Answers to Rothschild-Stiglitz Problem

Consider the following economy with one physical commodity per state of nature and three consumers, each of whom seek to maximize expected utility. For i = 1,2, consumer i is risk averse, with utility of certain consumption given by $u_i(x_i) = \log(x_i)$. For i = 1,2, consumer i is endowed with 1 unit of consumption when she does not have an accident, 0 units of consumption when she has an accident.

Consumer 1 is a "low risk" consumer, with a probability of an accident equal to 1/3. Consumer 2 is a "high risk" consumer, with a probability of an accident equal to ½. Consumer 1 having an accident and consumer 2 having an accident are independent events.

Consumer 3 is risk neutral, with utility of certain consumption given by $u_3(x_3) = x_3$, and has an endowment of 2 units of consumption in all states of nature. For parts (i) and (ii), assume that consumer 3 knows that consumer 1 is low risk and that consumer 2 is high risk, so information is symmetric.

(i) Define a competitive equilibrium for the economy with complete state-contingent commodity markets. Specify how many states of nature there are and the probability of each state.

(ii) Calculate the competitive equilibrium price vector and allocation for the economy with complete state-contingent commodity markets.

For parts (iii) and (iv), suppose that consumers 1 and 2 are in a Rothschild-Stiglitz world. That is, instead of consumer 3, there are many risk-neutral firms who cannot observe which consumer is low risk and which consumer is high risk. Firms compete by offering contracts, specifying consumption a policyholder receives when she has an accident and consumption she receives when she does not have an accident.

(iii) Find the pooling contract providing full insurance (consumption is independent of whether the policyholder has an accident) and yielding zero expected profits when both consumers accept the contract.

(iv) Show that the pooling contract of part (iii), call it α, is not an equilibrium. That is, find another contract, β, that would be chosen by the low risk consumer, would not be chosen by the high risk consumer, and yields expected profits for the firm offering the contract. Be as explicit as you can.

ANSWER: (i) There are 4 states of nature, corresponding to the four possible outcomes: consumer 1 does not have an accident and consumer 2 does not have an accident, and so on. Call the four states gg, gb, bg, and bb, where the first letter (g for good, b for bad) describes whether consumer 1 had an accident and the second letter describes whether consumer 2 had an accident. The probabilities of the four states are: (1/3, 1/3, 1/6, 1/6).

A competitive equilibrium is a price vector, $(p^{gg}, p^{gb}, p^{bg}, p^{bb})$, and an allocation, $\{x_i^s\}_{i=1,2,3;s=gg,gb,bg,bb}$ such that

(1) x_1 solves:

$$\begin{aligned}&\max\ 1/3\ \log(x_1^{gg}) + 1/3\log(x_1^{gb}) + 1/6\log(x_1^{bg}) + 1/6\log(x_1^{bb})\\ &\text{subject to}\quad p^{gg}x_1^{gg} + p^{gb}x_1^{gb} + p^{bg}x_1^{bg} + p^{bb}x_1^{bb} = p^{gg} + p^{gb}\\ &x_1 \geq 0.\end{aligned}$$

(2) x_2 solves:

$$\begin{aligned}&\max\ 1/3\ \log(x_2^{gg}) + 1/3\log(x_2^{gb}) + 1/6\log(x_2^{bg}) + 1/6\log(x_2^{bb})\\ &\text{subject to}\quad p^{gg}x_2^{gg} + p^{gb}x_2^{gb} + p^{bg}x_2^{bg} + p^{bb}x_2^{bb} = p^{gg} + p^{bg}\\ &x_2 \geq 0.\end{aligned}$$

(3) x_3 solves:

$$\begin{aligned}&\max\ 1/3\ (x_3^{gg}) + 1/3(x_3^{gb}) + 1/6(x_3^{bg}) + 1/6(x_3^{bb})\\ &\text{subject to}\quad p^{gg}x_3^{gg} + p^{gb}x_3^{gb} + p^{bg}x_3^{bg} + p^{bb}x_3^{bb} = 2(p^{gg} + p^{gb} + p^{bg} + p^{bb})\\ &x_3 \geq 0.\end{aligned}$$

