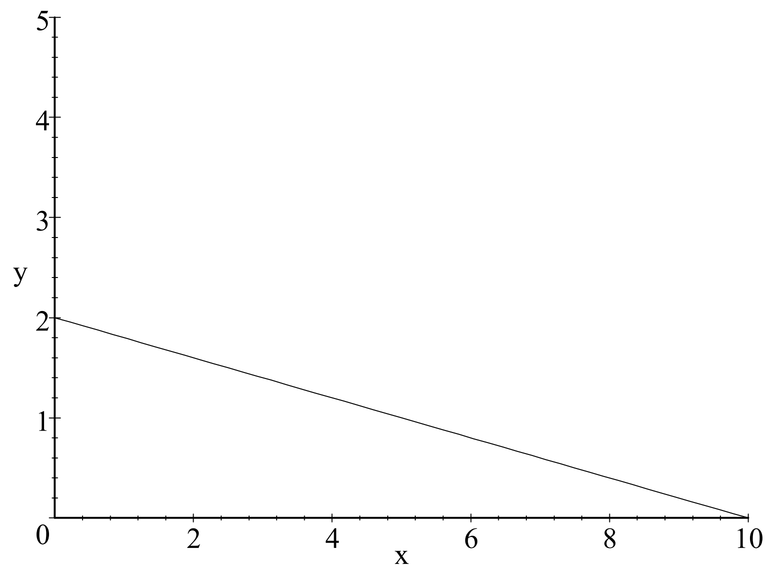
Utility Maximization

Given the consumer's income, M, and prices, p_x and p_y, the consumer's problem is to choose the affordable bundle that maximizes her utility.

The *feasible set (budget set)*: total expenditure cannot exceed income, so we have

$$p_x x + p_y y \leq M. \quad (1)$$

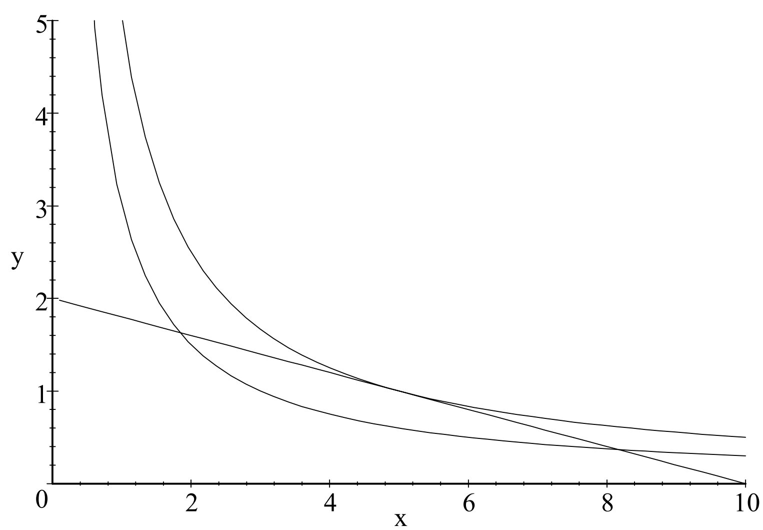
Since more is better, inequality (1) must hold with equality at the solution to the consumer's problem.



The Feasible Set, $p_x = 1, p_y = 5, M = 10$

The budget line, $p_x x + p_y y = M$, has a slope of $-p_x/p_y$, an x-intercept of M/p_x, and a y-intercept of M/p_y.

From the diagram, we see that utility maximization over the feasible set occurs at the point of tangency between an indifference curve and the budget line. (Notice that we need axiom 6 for the tangency to be a utility maximum.)



The slope of the indifference curve is $-MRS_{yx}$ and the slope of the budget line is $-p_x/p_y$. The optimal bundle is the point on the budget line where we have

$$MRS_{yx} = p_x/p_y. \tag{2}$$

Equation (2) has an economic interpretation: the internal rate of trade should equal the external or market rate of trade. Otherwise, there are further gains from trade between the consumer and the market. For example, if $MRS_{yx} = 1/4$ and $p_x = 1, p_y = 5$, then the consumer could increase her utility by choosing 1 unit less y, freeing up the income to buy 5 units more x. Since giving up 1 unit of y and gaining 4 units of x leaves her indifferent according to her internal rate of trade, gaining 5 units of good x makes her better off.

To see that the intuition from the diagram, leading to (2), is correct, *here the consumer's constrained optimization problem:*

$$\max u(x, y) \tag{3}$$
$$\text{subject to} \quad : \quad p_x x + p_y y = M$$

We could solve (3) by solving the constraint for y in terms of x, plugging it in to the objective, and solving the resulting unconstrained problem. This is called the substitution approach (not recommended).

