Pi_2^{B**} - \Pi_2^{A**} = \frac{1}{4}\frac{\theta}{2\theta - 1}(q_1^2\theta - 2\theta + 3 - 2q_1) \quad .$$

We verify that $\Pi_2^{B**} - \Pi_2^{A**} \geq 0$ if and only if $q_1 \geq q_{AB} := \frac{1 - \sqrt{(2\theta - 1)(\theta - 1)}}{\theta}$. Since $q_{AB} < q_A$ we have that in region 1, for $q_1 > q_{AB}$ the optimal period two profits are given by Π_2^{B**} and when $q_1 < q_{AB}$ the optimal profits are given by Π_2^{A**}.

In region 4, notice that $\underline{\Pi}_2^B = \overline{\Pi}_2^D$. So all we need to compute is $\Pi_2^{A**} - \underline{\Pi}_2^D$ which is given by

$$\frac{(q_1 - 1)}{4\theta}(-\theta^2 + 5q_1\theta^2 - 8q_1\theta + 4\theta - 4 + 4q_1) \quad . \tag{30}$$

Now we proceed to inspect equation (30), specifically we look at $f_{AD}(\theta) := -\theta^2 + 5q_1\theta^2 - 8q_1\theta + 4\theta - 4 + 4q_1$. Notice that $f'_{AD}(\theta) = 5\theta^2 - 8\theta + 4$ and this expression is always positive in our domain of θ. Furthermore, $f_{AD}(\theta) = 0$ when $q_1 = \frac{\theta^2 - 4\theta + 4}{5\theta^2 - 8\theta + 4}$ and from now on we define $q_D := \frac{\theta^2 - 4\theta + 4}{5\theta^2 - 8\theta + 4}$. We are now able to conclude that $q_1 < q_D$ if and only if $\Pi_2^{A**} > \underline{\Pi}_2^D$.

Appendix 4: Closed Resale Period 1

First we proceed to determine demand in period one within each regime and the bounds for q_1 that must hold for the regime to be in place. Then we proceed to find the profit maximizing choice of q_1.

Regime A

For regime A we have that the indifferent type between buying or not must evaluate the benefits of either buying and holding the unit for two periods or only buying in period two. This is, the consumer decides to buy in period one if and only if

$$2z - p_1 \geq z\theta - p_2 \iff z \geq \frac{p_1 - p_2}{2 - \theta} \quad .$$

Define $z_1^A := \frac{p_1 - p_2}{2 - \theta}$ as the indifferent type. By Proposition 1 we know every consumer with a higher valuation will buy. Therefore, demand is given by

$$q_1^A := 1 - z_1^A \quad . \tag{31}$$

For regime A to be in place, it must be the case that $z_2^{NB} < z_1^A$. We know that whenever regime A arises, the profit maximizing price in period two is interior. This means that $z_2^{NB} = \frac{1}{2}(1 - q_1^A)$ and $z_2^{NB} < z_1^A$ holds. Hence, the only restriction we have on q_1^A is that it must be in area 4 ($q_1^A \in [q_A, q_B^L]$).

In period two, we showed that regime A is valid for $q_1 \in [q_A, q_B^L]$. After plugging in p_2^{A*},

we compute total profits as a function of period one variables only

$$\Pi^A := 2q_1 - \theta q_1 + \frac{3}{4}\theta(q_1)^2 - 2(q_1)^2 + \frac{\theta}{4} \quad .$$

The unconstrained optimal choice of q_1 is given by

$$q_1^{A**} = \frac{2(2 - \theta)}{8 - 3\theta} \quad .$$

We verify that $q_1^{A**} > 0$ and that $q_1^{A**} \geq q_{ABD}$ for $\theta \geq \theta_A$ where $\theta_A = \frac{9 - \sqrt{17}}{4}$. For expositional clarity we explicitly we rewrite equation (3)

$$q_{ABD} := \begin{cases} \frac{1 - \sqrt{(1 + 2\theta^2 - 3\theta)}}{\theta} & \text{when} \quad 1 \leq \theta < \frac{6 + 2\sqrt{2}}{7} \\ \frac{\theta^2 - 4\theta + 4}{5\theta^2 - 8\theta + 4} & \text{when} \quad \frac{6 + 2\sqrt{2}}{7} \leq \theta \leq 2 \end{cases} \quad .$$

One can verify that $\theta_A < \frac{6 + 2\sqrt{2}}{7}$. So we need to inspect the sign of the derivative of Π^A for the functions that conform q_{ABD}. We define

$$M(\theta) := \sqrt{1 + 2\theta^2 - 3\theta} \tag{32}$$

which will be constantly used in this appendix. We obtain:

$$\left. \frac{\partial \Pi_1^A}{\partial q_1} \right|_{q_1 = \frac{1 - M(\theta)}{\theta}} = \frac{1}{2\theta} \left[(8 - 3\theta)M(\theta) - (2\theta^2 - 7\theta + 8) \right] \tag{33}$$

$$\left. \frac{\partial \Pi_1^A}{\partial q_1} \right|_{q_1 = \frac{\theta^2 - 4\theta + 4}{5\theta^2 - 8\theta + 4}} = \frac{1}{2} \left(\frac{-7\theta^3 + 16\theta^2 - 4\theta + 16}{5\theta^2 - 8\theta + 4} \right) \quad . \tag{34}$$

We analyze expressions (33) and (34) by parts in claims 14 and 15. Recall we are analyzing this function in the interval $\theta \in \left[\theta_A, \frac{6 + 2\sqrt{2}}{7} \right] \approx [1.22, 1.26]$.

