Mixed Strategy Nash Equilibrium

In the Matching Pennies Game, one can try to outwit the other player by guessing which strategy the other player is more likely to choose.

		player 2	
		H	T
player 1	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

However, by choosing the mixed strategy $(\frac{1}{2}, \frac{1}{2})$, either player can guarantee an expected payoff of zero, so no rational player should be consistently outwitted.

Both players choosing the mixed strategy $(\frac{1}{2}, \frac{1}{2})$ is the Mixed Strategy Nash Equilibrium of this game. Neither player can increase his/her payoff by changing his/her strategy.

Definition: A profile of mixed strategies, $\sigma = (\sigma_1, ..., \sigma_n)$, is a *mixed strategy Nash equilibrium (MSNE)* if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i})$$

for all $s_i \in S_i$ and all players i.

Here is an important fact that will help us compute the MSNE: It must put positive probability only on pure strategies that are best responses to the other players' mixed strategies σ_{-i}.

To see why players are only mixing over best responses, we can write the overall payoff of player i as

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i}).$$

Player i is choosing the probability weight, $\sigma_i(s_i)$, to put on the payoff $u_i(s_i, \sigma_{-i})$ for each strategy. The sum is highest when all of the weight is put on the highest $u_i(s_i, \sigma_{-i})$, so she should only put positive probability on best responses to σ_{-i}.

Put another way, each player must be **indifferent** between all of the pure strategies that she uses with positive probability in the MSNE, and no other pure strategy can be better.

A pure strategy NE is also a MSNE.

The indifference condition can be used to compute the MSNE for 2×2 games:

		player 2	
		P	A
player 1	P	$6, 6$	$2, 7$
	A	$7, 2$	$0, 0$

In the Chicken Game, suppose player 1 chooses the mixed strategy $(p, 1 - p)$ and player 2 chooses the mixed strategy $(q, 1 - q)$.

Player 1 must receive the same payoff from P and A. If player 1 chooses P, his payoff is

$$6q + 2(1 - q)$$

and if player 1 chooses A, his payoff is

$$7q + 0(1 - q).$$

Setting these payoffs equal to each other determines player 2's MSNE mixing probability, $q = \frac{2}{3}$.

Similarly, player 2 must receive the same payoff from P and A. Setting these payoffs equal to each other determines player 1's mixing probability, $p = \frac{2}{3}$.

Notice that each player's mixing probability is determined to make the **other** player indifferent between his two choices. This can be unintuitive, as the following examples illustrate.

Example: Malcolm Butler interception.

Example: A Simple Entry Game.

Two firms must choose whether or not to enter a market. The fixed cost of entry for firm i is c_i. Assume $c_1 < 1$ and $c_2 < 1$.

Monopoly revenues are 1, and duopoly revenues are 0. (Note–We can think of a lone entrant setting the monopoly price and if both firms enter, Bertrand competition leads to a price equal to marginal production cost.)

		firm 2	
		enter	don't enter
firm 1	enter	$-c_1, -c_2$	$1 - c_1, 0$
	don't enter	$0, 1 - c_2$	$0, 0$

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To compute the mixed strategy Nash equilibrium, let p_i denote the probability that firm i enters. If firm 1 enters, its payoff is

$$p_2(-c_1) + (1 - p_2)(1 - c_1).$$

For firm 1 to be indifferent between entering and not, both payoffs must be zero, yielding $p_2 = 1 - c_1$. Similarly, $p_1 = 1 - c_2$.

Notice that the firm with the higher cost is more likely to enter! Mixed strategy NE often yields unintuitive results, because mixing probabilities must make the **other** player(s) indifferent.

Imagine the game being played over and over again, with a new pair of players each time. It is very plausible that the play will evolve into the lower cost firm always entering and the other firm always staying out. This is a pure strategy NE.

It is also plausible that play evolves into the higher cost firm always entering and the lower cost firm being bullied into staying out, and that is also a NE.

But the play might instead settle down into a pattern of each player entering with a certain probability, so both entering and not entering must be best responses (must yield zero profits). Then the higher cost player must be entering with greater frequency than the lower cost player. If the lower cost player entered more often, then it would have the lower cost and the better chance of being the monopolist–if entering yields only zero profits then entering must yield the other player negative profits.

For games bigger than 2×2, then finding the MSNE is more complicated. You need to determine which strategies are being played with positive probability and which ones are not being played. Here is Figure 11.2 from the text (also discussed in Chapter 7). Player 1 decides whether to serve to the opponent's forehand side, the center (at the opponent's body), or the opponent's backhand side. Player 2 must decide whether to protect his forehand, his center, or his backhand.

		player 2		
		F	C	B
	F	0, 5	2, 3	2, 3
player 1	C	2, 3	0, 5	3, 2
	B	5, 0	3, 2	2, 3

To find the MSNE (there are no pure strategy NE), first note that player 1's strategy, F, is dominated by a mixture of C and B that puts very high probability on B. Thus, F is not rationalizable. Player 1 will be mixing between C and B.

			player 2	
		F	C	B
	F	0, 5	2, 3	2, 3
player 1	C	2, 3	0, 5	3, 2
	B	5, 0	3, 2	2, 3

Next, having eliminated player 1's strategy F, player 2's strategy F is now dominated by C, so player 2's strategy F is not rationalizable. Player 2 will be mixing between C and B.

We now know the MSNE is of the form $\sigma_1 = (0, p, 1-p)$ and $\sigma_2 = (0, q, 1-q)$.

Player 1's payoff from C is then $3(1-q)$, and his payoff from B is $3q + 2(1-q)$. Equating and solving for q, we have $q = \frac{1}{4}$. Therefore, $\sigma_2 = (0, \frac{1}{4}, \frac{3}{4})$.

Player 2's payoff from C is $5p + 2(1-p)$, and his payoff from B is $2p + 3(1-p)$. Equating and solving for p, we have $p = \frac{1}{4}$. Therefore, $\sigma_1 = (0, \frac{1}{4}, \frac{3}{4})$.

Here is a corollary of Nash's Theorem: *Every finite game (having a finite number of players and a finite set of strategies for each player) has at least one Nash equilibrium in pure or mixed strategies.*

Note: If the game is not finite, there may be no NE.

Consider the following two player game

$S_i = \{1, 2, 3, ...\}$ for $i = 1, 2$.

$u_i(s_1, s_2) = s_1 + s_2$ for $i = 1, 2$.