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Equations for the Final

The utility maximization problem is

$$\max u(x,y)$$
$$\text{subject to} \quad : \quad p_x x + p_y y = M$$

The Lagrangean approach transforms a constrained optimization problem into an unconstrained problem of choosing x, y, and the Lagrange multiplier, λ, to maximize

$$L = u(x,y) + \lambda[M - p_x x - p_y y].$$

By differentiating L with respect to x, y, and λ, and setting the derivatives equal to zero, the resulting first order conditions are:

$$\frac{\partial u}{\partial x} - \lambda p_x = 0, \frac{\partial u}{\partial y} - \lambda p_y = 0, \text{ and } M - p_x x - p_y y = 0.$$

$x^*(p_x, p_y, M)$ and $y^*(p_x, p_y, M)$ are known as the *generalized demand functions.*

The *production function* specifies the most output that can be produced with a given combination of inputs, based on the technology available to the firm.

$$x = f(K,L)$$

Technological efficiency occurs if the firm is on its "production frontier." That is, it is impossible to achieve more output with the same inputs. We typically assume that marginal products are positive

$$\frac{\partial f(K,L)}{\partial K} > 0 \text{ and } \frac{\partial f(K,L)}{\partial L} > 0.$$

A *production isoquant* is a curve describing the set of capital-labor combinations yielding the same output, according to the production function.

The *marginal rate of technical substitution* is defined to be the negative of the slope of the isoquant. The MRTS is the rate at which the firm would be willing to give up capital in exchange for labor. We assume diminishing MRTS.

In the *long run*, all inputs are variable, so the firm can choose any combination of capital and labor. In the *short run*, at least one input is fixed and cannot be varied.

Returns to Scale

For long run decisions, we may be interested in what happens as we vary all of the inputs simultaneously.

The production function exhibits *decreasing returns to scale* if, for $\theta > 1$, we have

$$f(\theta K, \theta L) < \theta f(K, L).$$

The production function exhibits *constant returns to scale* if, for $\theta > 1$, we have

$$f(\theta K, \theta L) = \theta f(K, L).$$

The production function exhibits *increasing returns to scale* if, for $\theta > 1$, we have

$$f(\theta K, \theta L) > \theta f(K, L).$$

Hold all but one of the inputs fixed (say, fix $K = \overline{K}$). The *total product of labor* is given by the function, $x = f(L; \overline{K})$. The average product of labor is defined as

$$AP_L = \frac{f(L; \overline{K})}{L}.$$

The marginal product of labor is defined as

$$MP_L = \frac{\partial f(L; \overline{K})}{\partial L}.$$

Cobb-Douglas example: $x = K^\alpha L^\beta$

$$\begin{aligned}
AP_L &= \frac{K^\alpha L^\beta}{L} = K^\alpha L^{\beta-1} \\
MP_L &= \frac{\partial K^\alpha L^\beta}{\partial L} = \beta K^\alpha L^{\beta-1} \\
AP_K &= \frac{K^\alpha L^\beta}{K} = K^{\alpha-1} L^\beta \\
MP_K &= \frac{\partial K^\alpha L^\beta}{\partial K} = \alpha K^{\alpha-1} L^\beta
\end{aligned}$$

$$MRTS = \frac{MP_L}{MP_K}.$$

Diminishing marginal returns (to labor) occur when the marginal product (of labor) eventually falls as L increases.

$$\frac{\partial MP_L}{\partial L} < 0$$

Cost minimization subject to an output constraint:

$$\begin{aligned} & \min wL + rK \\ \text{subject to } f(K,L) &= x \end{aligned} \tag{1}$$

Set up the Lagrangean,

$$Lagr. = wL + rK + \lambda[x - f(K,L)]$$

The first order conditions are

$$\begin{aligned} \frac{\partial Lagr.}{\partial L} &= 0 = w - \lambda \frac{\partial f}{\partial L} \\ \frac{\partial Lagr.}{\partial K} &= 0 = r - \lambda \frac{\partial f}{\partial K} \\ \frac{\partial Lagr.}{\partial \lambda} &= 0 = x - f(K,L) \end{aligned}$$

Solving, we have

$$\lambda = \frac{w}{MP_L} = \frac{r}{MP_K}. \tag{2}$$

This is the condition that the MRTS equals the input price ratio. It is also the condition that the marginal cost of producing output using any variable input must be the same. To solve for the generalized (conditional) input demand functions, $L^*(w,r,x)$ and $K^*(w,r,x)$, solve (1) and (2) for K and L.