CLAIM 14 Expression (33) is positive, hence we have that $q_1^{A*} := \frac{1 - M(\theta)}{\theta}$ for $\theta \in \left[\theta_A, \frac{6 + 2\sqrt{2}}{7} \right]$.

PROOF. It can be verified that $(8 - 3\theta)M(\theta)$ is positive and increasing for $\theta \in \left[\theta_A, \frac{6+2\sqrt{2}}{7}\right]$. In that same interval, $2\theta^2 - 7\theta + 8$ is positive and decreasing. Thus, we verify that equation (33) evaluated at $\theta = \theta_A$ is positive, which implies that (33) is positive for $\theta \in \left[\theta_A, \frac{6+2\sqrt{2}}{7}\right]$. ∎

CLAIM 15 Expression (34) is positive, hence we have that $q_1^{A*} := \frac{\theta^2 - 4\theta + 4}{5\theta^2 - 8\theta + 4}$ for $\theta \in \left[\frac{6+2\sqrt{2}}{7}, 2\right]$.

PROOF. One can easily verify that the denominator in expression (34) is positive $\frac{6+2\sqrt{2}}{7} \leq \theta \leq 2$. The denominator, $-7\theta^3 + 16\theta^2 - 4\theta + 16$ has a positive discriminant (58368) which implies that the polynomial has 3 roots. Since the cubic term is negative, the denominator is positive in the interval between the largest two roots which are $\frac{1+\sqrt{57}}{7}$ and 2. Notice that $\frac{1+\sqrt{57}}{7} < \frac{6+2\sqrt{2}}{7}$, the result follows. ∎

In short, we write the optimal quantity in regime A as:

$$
q_1^{A*} = \begin{cases} \frac{2(2-\theta)}{8-3\theta} & \text{when} & \theta \leq \frac{9-\sqrt{17}}{4} \\ \frac{1-M(\theta)}{\theta} & \text{when} & \frac{9-\sqrt{17}}{4} < \theta < \frac{6+2\sqrt{2}}{7} \\ \frac{\theta^2-4\theta+4}{5\theta^2-8\theta+4} & \text{when} & \frac{6+2\sqrt{2}}{7} \leq \theta \leq 2 \end{cases} \tag{35}
$$

where the last two pieces of q_1^{A*} correspond to the choice of q_{ABD}.

Regime B

For regime B we have two types of marginal consumers. The marginal consumer located exactly at z_1^B is making decision between buying and keeping or only buying in period two. There is another marginal type, that is considering the trade-off between buying and keeping or buying in both periods. Notice that the latter group is a segment of higher valuation consumers, and by Proposition 1 we know that we need to consider only that decision for the lower type. Thus, we have $q_1^B = 1 - z_1^B$.
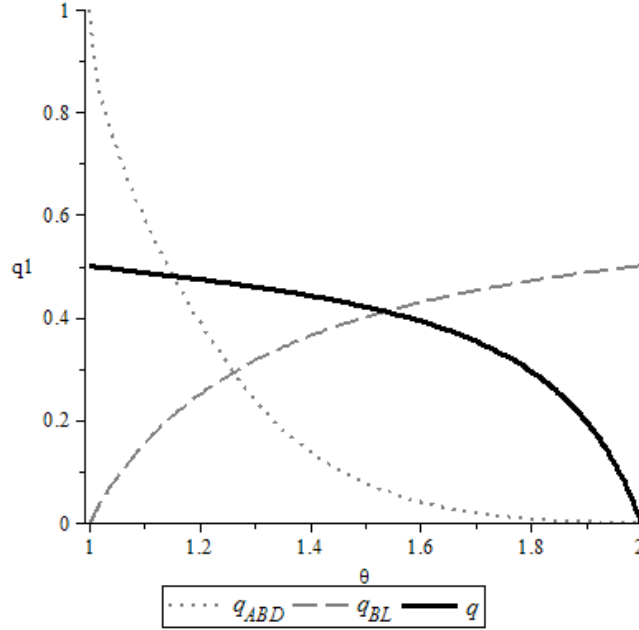
Figure 8: Period 1 Regime B q_1^{B**}

In areas II and III where regime B is in place, we require that $z_2^{NB}(p_2^{B**}) < 1 - q_1^B < z_2^B(p_2^{B**})$.

