Homework #3 Answers

1. (a) The marginal rate of technical substitution is the ratio of the marginal product of labor to the marginal product of capital. Since L is in the numerator and the denominator of the expression for x, use the quotient rule.

$$\begin{aligned} x &= \frac{5KL}{K+L} \\ MP_L &= \frac{\partial x}{\partial L} = \frac{(K+L)5K - 5KL}{(K+L)^2} = \frac{5K^2}{(K+L)^2} \\ MP_K &= \frac{\partial x}{\partial K} = \frac{(K+L)5L - 5KL}{(K+L)^2} = \frac{5L^2}{(K+L)^2} \\ MRTS &= \left(\frac{(K+L)5K - 5KL}{(K+L)^2}\right)\left(\frac{(K+L)^2}{(K+L)5L - 5KL}\right) = \frac{K^2}{L^2} \end{aligned}$$

(b) The average product of labor is given by

$$AP_L = \frac{x}{L} = \frac{5K}{K+L}.$$

The marginal product of labor was calculated in part (a), given by $MP_L = \frac{5K^2}{(K+L)^2}$.

(c) If we multiply the capital and labor inputs by θ, we have

$$f(\theta K, \theta L) = \frac{5(\theta K)(\theta L)}{\theta K + \theta L} = \frac{5KL\theta^2}{\theta(K+L)} = \theta\frac{5KL}{K+L} = \theta f(K, L).$$

Therefore, the production function exhibits constant returns to scale.

2. (a) First, we set up the Lagrangean for the cost minimization problem

$$Lagr. = wL + rK + \lambda[x - \frac{5KL}{K+L}].$$

The first order conditions are

$$\begin{aligned} \frac{\partial Lagr.}{\partial L} &= 0 = w - \lambda\frac{5K^2}{(K+L)^2} \\ \frac{\partial Lagr.}{\partial K} &= 0 = r - \lambda\frac{5L^2}{(K+L)^2} \\ \frac{\partial Lagr.}{\partial \lambda} &= 0 = x - \frac{5KL}{K+L}. \end{aligned}$$

Solving the first two equations for λ and setting them equal to each other, we have the condition that MRTS = w/r,

$$\frac{K^2}{L^2} = \frac{w}{r},$$

which can be written as

$$K = \sqrt{\frac{w}{r}} L = 2L. \tag{1}$$

To make the math easier, we have substituted $w = 4$ and $r = 1$. (After all, we are not asked to consider any input prices other than these anywhere in the assignment.) Plugging (1) into the third condition yields

$$x = \frac{10L^2}{2L + L} = \frac{10L}{3}.$$

Solving for L, we have

$$L = \frac{3x}{10}.$$

From (1), we have

$$K = \frac{6x}{10}.$$

(b) The long run total cost function is

$$LRTC = wL^* + rK^* = 4(\frac{3x}{10}) + (\frac{6x}{10}) = \frac{9x}{5}.$$

Long run average cost is therefore

$$LRAC = \frac{LRTC}{x} = \frac{9}{5}.$$

Notice that long run average cost is independent of x, because of constant returns to scale.

3. Now we have a short run situtation in which capital is fixed at 4 units, $\overline{K} = 4$.

(a) If capital is fixed at 4, $\overline{K} = 4$, then we have

$$x = \frac{20L}{4 + L}.$$

Solving for L, we have $4x + xL = 20L$, so

$$L = \frac{4x}{20 - x}.$$

Therefore, the short run total cost and average total cost functions are

$$SRTC = 4(\frac{4x}{20 - x}) + 4 \text{ and } SRATC = \frac{16}{20 - x} + \frac{4}{x}.$$

The short run average variable cost function is

$$SRAVC = \frac{wL}{x} = \frac{16}{20 - x}.$$

Taking the derivative of SRTC, and using the quotient rule and chain rule, we have the short run marginal cost function,

$$SRMC = \frac{(20-x)16 - 16x(-1)}{(20-x)^2} = \frac{320}{(20-x)^2}.$$

(b) First we must find the minimum SRATC, by solving

$$\frac{\partial SRATC}{\partial x} = 0 = 16(20-x)^{-2} - 4x^{-2}.$$

Simplifying, we have

$$\frac{16}{(20-x)^2} = \frac{4}{x^2}.$$

Taking the square root of both sides, we have

$$\frac{4}{(20-x)} = \frac{2}{x}.$$

Cross multiplying, we have $4x = 40-2x$, so $x = \frac{20}{3}$. This is the x that minimizes SRATC. To find that minimum cost, we evaluate SRATC at $x = \frac{20}{3}$:

$$\begin{aligned} SRATC &= \frac{16}{20-x} + \frac{4}{x} = \frac{16}{40/3} + \frac{4}{20/3} = \frac{9}{5} \\ SRMC &= \frac{320}{(20-x)^2} = \frac{320}{(20-\frac{20}{3})^2} = \frac{9}{5}. \end{aligned}$$

Thus, when evaluated at the x that minimizes SRATC, the following functions all take the value $\frac{9}{5}$: SRATC, SRMC, and LRAC (from problem 2).