Deriving the Total Cost Function

Since the conditional input demand functions tell us the amounts of K and L to choose in order to produce x units of output, evaluating the cost of these inputs gives us the total cost of producing x, assuming the firm chooses its inputs optimally to minimize costs.

$$TC^* = wL^* + rK^*$$

Since TC^* is computed based on all inputs being variable, this is the long run total cost function, LRTC. We can also define *long run average cost and long run marginal cost*:

$$LRAC = \frac{LRTC}{x} \quad \text{and} \quad LRMC = \frac{d(LRTC)}{dx}$$

Suppose that we have a fixed amount of capital, $\overline{K}$. To find the conditional labor demand, we invert the short run production function by solving $x = f(L;\overline{K})$ for L. This gives us $L(x;\overline{K})$, which does not depend on input prices, since this amount of labor is *required* in order to produce x units of output.

Then the short run total cost function is given by

$$SRTC(x; \overline{K}, w, r) = wL(x; \overline{K}) + r\overline{K}.$$

We can also define the following:

$$\begin{aligned}
SRTC(x; \overline{K}, w, r) &= wL(x; \overline{K}) + r\overline{K} \\
SRVC &= wL(x; \overline{K}) \\
FC &= r\overline{K} \\
SRATC &= \frac{wL(x; \overline{K})}{x} + \frac{r\overline{K}}{x} \\
SRAVC &= \frac{wL(x; \overline{K})}{x} \\
AFC &= \frac{r\overline{K}}{x} \\
SRMC &= \frac{d(SRTC)}{dx} \\
&= \frac{d(SRVC)}{dx} = w\frac{dL(x; \overline{K})}{dx}
\end{aligned}$$

Profit Maximization by a Competitive Firm

Having derived the total cost function (either long run or short run), we can now solve for the profit-maximizing output level, x^*. Given x^*, we can then compute the unconditional demand for inputs such as capital and labor.

A perfectly competitive firm's profit function is $\pi(x) = p_x x - TC(x)$, and the condition for (interior) profit maximization is

$$p_x = MC(x).$$

To make sure that we have a profit maximum (and not a minimum!), use the second-order condition

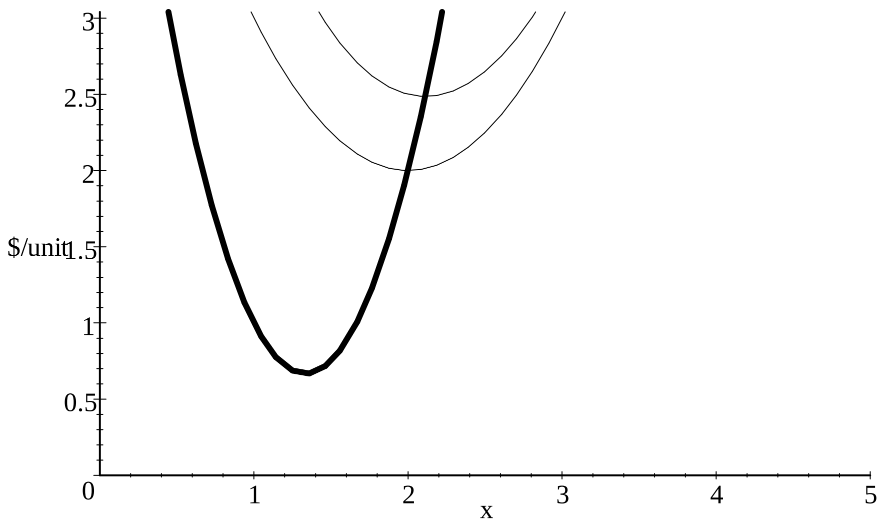
$$\frac{d^2\pi}{dx^2} = -\frac{dMC(x)}{dx} < 0,$$

which says that the marginal cost curve should be upward sloping (increasing in x).