For regime B we will require that $q_1 \in [\max\{q_{ABD}, q_B^L\}, q_{BC}]$. First we compute overall profits given $p_2^{B*}(q_1)$

$$\Pi^B := \left[(2 - \theta)(1 - q_1) + p_2^{B*}(q_1) \right] q_1 + \Pi_2^{B*}(q_1) \tag{36}$$

and after taking first order conditions we obtain the unconstrained optimal solution to be

$$q_1^{B**} := \frac{2(2\theta - 1)(\theta - 2)}{7\theta^2 - 19\theta + 8} \quad .$$

Figure 8 shows clearly that for a given range of θ we have an interior solution. We define $\underline{\theta}_B$ as the solution to $q_1^{B**} = \frac{1 - M(\theta)}{\theta}$ which is computed numerically to be approximately 1.1523. Furthermore we verify numerically that $\theta \in [1, \underline{\theta}_B] \implies q_1^{B**} - \frac{1 - M(\theta)}{\theta} < 0$ and $\theta \in [\underline{\theta}_B, 2] \implies q_1^{B**} - \frac{1 - M(\theta)}{\theta} > 0$. In order to verify if the solution is at this boundary, we

43

require the derivative of (36) evaluated at $q_1 = \frac{1-M(\theta)}{\theta}$ to be positive.

CLAIM 16 If $\theta \in [1,\underline{\theta}_B]$ then $\left.\frac{\partial \Pi_1^A}{\partial q_1}\right|_{q_1=(\frac{1-\sqrt{(1+2\theta^2-3\theta)}}{\theta})} < 0$. Hence, $q_1^{B*} = \frac{1-M(\theta)}{\theta}$ in the given interval of θ.

PROOF. We compute

$$\left.\frac{\partial \Pi_1^A}{\partial q_1}\right|_{q_1=\frac{1-\sqrt{(1+2\theta^2-3\theta)}}{\theta}} = \frac{1}{2}\frac{M(\theta)f(\theta)+g(\theta)}{\theta(2\theta-1)} \tag{37}$$

where $f(\theta) := \left(19\theta - 7\theta^2 - 8\right)$, and $g(\theta) := \left(-4\theta^3 - 17\theta^2 + 23\theta - 8\right)$. One can verify that $f(\theta) > 0$ for $\theta \in [1,\underline{\theta}_B]$ and hence $M(\theta)f(\theta) > 0$. Furthermore, $f'(\theta) > 0$ for our domain of θ. Since $M(\theta)$ is also increasing in θ, then $M(\theta)f(\theta)$ is positive and increasing. Now we turn to analyze $g(\theta)$. Notice that the discriminant of $g(\theta)$ is $-62527 < 0$ which implies that the polynomial has only one real root given by[16] $\theta_g := \frac{1}{12}(K + 565/K - 17) \approx -5.384$. Since the cubic term in $g(\theta)$ is negative, this function is negative for $\theta > \theta_g$. Furthermore, one can show that $g(\theta)$ is decreasing in our interval of interest. All we need to verify is that expression (37) is negative for both $\theta = 1$ and $\theta = \underline{\theta}_B$. We find that expression (37) evaluated at $\theta = 1$ is approximately -1 and 0 when evaluated at $\theta = \underline{\theta}_B$. Thus we have shown that the monopolist would wish to decrease the quantity, but that would place the firm out of regime B, hence the lower boundary constraint is active. ∎

Now we define $\overline{\theta}^B$ as the solution to $q_1^{B**} = \frac{2(\theta-1)}{3\theta-2}$ which we numerically compute to be 1.537.

CLAIM 17 For $\theta \in [\overline{\theta}^B, 2]$ we have that $q_1^{B*} = \frac{2(\theta-1)}{3\theta-2}$.

A similar analysis as in the previous claim guarantees that the constraint binds. In

---

[16]Where $K = (397 + 6 \cdot \sqrt{4317})^{\frac{1}{3}}$.

short, the optimal choice of q_1 in regime B is given by:

$$q_1^{B*} := \begin{cases} \frac{1-M(\theta)}{\theta} & \text{when} & \theta \leq \underline{\theta}_B \\ \frac{2(2\theta-1)(\theta-2)}{7\theta^2-19\theta+8} & \text{when} & \underline{\theta}_B < \theta < \overline{\theta}^B \\ \frac{2(\theta-1)}{3\theta-2} & \text{when} & \overline{\theta}^B \leq \theta \leq 2 \end{cases} . \tag{38}$$

Regime C

In regime C, the consumer who purchases in period one with the lowest valuation is indifferent between not buying at all and only purchasing in period one. This is

$$2z - p_1 \geq 0 .$$

Solving for the quantity demanded as function of the firm's choices we obtain

$$q_1^C := 1 - \frac{p_1}{2} . \tag{39}$$

The choice of q_1 has to be from the interval $[q_{BC}, 1]$ for the regime to be in place. We evaluate total profits in regime C for p_2^{C*} and obtain that

$$\Pi^C := 2(1 - q_1) + \frac{1}{4}(\theta - 1) . \tag{40}$$

After taking first order conditions, we obtain the unconstrained maximum to equation (40) to be

$$q_1^{C**} = 1/2 .$$

However, $q_1^{C**} < q_{BC}$. We evaluate the derivative to (40) and obtain

$$\left. \frac{\partial \Pi_1^C}{\partial q_1} \right|_{q_1=q_{BC}} = \frac{2}{\theta} \left(\sqrt{\theta(2\theta - 1)} - 3\theta \right) . \tag{41}$$