The **Lagrangean approach** transforms a constrained optimization problem, such as (3), into an unconstrained problem of choosing x, y, and the Lagrange multiplier, λ, to maximize

$$L = u(x, y) + \lambda[M - p_x x - p_y y]. \tag{4}$$

By differentiating L with respect to x, y, and λ, and setting the derivatives equal to zero, the resulting first order conditions are:

$\frac{\partial u}{\partial x} - \lambda p_x = 0, \frac{\partial u}{\partial y} - \lambda p_y = 0,$ and $M - p_x x - p_y y = 0.$

Solving, we are able to formally derive (2).

Axiom 6 guarantees that the second-order conditions are satisfied. We still need to make sure there is no corner solution with $x = 0$ or $y = 0$.

An Example

$u(x, y) = xy$

$M = 10$

$p_x = 1, p_y = 1$

The Lagrangean expression is

$$L = xy + \lambda[10 - x - y].$$

The first order conditions are

$$\begin{aligned} y - \lambda &= 0, \\ x - \lambda &= 0, \\ 10 - x - y &= 0. \end{aligned}$$

To solve, first solve the first two equations for λ and set the expressions equal to each other.

$$\lambda = y = x.$$

Now solve the third equation (the budget constraint) for either x or y in terms of the other.

$$y = 10 - x.$$

Finally, substitute this equation into the earlier equation:

$$10 - x = x.$$

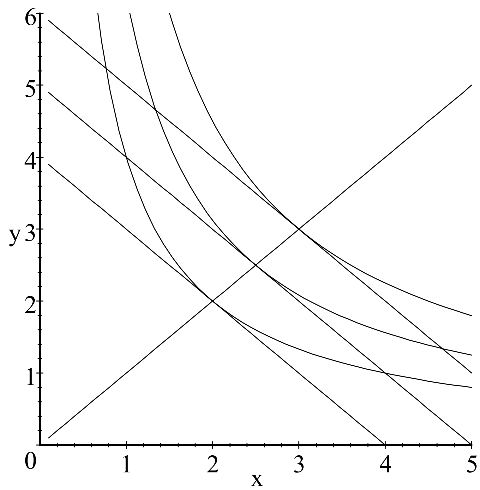
Solving, we get $x = 5$. Plug this into the budget constraint to get $y = 5$. You can check that the marginal rate of substitution equals the price ratio at the point (5,5).

Derivation of the Demand Function

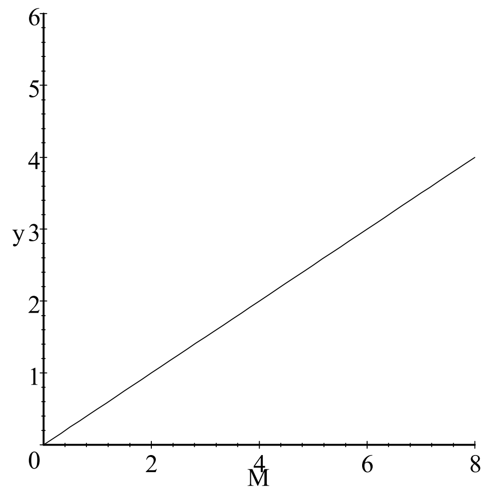
The solution to utility maximization problem (3) gives the consumer's choice of x and y, as a function of prices and income, which we denote by $x^*(p_x, p_y, M)$ and $y^*(p_x, p_y, M)$. These are known as the *generalized demand functions*.

We will now explore these functions in more detail, first graphically and then by computing an example.

The demand for, say, good y as a function of income, holding prices constant, is called the *Engel Curve*. This is related to the *income-consumption curve*, the set of consumption bundles chosen as income varies, holding prices constant.



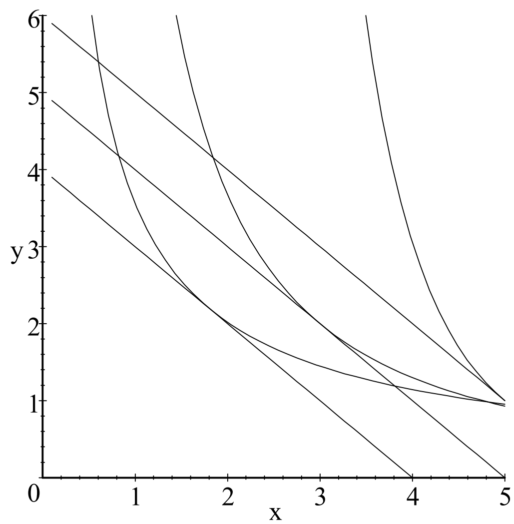
The Income-Consumption Curve



The Engel Curve

In the above diagrams, we have $p_x = 1$ and $p_y = 1$. The three budget lines correspond to incomes of 4, 5, and 6.