The interpretation of $p_x = MC(x)$, is that the firm is choosing the output level at which marginal cost equals the price, not that the firm is choosing the price.

At any price, the firm will choose x so that $p_x = MC(x)$. Thus, a competitive firm's marginal cost curve is its supply curve.

The Short Run Supply Curve



Competitive Firm's Short Run Supply Curve

If p_x is greater than the minimum SRATC, the firm receives positive economic profits by selecting the profit maximizing x.

If p_x is greater than the minimum SRAVC but less than the minimum SRATC, the firm receives negative profits, but still should produce.

If p_x is less than the minimum SRAVC, the firm cannot even cover its variable costs at any level of output, and should shut down.

Notice that the decision of what to produce, and whether to shut down, does not depend on the level of fixed costs.

Complete solution of the firm's problem in the *short run*: an example $f(K, L) = K^{1/2}L^{1/2}$, where K is fixed at $\overline{K}$.

Approach #2: Set SRMC equal to the price.
From $x = \overline{K}^{1/2}L^{1/2}$, we solve for L:

$$L^{1/2} = \frac{x}{\overline{K}^{1/2}}, \text{ which implies } L = \frac{x^2}{\overline{K}}.$$

Therefore, we have

$$SRTC = w\frac{x^2}{\overline{K}} + r\overline{K}.$$

Taking the derivative to get SRMC, and setting it equal to the price, we have

$$SRMC = \frac{2wx}{\overline{K}} = p_x.$$

Notice that $SRAVC = wx/\overline{K}$, so the minimum SRAVC occurs at x=0. We do not have to worry about shutting down.

Demand for Inputs

Since profits are

$$\pi = p_x f(K, L) - wL - rK,$$

the profit maximizing choice of L satisfies

$$\frac{\partial \pi}{\partial L} = 0 = p_x MP_L - w.$$

Thus, the firm hires labor until the amount of revenues generated by one more labor hour equals the hourly wage. Treating K as fixed, so MP_L depends only on L, the equation for the firm's labor demand curve is $w = p_x MP_L$.

Similarly, holding L fixed, the firm's demand for capital is $r = p_x MP_K$.

The Firm's Long Run Profit Maximization Problem

example: $f(K, L) = K^\alpha L^\alpha$

Derive the LRMC curve.

First derive LRTC by setting up the Lagrangean expression for the cost minimization problem, and getting as first order conditions:

$$x = K^\alpha L^\alpha \text{ and}$$

$$\frac{MP_L}{MP_K} = \frac{\alpha K^\alpha L^{\alpha-1}}{\alpha K^{\alpha-1} L^\alpha} = \frac{w}{r},$$

which simplifies to

$$L = \frac{rK}{w}.$$

Now plug into the production function to get

$$x = K^\alpha (\frac{rK}{w})^\alpha = K^{2\alpha} (\frac{r}{w})^\alpha.$$

Simplifying, we have

$$x^{1/\alpha} = (\frac{rK^2}{w}).$$

Solving for K as a function of x, we have

$$K = (\frac{w}{r})^{1/2} x^{1/2\alpha}.$$

From the MRTS condition, we have

$$L = (\frac{r}{w})^{1/2} x^{1/2\alpha}.$$

Therefore, the LRTC function is

$$\begin{aligned} LRTC &= w(\frac{r}{w})^{1/2} x^{1/2\alpha} + r(\frac{w}{r})^{1/2} x^{1/2\alpha} \\ &= 2(wr)^{1/2} x^{1/2\alpha} \end{aligned}$$

Differentiating with respect to x, we have the marginal cost function,

$$LRMC = \frac{(wr)^{1/2}}{\alpha} x^{\frac{1}{2\alpha}-1}.$$

Case 1: $\alpha < \frac{1}{2}$. This is the case of decreasing returns to scale, and LRMC is upward sloping. The formula for the supply function is

$$p_x = \frac{(wr)^{1/2}}{\alpha} x^{\frac{1}{2\alpha}-1}.$$

Given prices, we solve for x.