Expression (41) is negative if and only if $\sqrt{\theta(2\theta-1)} - 3\theta < 0 \iff 2\theta(\theta-5) < 0$ which is the case. We conclude that the optimal choice in regime C is always the lower boundary. We establish that

$$q_1^{C*} = q_{BC} \quad .$$

Regime D

In regime D, the lowest valuation consumer that buys in period one is indifferent between buying twice and only buying in period two. Thus, for all types greater than or equal to z_1 we have that

$$z - p_1 + z\theta - p_2 \geq z\theta - p_2 \quad .$$

Hence demand in period one is given by

$$q_1^D := 1 - p_1 \quad . \tag{42}$$

Regime D only takes place for $\theta > \frac{6+2\sqrt{2}}{7}$ and $q_1 \in [q_{ABD}, q_B^L]$. We evaluate total profits with the optimal period two choice p_2^{D*} which results in

$$\Pi^D = \frac{(1-q_1)(\theta^2 - q_1\theta - \theta - 1 + q_1)}{\theta} \quad . \tag{43}$$

We take first order conditions and solve for the unconstrained maximizer and obtain

$$q_1^{D**} = \frac{1}{2} \frac{\theta^2 - 2\theta + 2}{\theta^2 - \theta + 1} \quad . \tag{44}$$

There are two boundaries that we must inspect to see if the interior solution is within the set of possible choices. We look at the lower boundary and compute $q_B^L - q_1^{D**}$, which equals $\frac{1}{2} \frac{\theta(\theta^2-2)}{(3\theta-2)(\theta^2-\theta+1)}$ and has three roots: $\{-\sqrt{2}, 0, \sqrt{2}\}$. Since the denominator is positive for $\theta \in [1,2]$ we know that for $\theta > \sqrt{2}$ it holds that $q_B^L < q_1^{D**}$.
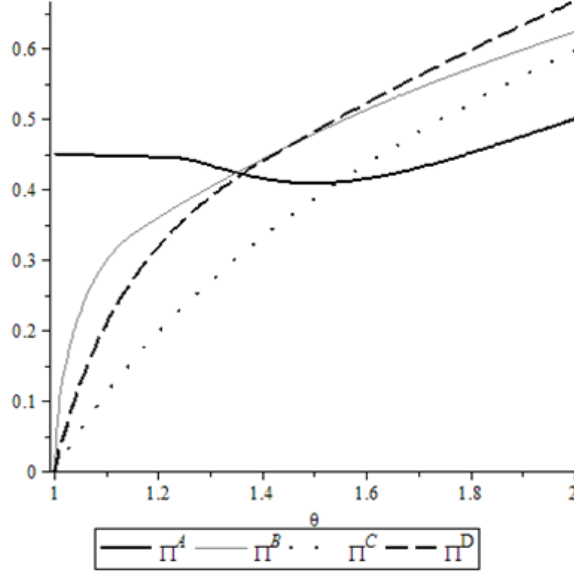
Figure 9: Optimal Profits in Each Regime

For the upper boundary, we will verify that $q_1^{D**} > q_{ABD}$ for $\theta \in \left[\sqrt{2}, 2\right]$. To see why this holds, notice that $\frac{\partial q_1^{D**}}{\partial \theta} = \frac{1}{2} \frac{\theta(\theta-2)}{(\theta^2-\theta+1)^2} < 0$. Now we compare $q_1^{D**}(2) = 1/3 > q_{ABD}(2) = 0$ and the result follows.

The optimal choice of q_1 in regime D is given by

$$
q_1^{D*} = \begin{cases} q_B^L & \text{when} \quad \sqrt{2} \geq \theta \\ q_1^{D**} & \text{when} \quad \theta \geq \sqrt{2} \end{cases} .
\tag{45}
$$

Period One Comparison of Profits

For this subsection it is useful to follow Figure 9. Graphically it can be seen that regime C is never optimal.

We proceed to evaluate the profit function in each regime with the optimal choices. Recall that $M(\theta) := \sqrt{(1 + 2\theta^2 - 3\theta)}$.

For the region in which regime A is in place, we have

$$
\Pi^{A*} := \begin{cases}
\frac{1}{4}\frac{(\theta-4)^2}{8-3\theta} & \text{if} & 1 \leq \theta \leq \frac{9-\sqrt{17}}{4} \\[2mm]
\frac{M(\theta)g_A(\theta)+f_A(\theta)}{4\theta^2} & \text{if} & \frac{9-\sqrt{17}}{4} < \theta < \frac{6+2\sqrt{2}}{7} \\[2mm]
\frac{2\theta(\theta^4+5\theta^3-26\theta^2+36\theta-16)}{\left(5\theta^2-8\theta+4\right)^2} & \text{if} & \frac{6+2\sqrt{2}}{7} \leq \theta \leq 2
\end{cases}
\tag{46}
$$

where $g_A(\theta) := 4\theta^2 - 14\theta + 16$ and $f_A(\theta) := 7\theta^3 - 29\theta^2 + 38\theta - 16$.

