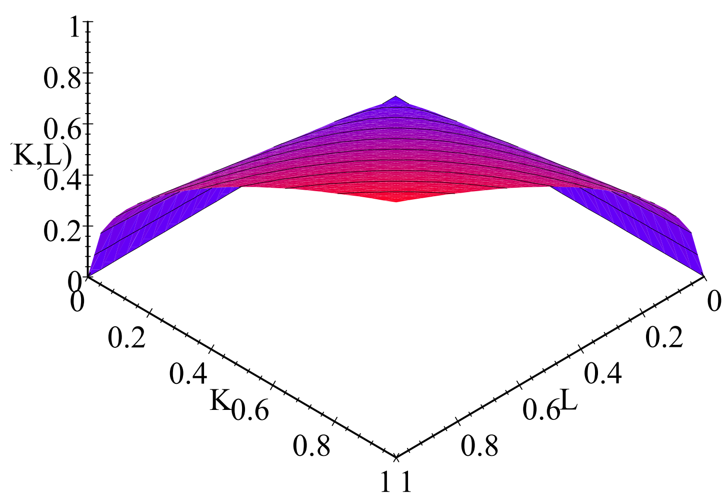
Total, Average, and Marginal Physical Products

Hold all but one of the inputs fixed (say, fix $K = \overline{K}$). Perhaps we are in a short run situation, or perhaps we are just focusing on the effect of changing L.

The *total product of labor* is given by the function, $x = f(L; \overline{K})$. We can graph this as a cross-section of the production function.



Cobb-Douglas Production Function

The average product of labor is defined as

$$AP_L = \frac{f(L; \overline{K})}{L}.$$

The marginal product of labor is defined as

$$MP_L = \frac{\partial f(L; \overline{K})}{\partial L}.$$

Cobb-Douglas example: $x = K^\alpha L^\beta$

$$\begin{aligned}
AP_L &= \frac{K^\alpha L^\beta}{L} = K^\alpha L^{\beta-1} \\
MP_L &= \frac{\partial K^\alpha L^\beta}{\partial L} = \beta K^\alpha L^{\beta-1} \\
AP_K &= \frac{K^\alpha L^\beta}{K} = K^{\alpha-1} L^\beta \\
MP_K &= \frac{\partial K^\alpha L^\beta}{\partial K} = \alpha K^{\alpha-1} L^\beta
\end{aligned}$$

Connection between MRTS and Marginal Products

We can get an expression for the MRTS by treating output as fixed and totally differentiating the equation

$$f(K, L) = \overline{x}.$$

By holding x fixed, we are remaining on the same isoquant as we vary K and L. This yields

$$\frac{\partial f}{\partial L} dL + \frac{\partial f}{\partial K} dK = 0.$$

Rearranging, we have

$$-\frac{dK}{dL} \mid_{f(K,L)=\overline{x}} = \frac{\frac{\partial f}{\partial L}}{\frac{\partial f}{\partial K}}. \quad (1)$$

The left side of (1) is the MRTS, and the right side is the ratio of marginal products.

$$MRTS = \frac{MP_L}{MP_K}. \quad (2)$$

Diminishing Marginal Returns

Diminishing marginal returns (to labor) occur when the marginal product (of labor) eventually falls as L increases.

$$\frac{\partial MP_L}{\partial L} < 0$$

That is, labor is less and less productive at the margin, as L increases. It can be shown that with CRS or DRS, we must have diminishing marginal returns to each input.

Cobb-Douglas example: $x = K^\alpha L^\beta$

$$\begin{aligned} MP_L &= \frac{\partial K^\alpha L^\beta}{\partial L} = \beta K^\alpha L^{\beta-1} \\ \frac{\partial MP_L}{\partial L} &= (\beta-1)\beta K^\alpha L^{\beta-2} \end{aligned}$$

Thus, we have diminishing marginal returns to labor when $\beta < 1$. Constant returns to scale and diminishing marginal returns can easily coexist.