Case 2: $\alpha = \frac{1}{2}$. This is the case of constant returns to scale, and we have $LRMC = 2(wr)^{1/2}$.

The marginal cost curve is flat, and equal to LRAC, independent of x. In other words, the firm's supply curve is flat.

Case 3: $\alpha > \frac{1}{2}$. This is the case of increasing returns to scale, and LRMC is downward sloping. The firm is not profit maximizing by choosing x such that $LRMC = p_x$. In fact, no matter what p_x is, LRAC will fall below p_x for large enough x. Higher and higher x will only increase the profit per unit, so overall profits are infinite! There is an inconsistency between *increasing returns to scale* and *perfect competition.*

Short Run Market Equilibrium

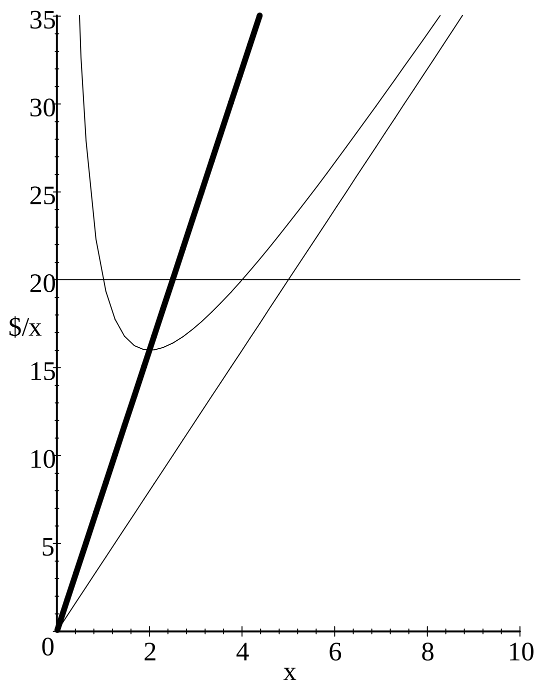
The market supply curve is found by horizontally adding the supply curves of individual firms. If there are m firms, we have

$$X^s(p_x) = \sum_{j=1}^{m} x_j(p_x).$$

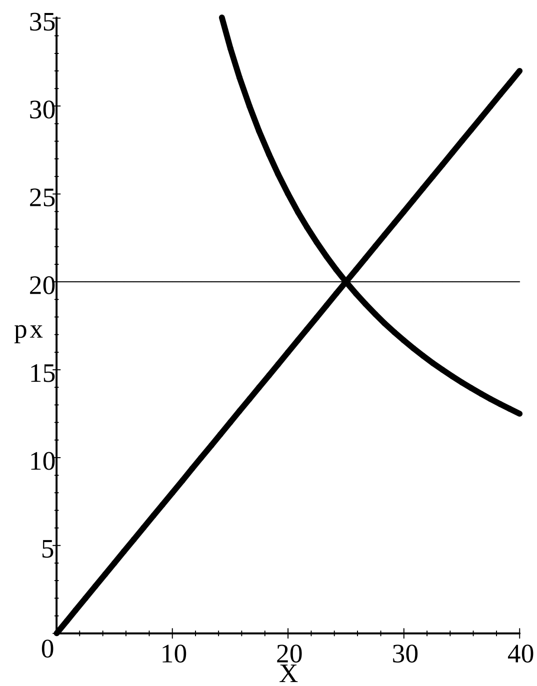
Just as we can talk about the elasticity of demand, the price elasticity of *supply* is defined as

$$\varepsilon^s = \frac{dX^s}{dp_x}\frac{p_x}{X^s}.$$

A *short run market equilibrium* is defined to be a price and a quantity of output, where (1) demand is derived from utility maximization, (2) supply is derived from firms choosing variable inputs and output to maximize profits, and (3) supply equals demand.