For the region in which regime B is in place, profits are given by

$$
\Pi^{B*} := \begin{cases}
\frac{M(\theta)g_B(\theta)+f_B(\theta)}{4\theta^2(2\theta-1)} & \text{if} & 1 \leq \theta \leq \underline{\theta}_B \\[2mm]
\frac{3\theta^4-6\theta^3-6\theta^2+12\theta-4}{(2\theta-1)(7\theta^2-19\theta+8)} & \text{if} & \underline{\theta}_B < \theta < \overline{\theta}^B \\[2mm]
\frac{(1+2\theta)(\theta-1)\theta}{(3\theta-2)^2} & \text{if} & \overline{\theta}^B \leq \theta \leq 2
\end{cases}
\tag{47}
$$

where $g_B(\theta) := 8\theta^3 - 34\theta^2 + 46\theta - 16$ and $f_B(\theta) := -71\theta^3 + 107\theta^2 - 70\theta + 16$.

We compute regime C profits in which the optimal period one quantity is always at the lower corner of the region in which the regime occurs. They are given by:

$$
\Pi^{C*} := \frac{24\sqrt{\theta(2\theta-1)} + \theta^2 - 33\theta + 8}{4\theta} \quad .
\tag{48}
$$

Finally, regime D optimal profits are given by:

$$
\Pi^{D*} := \begin{cases}
\frac{\theta(2\theta^2-\theta-1)}{(3\theta^2-2)^2} & \text{if} & \frac{6+2\sqrt{2}}{7} \leq \theta \leq \sqrt{2} \\[2mm]
\frac{\theta^3}{4(\theta^2-\theta+1)} & \text{if} & \sqrt{2} < \theta \leq 2
\end{cases} \quad .
\tag{49}
$$

Profit Comparison

We will compare profits to finish our characterization of the equilibrium. The cutoff values for θ separating the induced period two regimes are the solutions to higher order polynomials in θ, and can only be solved numerically. It is useful to keep track of the relative magnitudes of the cutoffs derived so far:

$$1 < \underline{\theta}_B < \theta_A < \frac{6 + 2\sqrt{2}}{7} < \sqrt{2} < \bar{\theta}^B < 2$$

It can be easily shown that the each optimal profit expression Π^{i*} ($i \in \{A, B, C, D\}$) is continuous in θ in the interval $[1, 2]$. We also have that $\Pi^{i*}(1) = 0$ for Π^{i*} ($i \in \{B, C, D\}$) and $\Pi^{A*}(1) > 0$.

CLAIM 18 $\frac{\mathrm{d}\Pi^{i*}}{\mathrm{d}\theta} > 0$ for $i \in \{B, C, D\}$ and $\theta \in [1, 2]$ and regime C is never optimal in equilibrium.

PROOF. We first proceed to show that $\frac{\mathrm{d}\Pi^{i*}}{\mathrm{d}\theta} > 0$. One can easily verify that $\Pi^{B*}(1) < \Pi^{B*}(2)$, $\Pi^{C*}(1) < \Pi^{C*}(2)$, and $\Pi^{D*}\left(\frac{6+2\sqrt{2}}{7}\right) < \Pi^{D*}(2)$. In words, each equilibrium profit function evaluated at the lowest θ consistent with that regime, is below the value associated with $\theta = 2$. We will now show that

$$\frac{\mathrm{d}\Pi^{C*}}{\mathrm{d}\theta} = \frac{1}{4} \frac{\left[12\theta - 8\sqrt{\theta(2\theta - 1)} + \theta^2 \sqrt{\theta(2\theta - 1)} \right]}{\theta^2 \sqrt{\theta(2\theta - 1)}} \tag{50}$$

is positive. First notice that $\theta\sqrt{2} > \sqrt{\theta(2\theta - 1)}$. We then substitute $8\sqrt{2}\theta$ for $8\sqrt{\theta(2\theta - 1)}$, which means that the numerator in (50) is greater than $(12 - 8\sqrt{2})\theta + \theta^2\sqrt{\theta(2\theta - 1)}$ which in turn is positive. We find that the expressions $\frac{\mathrm{d}\Pi^{i*}}{\mathrm{d}\theta}$ for $i \in \{B, D\}$ have no roots in $\theta \in [1, 2]$ so this means that there are no critical points. Given that we showed that each function starts at a lower value than at which it ends, together with the fact that such functions are continuous with no critical points in the interval in question, we conclude that each Π^{i*} for $i \in \{B, D\}$ is increasing.

Additionally, with straightforward computations one can verify that $\Pi^{C*}(1) < \Pi^{B*}(1), \Pi^{C*}\left(\frac{6+2\sqrt{2}}{7}\right) < \Pi^{D*}\left(\frac{6+2\sqrt{2}}{7}\right)$ and that $\Pi^{C*}(2) < \Pi^{B*}(2) < \Pi^{D*}(2)$. We find no roots for $\Pi^{C*} - \Pi^{B*}$ for $\theta \in (1, 2)$ nor $\Pi^{C*} - \Pi^{D*}$ for $\theta \in \left(\frac{6+2\sqrt{2}}{7}, 2\right)$, thus $\Pi^{C*} < \min\{\Pi^{B*}, \Pi^{D*}\}$. ∎

We have established that regime C is never in place so we inspect for the relationship between Π^{i*} for $i \in \{A, B, D\}$. There exists θ_{AB} such that $\Pi^{A*}(\theta_{AB}) = \Pi^{B*}(\theta_{AB}) > \Pi^{D*}(\theta_{AB})$.