(4) Market clearing:

$$\begin{aligned}x_1^{gg} + x_2^{gg} + x_3^{gg} &= 4\\ x_1^{gb} + x_2^{gb} + x_3^{gb} &= 3\\ x_1^{bg} + x_2^{bg} + x_3^{bg} &= 3\\ x_1^{bb} + x_2^{bb} + x_3^{bb} &= 2\end{aligned}$$

(ii) Solving for consumer 1's demand functions, pick a commodity, say bb, and there will be a marginal rate of substitution condition for each of the other three commodities in relation to bb:

$$\frac{1/3x_1^{bb}}{1/6x_1^{gg}} = \frac{p^{gg}}{p^{bb}},\quad \frac{1/3x_1^{bb}}{1/6x_1^{gb}} = \frac{p^{gb}}{p^{bb}},\quad \frac{x_1^{bb}}{x_1^{bg}} = \frac{p^{bg}}{p^{bb}}.$$

Cross-multiplying, we have:

$$p^{gg}x_1^{gg} = p^{gb}x_1^{gb} = 2p^{bb}x_1^{bb} \ \text{ and }\ p^{bg}x_1^{bg} = p^{bb}x_1^{bb}.$$

Substituting the above equations into the budget constraint, we have: $6p^{bb}x_1^{bb} = p^{gg} + p^{gb}$.
This yields the demand functions:

$$x_1^{bb} = (p^{gg} + p^{gb})/6p^{bb} = x_1^{bg} \text{ and}$$
$$x_1^{gg} = (p^{gg} + p^{gb})/3p^{bb} = x_1^{gb}.$$

Consumer 2 has the same marginal rate of substitution equations and a similar budget equation. Similar analysis yields the demand functions:

$$x_2^{bb} = (p^{gg} + p^{bg})/6p^{bb} = x_2^{bg} \text{ and}$$
$$x_2^{gg} = (p^{gg} + p^{bg})/3p^{bb} = x_2^{gb}.$$

Consumer 3's marginal rate of substitution conditions yield:

$$p^{gg}/p^{bb} = 2,\quad p^{gb}/p^{bb} = 2,\quad \text{and}\quad p^{bg}/p^{bb} = 1.$$

Normalizing $p^{bb} = 1/6$, we have p = (1/3, 1/3, 1/6, 1/6). Intuitively, because consumer 3 is risk neutral and we have an interior solution, prices are proportional to probabilities. Plugging these prices into the demand function for consumers 1 and 2, we get the consumption bundles x_1 = (2/3, 2/3, 2/3, 2/3) and x_2 = (½, ½, ½, ½). Not surprisingly, both consumers receive full and fair insurance. We have not yet used market clearing conditions, but we will do so to determine consumer 3's allocation: $x_3 = (17/6, 11/6, 11/6, 5/6)$.

(iii) Now suppose firms cannot tell who is low risk and who is high risk. To find the pooling contract yielding full insurance and zero expected profits, we find the point on the pooled fair odds line where $W_1 = W_2$. To find the equation for the fair odds line, notice that the probability that a consumer taken at random has an accident is ½ (1/3) + ½ (½) = 5/12. Then the slope of the fair odds line is - 7/5 and the equation can be written as $W_2 = (-7/5)\, W_1 + b$. Since we know that (1,0) is on the line, the intercept, b must equal 7/5, so the full equation is:

$$W_2 = (-7/5)\, W_1 + 7/5.$$

Setting $W_1 = W_2$ and solving, we have $W_1 = W_2 = 7/12$.

(iv) At the point (7/12, 7/12), the low risk consumer's MRS is 2 and the high risk consumer's MRS is 1. Therefore, the low risk's indifference curve is steeper than the pooled fair odds line and the high risk's indifference curve is flatter. Any contract on the pooled fair odds line below (7/12,7/12) will be accepted by the low risk consumer and not by the high risk consumer, and will therefore be profitable. For any sufficiently small and positive ϵ, $(7/12 + \epsilon, 7/12 - 7\epsilon/5)$ works. Many other solutions are possible.