Firm's Supply Curve



Market Supply Curve

Long Run Equilibrium

In the long run, firms can adjust all inputs. More importantly, new firms can enter the market in search of profit opportunities, and existing firms can exit the market if they are receiving negative profits.

To make things simple, we will assume that there is free entry and exit, and that all firms have the same technology and therefore the same cost functions.

We will also assume that good x is a constant cost industry, meaning that input prices do not change as the industry (market) output varies. This assumption is needed to insure that an individual firm's cost curves do not change as the market expands or contracts.

If firms are receiving profits, entry will occur, driving the price down towards zero profits. If firms are making losses, exit will occur, raising the price up towards zero profits.

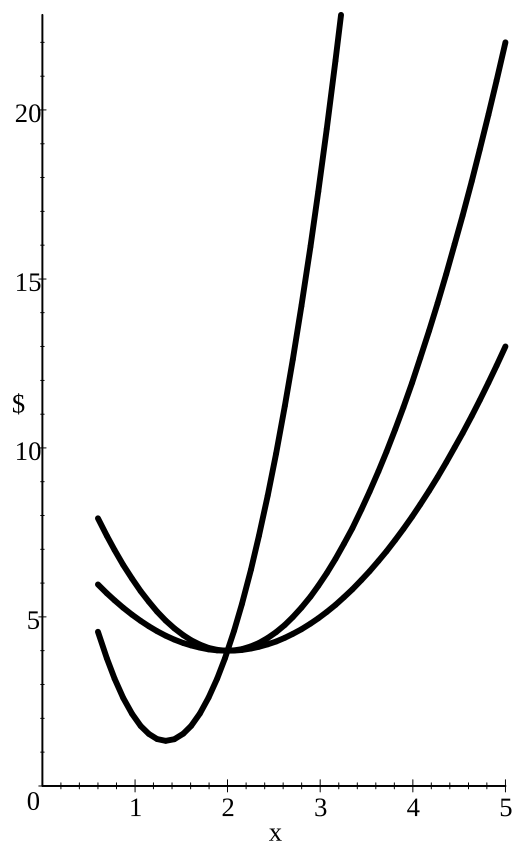
In long run equilibrium, all firms receive zero economic profits. Think of this as "normal" profits. Remember that firms that own their own capital should receive a market rate of return on their capital in order to be breaking even (including opportunity cost).

$$p_x^{**} = \min LRAC$$

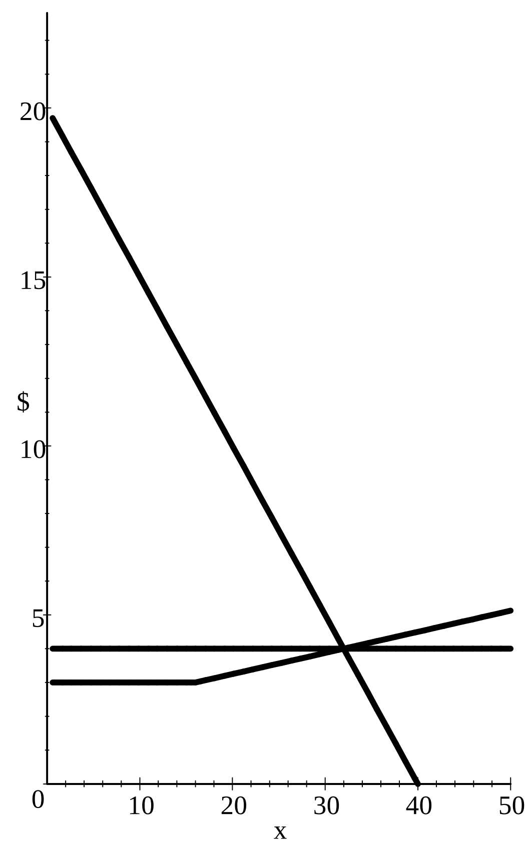
Being in long run equilibrium also entails being in short run equilibrium.