We solve numerically for θ such that $\Pi^{A*} - \Pi^{B*} = 0$, and obtain that $\theta_{AB} \approx 1.342$. Furthermore, $\theta_{AB} \in \left[\frac{6+2\sqrt{2}}{7}, \sqrt{2}\right]$. For all values of $\theta > \theta_{AB}$ we have that profits in regime B higher than in A.

There exists θ_{BD} such that $\Pi^{B*}(\theta_{BD}) = \Pi^{D*}(\theta_{BD}) > \Pi^{A*}(\theta_{BD})$. We solve numerically for θ such $\Pi^{B*} - \Pi^{D*} = 0$, and obtain that $\theta_{BD} \approx 1.469$. Furthermore, $\theta_{BD} \in [\ \sqrt{2} < \bar{\theta}^B]$, hence for values of $\theta > \theta_{BD}$ regime D yields higher profits than B and A.

We are now able to fully characterize the optimal choice of q_1. This is:

$$q_1^*(\theta) := \begin{cases} \frac{2(2-\theta)}{8-3\theta} & \text{when} & 1 \leq \theta \leq \frac{9-\sqrt{17}}{4} \\ \frac{1-\sqrt{(1+2\theta^2-3\theta)}}{\theta} & \text{when} & \frac{9-\sqrt{17}}{4} < \theta \leq \frac{6+2\sqrt{2}}{7} \\ \frac{\theta^2-4\theta+4}{5\theta^2-8\theta+4} & \text{when} & \frac{6+2\sqrt{2}}{7} < \theta \leq \theta_{AB} \\ \frac{2(2\theta-1)(\theta-2)}{7\theta^2-19\theta+8} & \text{when} & \theta_{AB} < \theta \leq \theta_{BD} \\ \frac{1}{2}\frac{\theta^2-2\theta+2}{\theta^2-\theta+1} & \text{when} & \theta_{BD} < \theta \leq 2 \end{cases} \qquad (51)$$

The first three pieces of this function are all the optimal q_1^{A*} function, then we have the interior optimal for regime B, and final part of the piecewise function corresponds to the interior optimal choice in regime D.

Appendix 5: Resale Market Proofs

Proof of proposition 7

PROOF. Two conditions must be satisfied for a consumer to repeat purchase:

$$z\theta_2 - p_2 + p_r^* > p_r^* \qquad (52)$$
$$z\theta_2 - p_2 + p_r^* > z \qquad .$$

The conditions for non-owner of the period one good to buy a new good in period two are:

$$z\theta_2 - p_2 > 0 \tag{53}$$

$$z\theta_2 - p_2 > z - p_r^* \ .$$

It is clear that these two conditions are equivalent by simply subtracting p_r^*. In order to determine the unique cutoff type (z_2) notice that $\theta_2 - p_2 \geq z - p_r^* \geq -p_r^*$. Rearranging we obtain:

$$z \geq \frac{p_2 - p_r}{\theta_2 - 1} \geq \frac{p_2 - p_r - z}{\theta_2 - 1} \ .$$

Since we are considering quality upgrades $(\theta > 1)$ it is enough for types to satisfy the inequality $z \geq \frac{p_2 - p_r}{\theta_2 - 1} = z_2$. ∎

Proof of proposition 8

PROOF. Regardless of which case occurs, supply of the used good is always given by $1 - z_1$. When $z_1 < z_2$ its as if some consumers were selling and buying back their used unit. When $z_1 \geq z_2$ all used units swap hands to new consumers. Demand for used units is given by $z_2 - z_r$. Thus, we have that p_r solves $z_2 - z_r = 1 - z_1$. The result follows. ∎

Proof of proposition 12

PROOF. The profit expression for closed resale markets in the pertinent region is given by Π_D^* . Hence we solve $\frac{\theta^3}{4(\theta^2 - \theta + 1)} = \frac{\theta}{4}\left(\frac{8\theta + 1}{8\theta - 3}\right)$ and obtain a cubic polynomial. We find a root at $\theta = (7 + \sqrt{65})/8$. ∎

Appendix 6: Proof of Proposition 9

In this appendix we will look at all possible subgames and verify that, indeed, it never pays to induce $p_r = 0$. We separate the analysis into the positive resale price and the zero resale price cases.

Case 1: Positive Resale Price

Let $z_1 \in [0,1]$. Consider the case in which $p_2 > q_1(\theta - 1)$, so that $p_r > 0$. In period two, the stage profits are given by

$$\Pi_2^+ = p_2 \left(\frac{\theta - q_1 - p_2}{\theta} \right) \quad . \tag{54}$$

We take first order conditions and solve for the optimal price to be given by

$$p_2^+(q_1; \theta) = \frac{\theta - q_1}{2} \quad . \tag{55}$$

However, for a positive resale price to be in place we require that $p_2^+(q_1; \theta) > q_1(\theta - 1)$ which happens if and only if $q_1 < \frac{\theta}{2\theta - 1}$. We consider such subgames and this yields the solution present in Proposition 9.