If the quantity demanded increases as income increases (Engel curve slopes up), the good is a *normal good*. Otherwise (if the Engel curve slopes down), the good is an *inferior good*.

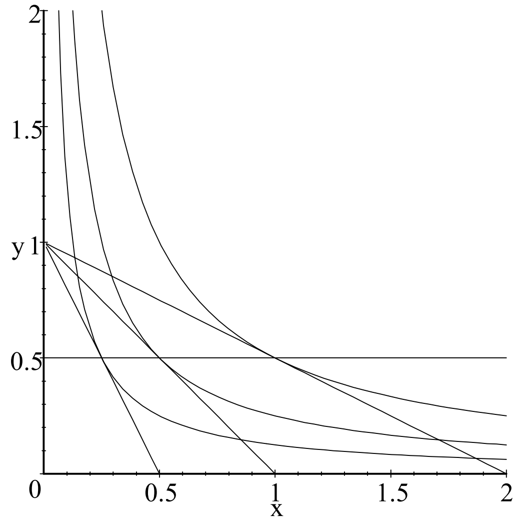


An Inferior Good

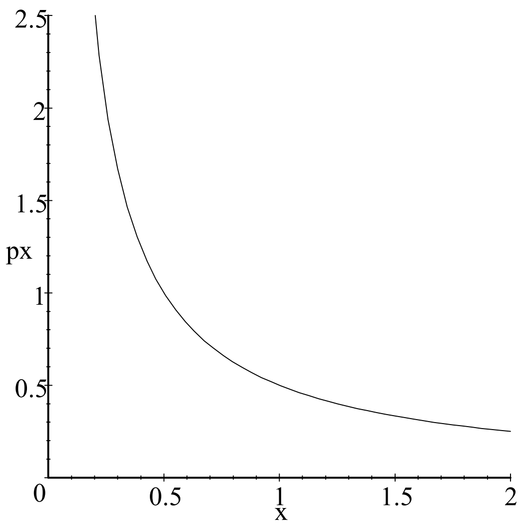
The set of consumption bundles chosen as, say, p_x varies, holding M and p_y constant, is called the *price-consumption curve*.

The demand for x, as a function of p_x, holding M and p_y constant, is called the *ordinary demand function*.

For the following diagrams, fix $M = 1$ and $p_y = 1$.

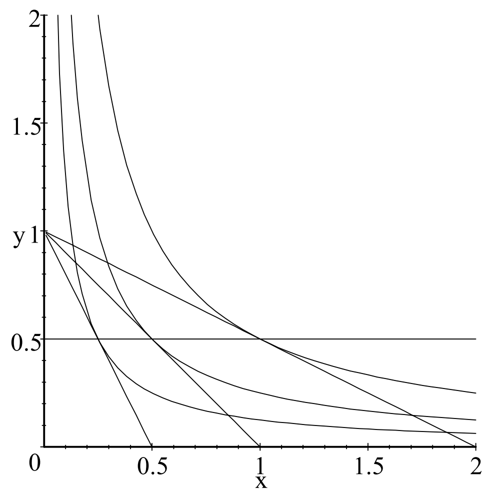


The Price-Consumption Curve

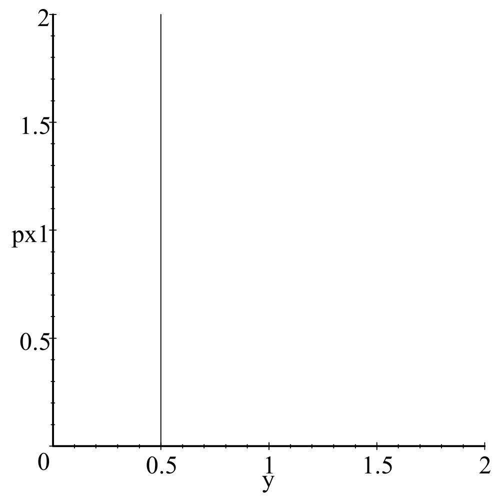


The Ordinary Demand Curve

The *cross-price demand function* is the demand for (say) good y, as a function of p_x, holding M and p_y constant.



Price-Consumption



Cross-Price Demand

Good y is a *gross substitute* for x if the quantity of y demanded increases as p_x increases, holding M and p_y constant (cross-price demand function is upward sloping). Example: red wine and white wine

Good y is a *gross complement* for x if the quantity of y demanded decreases as p_x increases, holding M and p_y constant (cross-price demand function is downward sloping). Example: red wine and steak