With constant returns to scale, any output level is efficient, so long run equilibrium does not pin down the output of each firm or the number of firms.

With a U-shaped LRAC curve, there is a single value of x^{**} that minimizes LRAC, and all firms are forced to produce x^{**} in order to break even. The number of firms that the market can support is X^{**}/x^{**}.



U-shaped LRAC



Long Run Eq.

Consumer and Producer Surplus

The height of the (inverse) demand curve at a particular quantity, X, is the price that some consumer is willing to pay for that unit of output. It is the market's *marginal willingness to pay or marginal benefit* to society of consumption.

That individual consumer receives a surplus equal to her willingness to pay minus the price she pays.

The total *consumer surplus* is the sum of the surplusses generated by all X units, which is the area below the demand curve and above the price line, over the range from 0 to X.

The height of the (inverse) supply curve at a particular quantity, X, is the price at which some firm is willing to sell that unit of output. Since it also is the marginal cost of production for the firm selling the output, the height up to the supply curve is society's *marginal cost* of providing the consumption.

That individual firm receives a surplus from that unit equal to the price received minus the marginal cost of production.

The total *producer surplus* is the sum of the surplusses generated by all X units, which is the area below the price line and above the supply curve, over the range from 0 to X.

The total benefit to society of X units of output is the sum of the consumer surplus and the producer surplus.

$$\begin{aligned} \text{Total Surplus} &= \text{Cons. Surplus} + \text{Producer Surplus} \\ &= \int (\text{Value} - \text{price}) + (\text{price} - \text{cost}) \\ &= \text{Value to buyers} - \text{Cost to sellers.} \end{aligned}$$

To choose the output that maximizes the total benefit to society (net of costs), we want the marginal benefit to consumers to equal the marginal cost of production. This is exactly where the demand curve and the supply curve intersect.

Thus, efficiency requires the output to be where the demand and supply curves intersect. It also requires the consumption to be done by the consumers with the highest willingness to pay, and the production to be done by the firms with the lowest costs. A market system yields an efficient outcome.

Monopoly

Reasons for monopoly: (1) patent, government protection, (2) ownership of a key input, (3) natural monopoly, because LRAC falls until it crosses the market demand curve.

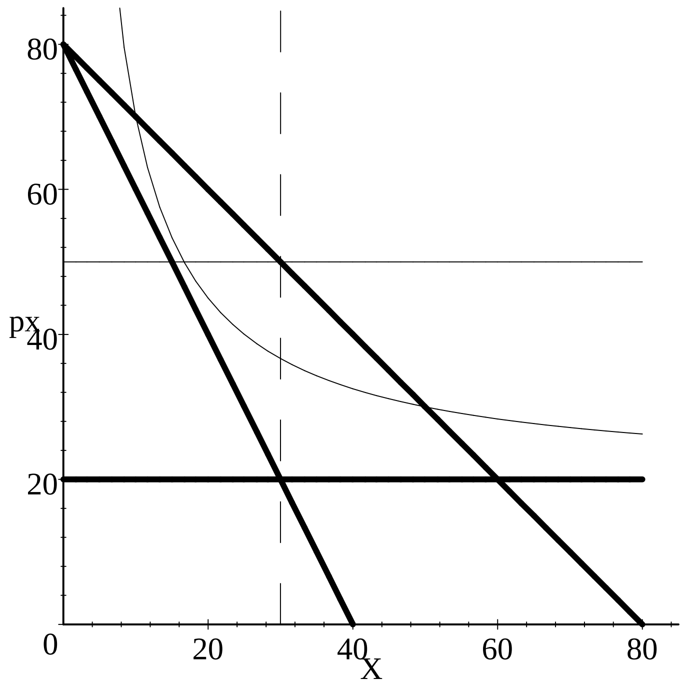
$$\max \pi = p_x x - TC(x)$$

yields the first order condition

$$p_x + x\frac{dp_x}{dx} - MC(x) = 0.$$

The first two terms comprise marginal revenue. When a monopolist produces more output, it takes into account the fact that the price it can charge is lower.