Case 2: Zero Resale Price

So now we turn to subgames in which $q_1 > \frac{\theta}{2\theta - 1}$. The period two profits we maximize are

$$\Pi_2^- = p_2 \left(1 - \frac{p_2}{\theta - 1} \right) \quad .$$

Taking first order conditions and solving for the optimal period two price we obtain

$$p_2^-(q_1, \theta) = \frac{\theta - 1}{2} \quad .$$

CLAIM 19 For $q_1 < \frac{1}{2}$ we have that the positive resale price regime is optimal and for $q_1 > \frac{\theta}{2\theta - 1}$ we have that $p_r = 0$ is optimal.

PROOF. We verify by simple algebraic computations that $\frac{\partial \Pi_2^+}{\partial p_2}\big|_{p_2 = (\theta - 1)q_1} > 0 \iff q_1 < \frac{\theta}{2\theta - 1}$ and that $\frac{\partial \Pi_2^-}{\partial p_2}\big|_{p_2 = (\theta - 1)q_1} < 0 \iff q_1 > \frac{1}{2}$. The result follows. ∎

Now we inspect for $q_1 \in \left[\frac{1}{2}, \frac{\theta}{2\theta - 1} \right]$ and verify that $\Pi_2^+(p_2^+) \geq \Pi_2^-(p_2^-) \iff q_1 \leq$

$\theta - \sqrt{\theta^2 - \theta}$. We now show that $\theta - \sqrt{\theta^2 - \theta} < \frac{\theta}{2\theta - 1}$ which occurs if and only if

$$\sqrt{\theta(\theta - 1)} > \frac{2\theta(\theta - 1)}{2\theta - 1} \iff 4\theta(\theta - 1) > (2\theta - 1)^2 \iff 1 > 0 \ .$$

Now we proceed to period one. First the positive resale case. For this, we substitute (55) into (10) which yields

$$\Pi^+(q_1) = \frac{1}{4} \frac{4\theta q_1 - 8\theta q_1^2 + 3q_1^2 + \theta^2}{\theta} \ .$$

Taking first order conditions we obtain that

$$q_1^+(\theta) = \frac{2\theta}{8\theta - 3} \ .$$

One can verify that $q_1^+(\theta) < 1/2$. Hence we have that the interior solution is in place and this yields the profit level given by

$$\Pi^+(\theta) = \frac{\theta}{4} \left(\frac{8\theta + 1}{8\theta - 3} \right) \ .$$

Now we turn to the subgames such that $q_1 > \frac{\theta}{2\theta - 1}$. Period one profits are given by

$$\Pi^-(q_1) := q_1(1 - q_1) + \frac{\theta - 1}{4} \ .$$

Taking first order conditions we obtain that the unconstrained maximizing choice is at $q_1 = 1/2$. However, we verify that $1/2 < \frac{\theta}{2\theta - 1}$. According to claim 19 we know that this constraint binds. Thus we have that the optimal choice in the zero resale price subgames is given by:

$$q_1^- = \theta - \sqrt{\theta^2 - \theta}$$

and total profits are

$$\Pi^-(\theta) := (2\theta - 1)\sqrt{\theta^2 - \theta} - \left(2\theta^2 - \frac{9}{4}\theta + \frac{1}{4}\right) \ .$$

.

CLAIM 20 $\Pi^-(\theta) < \Pi^+(\theta)$.

PROOF. We will show that both profit expressions are increasing and continuous in θ, thus it suffices to evaluate them at the boundaries and verify that $\Pi^-(1) < \Pi^+(1)$ and $\Pi^-(2) < \Pi^+(2)$. First, it is straightforward to notice that both $\Pi^-(\theta)$ and $\Pi^+(\theta)$ are continuous for $\theta \in (1,2]$. Also, $\frac{d\Pi^-(\theta)}{d\theta} := \frac{16\theta^2 - 16\theta + 2 - (16\theta - 9)\sqrt{\theta^2 - \theta}}{4\sqrt{\theta^2 - \theta}}$. One can verify that the denominator of $\frac{d\Pi^-(\theta)}{d\theta}$ is positive: $16\theta^2 - 16\theta + 2 - (16\theta - 9)\sqrt{\theta^2 - \theta} > 0 \iff \left(\frac{16\theta^2 - 16\theta + 2}{(16\theta - 9)}\right)^2 > \theta^2 - \theta \iff \frac{32\theta^3 - 49\theta^2 + 17\theta + 4}{(16\theta - 9)^2} > 0 \iff g(\theta) := 32\theta^3 - 49\theta^2 + 17\theta + 4 > 0$. We compute the discriminant of g which is equal to -414191, and since this value is negative, we conclude that it only has one real root, call it θ_1. Given that the coefficient associated to the cubic term is positive, $g(\theta) > 0$ for all $\theta > \theta_1$. We apply Sturm's theorem and verify that $g(\theta)$ has one root in the interval $[-\infty, 1]$, and thus no roots in $[1, 2]$. A numerical approximation for the root is given by $\theta_1 \approx -0.15$. It follows that $\frac{d\Pi^-(q_1^-, p_1^-)}{d\theta} > 0$. Now we turn to examine $\frac{d\Pi^+(\theta)}{d\theta} = \frac{1}{4}\frac{64\theta^2 - 48\theta - 3}{(8\theta - 3)^2}$. The numerator has two roots: $\left\{\frac{3}{8} - \frac{\sqrt{3}}{4}, \frac{3}{8} + \frac{\sqrt{3}}{4}\right\}$. Since $\frac{3}{8} + \frac{\sqrt{3}}{4} < 1$ it follows that $\frac{d\Pi^+(\theta)}{d\theta} > 0$. One can compute and verify that $\Pi^-(1) = 0 < 9/20 = \Pi^+(1)$ and $\Pi^-(2) = 3\sqrt{2} - 15/4 < 17/26 = \Pi^+(2)$. ∎

Appendix 7: The Renter's Problem

In this setting, the monopolist can simultaneously rent goods of both qualities in period two, thus we allow for some consumers to rent a good of quality θ and others an old unit. The firm chooses the rental prices and consumers decide in each period weather or not to rent. Let p_1 denote the rental price in period 1, p_2^1 denote the rental price in period 2 of a

good of quality 1, and p_2^θ the period two rental price of a good of quality θ.

Consider any subgame induced by any choice of p_1. Notice the in period two, previous choices by the consumer do not affect her buying decision, it is water under the bridge; every consumer starts with zero holdings. In period 2 a consumer decides to rent a period 1 unit if and only if

$$z - p_2^1 > \max\{z\theta - p_2^\theta, 0\}$$

and a period 2 unit if and only if

$$z\theta - p_2^\theta > \max\{z - p_2^1, 0\}$$

and decides not to rent at all if and only if

$$0 > \max\{z\theta - p_2^\theta, z - p_2^1\} \ .$$

The consumer who is indifferent between renting a period 1 unit and a period 2 unit in period 2 satisfies $z - p_2^1 = z\theta - p_2^\theta > 0$. We solve for such cutoff type and obtain

$$z_2^\theta := \frac{p_2^\theta - p_2^1}{\theta - 1} \ .$$

Since $z \geq 0$ we require that $p_2^\theta > p_2^1$ holds, but it is evident that $p_2^\theta < p_2^1$ cannot improve profits. Demand for the period 2 units is given by $q_2^\theta = 1 - z_2^\theta$. Demand in period 2 for period 1 units is given by those types below z_2^θ but that still find $z - p_2^1 > 0$. Thus the cutoff renting type of period 1 units in period 2 is simply $z_2^1 = p_2^1$ and demand for period 1 units in period 2 is given by $q_2^1 = z_2^\theta - z_2^1$. The firm faces the following problem

$$\max_{p_2^1, p_2^\theta} \ p_2^1(z_2^\theta - z_2^1) + p_2^\theta(1 - z_2^\theta) \ . \tag{56}$$

The solution to problem (56) is given by $p_2^{1*} = 1/2$ and $p_2^{\theta*} = \theta/2$ which clearly satisfies our

requirement that $p_2^\theta > p_2^1$. As a result, $z_2^{\theta*} = z_2^{1*} = 1/2$ which means that no consumer in period two is renting an old unit.

In period 1, it is straightforward to see that the optimal rental price is $p_1^{1*} = 1/2$. Total profits for the renter are given by

$$\Pi^{\text{Rent}} = \frac{1+\theta}{4} \quad . \tag{57}$$

Appendix 8: The Seller Monopolist with Commitment Power and an Open Resale Market

The monopolist's problem (from 12) is given by

$$\max_{q_1,p_2} \quad q_1 \left[1 - q_1 + p_r(q_1, p_2)\right] + p_2 \left[1 - \frac{p_2 - p_r(q_1, p_2)}{\theta - 1}\right] \quad .$$

We proceed to solve the profit maximizing problem assuming that $p_r > 0$. The solution to this problem is given by $q_1^{\text{Comm}} = \frac{\theta}{4\theta-2}$, $q_2^{\text{Comm}} = \frac{\theta-1}{2\theta-1}$, $p_1^{\text{Comm}} = 1$, $p_2^{\text{Comm}} = \frac{\theta}{2}$. The resulting resale price is given by $p_r^{\text{Comm}} = \frac{\theta}{4\theta-2}$. The total profit level is given by $\Pi^{\text{Comm}} = \frac{\theta^2}{4\theta-2}$.

Now we turn to the possibility of the firm inducing $p_r = 0$ which could occur only in cases in which $p_2 \leq (\theta - 1)q_1$. As it turns out, with this restriction the solution is exactly where $p_2 = (\theta - 1)q_1$. The optimal choices are $q_{1-}^{\text{Comm}} = 1/2$, $q_{2-}^{\text{Comm}} = 1/2$, $p_{1-}^{\text{Comm}} = 1/2$, $p_{2-}^{\text{Comm}} = \frac{1}{2}(\theta - 1)$ and total profits are $\Pi_{-}^{\text{Comm}} = \frac{\theta}{4}$. Clearly, $\Pi^{\text{Comm}} > \Pi_{-}^{\text{Comm}}$.