We can think of the monopolist as choosing p_x or as choosing x.



Monopoly

The monopoly quantity satisfies $MR(x) = MC(x)$. The monopolist charges the price the market will pay for that quantity.

example:

$$\begin{aligned} X^d &= 80 - p_x \\ LRTC &= 500 + 20x. \end{aligned}$$

First, we convert the demand curve into an inverse demand, price as a function of x.

$$p_x = 80 - x.$$

Next, derive the total revenue function.

$$TR(x) = (80 - x)x.$$

Finally, set marginal revenue equal to marginal cost and solve for x.

$$\begin{aligned} MR(x) &= 80 - 2x = 20 = MC(x). \\ x &= 30, \text{ which implies } p_x = 50. \end{aligned}$$

The monopolist receives profit equal to $50(30) - 500 - 20(30) = 400$. There is a deadweight loss triangle, since the socially optimal quantity is 60, where demand and marginal cost intersect.

What is wrong with monopoly?

The monopoly keeps prices high by restricting output. Deadweight loss.

The monopoly may waste additional resources by trying to maintain their market power. (Lobbying governments, erecting barriers to entry)

What can society do about monopoly?

Antitrust Laws allow for lawsuits if one firm is monopolizing an industry, or if a group of firms conspires to restrain competition (for example, if they were to form a cartel). Mergers to form monopolies can be prevented. Antitrust laws will not help the problem of natural monopoly, where we want there to be just one firm. Competition or breakup would lead to duplication of costs.

1. Rate of return regulation. Allow the firm to receive a normal rate of return on the firm's capital, so that economic profits are zero. Choose x such that $LRAC = p_x$.

Problems with rate of return regulation:

(a) A slightly inefficient quantity is chosen, since the efficient quantity would be where $MC = p_x$.

(b) Cost of having a regulatory commission, time delays, danger of the firm capturing the regulation process (revolving door theory).

(c) The firm might be able to shift costs from the nonregulated part of the firm to the regulated part of the firm (where the costs are added to the capital base).

(d) If the rate of return on capital is set too high, there is an incentive to overinvest in capital.

2. Force the firm to price at marginal cost, yielding an efficient amount of output.

Problems:

(a) Since marginal cost is below average cost, the firm makes negative profits. This scheme is not viable.

(b) Cost of having a regulatory commission, time delays, danger of the firm capturing the regulation process (revolving door theory).

3. Marginal cost pricing, but with a subsidy to the firm, allowing it to break even.

Problems:

(a) It isn't fair for noncustomers to pay towards the subsidy. On the other hand, if the customers must pay, we must raise p_x until the firm can break even, and we are back with rate of return regulation.

4. Have firms bid for the right to become the (unregulated) monopolist for a period of time. That way, the monopoly profits go back to the government.

Problems:
(a) Pinning down and committing to the details of the agreement is difficult.
(b) After the first auction, the incumbent insider has an advantage.
(c) Now the monopolist will drastically raise the price and restrict the supply, so we are back to the huge deadweight loss of an unregulated monopoly. Rather than bidding a flat amount of money, the firms should bid in terms of the price they will charge.

Price discrimination occurs when a firm takes advantage of market power to sell the same product in two different markets at two different prices. Sometimes price discrimination can be subtle, such as coupon pricing or the pricing of trade-ins.

Oligopoly

Game Theory is designed to address strategic interactions in oligopoly.

A *game* is a set of players, a set of feasible strategies for each player, and an outcome function which specifies each player's payoff as a function of the strategies selected.

A *Nash equilibrium* of a game is a choice of strategies, one for each player, for which no player can receive a higher payoff by deviating to another strategy, holding the other players' strategies constant.

Example: Prisoner's Dilemma

		Pepsi	
		high	low
Coke	high advertising	1,1	3,0
	low advertising	0,3	2,2

Equilibrium is (high,high). No matter what the other player does, you are better off with high.

This is not purely confrontational. Both players could benefit by signing a binding contract.

Cournot (Quantity) Competition

For $i = 1, 2$, the strategic choice for firm i is its output quantity, x_i. Given the outputs of the two duopolists, the price is determined by the inverse demand curve, $p_x(x_1 + x_2)$.

Given how profits depend on x_1 and x_2, we can solve for the Nash equilibrium, where each firm's output is a best response to the other firm's output.

Example: Each firm has the cost function, $TC = 20x_i$, so average and marginal cost equals 20.

The market demand curve and inverse demand curves are

$$\begin{aligned} x &= 80 - p_x \\ p_x &= 80 - x = 80 - x_1 - x_2. \end{aligned}$$

Let us look at firm 1's optimization problem. Profits are

$$\pi_1 = (80 - x_1 - x_2)x_1 - 20x_1.$$

Firm 1's best response to x_2 (or what it believes will be x_2) is found by differentiating π_1 with respect to x_1.

$$\frac{\partial \pi_1}{\partial x_1} = 0 = 80 - 2x_1 - x_2 - 20.$$

Solving for x_1, we have

$$x_1 = \frac{60 - x_2}{2}.$$

This is firm 1's *reaction function*, because it shows how firm 1 optimally reacts to expectations of its rival's strategy.

The same procedure allows us to calculate the reaction function for firm 2:

$$x_2 = \frac{60 - x_1}{2}.$$

The *Nash equilibrium* occurs when neither firm has an incentive to change its strategy, so we are on both reaction functions. Solving, we have

$$\begin{aligned} x_1 &= 20, \quad x_2 = 20, \text{ and therefore} \\ p_x &= 40, \quad \pi_1 = \pi_2 = 400. \end{aligned}$$

Notice that the price is between the monopoly price, 50, and the competitive price, 20.

Analysis of a Cartel

If the two firms could conspire, they could increase profits by reducing their output to 15 (half the monopoly output). Then we would have $p_x = 50$, $\pi_1 = \pi_2 = 450$.

However, this is not a Nash equilibrium, and there is a tendency to cheat. From (2), firm 1's best response to $x_2 = 15$ is 22.5, which would yield $\pi_1 = 506.25$.

Repeated Quantity Competition

Oligopolies usually compete repeatedly over time, which changes the game. A strategy now specifies your output, as a function of the observed *history* of outputs. This opens the possibility of rewards and punishments.

Here is a strategy for firm 1, where $x_1(t)$ is firm 1's output in round t:

$$\begin{aligned} x_1(t) &= 15 \quad \text{if} \quad x_2(1) = \cdots = x_2(t-1) = 15 \\ x_1(t) &= 20 \quad \text{otherwise.} \end{aligned}$$

In other words, firm 1 produces the cartel output as long as firm 2 has never cheated, but reverts to the one-shot Nash equilibrium if firm 2 ever cheats.

If both firms adopt the above "trigger" strategy, that is a Nash equilibrium. A firm that deviates to an output other than 15 at best receives profits of 506.25 during the round it cheats. However, its profits are reduced from 450 to 400 every round afterwards.

Price Competition

Now firm i's strategy is its choice of price, p_x^i. Given the prices, the firm offering the lower price supplies whatever is demanded at that price. If both firms set the same price, they split the market.

For our example, the solution is marginal cost pricing, $p_x^1 = p_x^2 = 20$. Raising your price loses all customers, and lowering your price yields negative profits. Fierce competition to undercut your rival.

If the two firms frequently bid against each other, tacit collusion is again possible.

Math Fact: the quotient rule for derivatives.

$$\frac{d\left(\frac{u(x)}{v(x)}\right)}{dx} = \frac{v(x)\frac{du(x)}{dx} - u(x)\frac{dv(x)}{dx}}{[v(x)]^2}.$$